2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

State-Feedback Stabilizability in Switched Homogeneous Systems

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Abstract— **The present article is concerned with state-feedback stabilizability of discrete-time switched homogeneous systems. Necessary and sufficient conditions for state-feedback exponential stabilizability are presented. It is shown that, a switched homogeneous system is state-feedback exponentially stabilizable if and only if an associated sequence converges to zero. Equivalently, a switched homogeneous system is state-feedback exponentially stabilizable if and only if an associated dynamic programming equation admits a solution on a given convex set. This unique solution of that associated dynamic programming equation is shown to be the optimal cost functional of a related infinite-horizon quadratic regulator problem (for the switched homogeneous system) whose solution is also presented. A numerical example illustrates the results reported in the paper.**

I. Introduction

In this paper, the term switched system is used to refer to a class of dynamical system described by a differential or difference equation whose right hand side is dynamically selected from a given finite set of (right hand side) functions, and this selection is governed by a function (of the time) usually refereed to as switching signal.

Switched systems are used to model processes exhibiting significatively different behaviors depending on a state that takes discrete finite values and which describes the mode of operation of the process. Such processes appear in different engineering areas. For instance: In power electronics, [14], [13] various types of power converters are naturally modeled in that manner. In control systems, control schemes are being considered [2] in which a master controller switches between a given finite set of available controllers to close the loop with a given plant. In control systems, switched systems are also being considered [6] to model the complex behavior of processes that are subject to the occurrence of faults.

Some of the recent research, in the area of switched systems, is documented in various survey papers [4], [1], [10] and monographs [13], [9], [5], [11]. Among the important issues that are being studied, some are concerning with certain stability problems related to these systems. In [4], three basic problems concerning the stability of switched systems are recognized: (1) The problem of finding conditions for stability, for arbitrary switching signals. (2) The problem of finding conditions for stability, for switching signals of some given class. (3) The problem of finding conditions for the existence of a switching signal that stabilizes the system. The topic of the present paper is related to (but different from) problem (3). It is concerned with the problem of finding conditions for the existence of a state-feedback that stabilizes the system. More specifically, the present work is devoted to the problem of finding necessary and sufficient conditions to determine the state-feedback exponential stabilizability in a class of discrete-time switched homogeneous systems. The contribution of this work, which extends results reported in [7], is in providing complete and general solutions for that problem that (to our knowledge) had remained unsolved. We present different (but equivalent) necessary and sufficient conditions for the existence of a state-feedback that exponentially stabilizes the switched homogeneous system.

It may be appropriate to mention that in [3] and [12] switched homogeneous systems are also considered, although they are concerned (or related) with the aforementioned basic problem (1), and the problem to which our work is devoted is not addressed in those papers.

The present article is organized as follows. In section II mathematical preliminaries and definitions are introduced. Necessary and sufficient conditions for state-feedback exponential stabilizability of switched homogeneous systems are presented and proved in section III. In this section, a sequence is associated to each switched homogeneous system. We then prove that the state-feedback exponential stabilizability of the switched system is equivalent to the property of convergency (to zero) of the associated sequence. We also prove that the state-feedback exponential stabilizability of the switched system is equivalent to the solvability of an associated dynamic programming equation (on some specific convex set). Results regarding the solvability of the associated dynamic programming equation are in section IV. Some remarks on Lyapunov functions are included in section V. The complete solution of a related infinite-horizon quadratic regulator problem is presented in section VI. A numerical example, included in section VII, illustrates on the results reported in this work. Summary and concluding remarks are in section VIII.

Most of the notation used through the paper is standard. $\mathbb{Z}^+$ denote the non-negative integers. For $k \in \mathbb{Z}^+$, we use $\mathbb{Z}^{[0,k]}$ to also denote the set $\mathbb{Z}^{[0,k]} = \{0, \ldots, k\}$. We use l_+^n to denote the set of all the sequences $\{x_k\} \subset \mathbb{R}^n$, $k \in \mathbb{Z}^+$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its euclidian norm.

II. Preliminaries

Let $N \in \mathbb{Z}^+$, $N > 0$, be given. We will denote by $\mathcal{Q}$ the set $\mathcal{Q} = \{1, \ldots, N\}$. Let us introduce the following sets of

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control functions (or switching signals)

$$\mathcal{Q}_k = \{q : q : \mathbb{Z}^{[0,k-1]} \longrightarrow \mathcal{Q}\}, \ k \in \mathbb{Z}^+, \ k > 0,$$
$$\mathcal{Q}_\infty = \{q : q : \mathbb{Z}^+ \longrightarrow \mathcal{Q}\}.$$

Let the continuous functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $i \in \mathcal{Q}$, satisfying

$$f_i(0) = 0, \text{ and}$$
$$f_i(\lambda x_0) = \lambda f_i(x_0), \ \forall \lambda \in \mathbb{R}^+, \ \forall x_0 \in \mathbb{R}^n, \ \forall i \in \mathcal{Q} \quad (1)$$

be given. The present article is concerned with the dynamical homogeneous system described by

$$x(k+1) = f_{q(k)}(x(k)), \ k \in \mathbb{Z}^+,$$
$$x(0) = x_0 \in \mathbb{R}^n, \ q \in \mathcal{Q}_\infty. \quad (2)$$

The motion of this controlled dynamical system will be denoted by $x(\cdot\,; x_0, q)$.

To each mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ we associate the diagonal (or static) operator $\mathcal{K}_\kappa : l_+^n \longrightarrow \mathcal{Q}_\infty$ defined by

$$\mathcal{K}_\kappa(x)(k) = \kappa(x(k)), \ k \in \mathbb{Z}^+.$$

It is clear that if we also associate to each mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ the (closed-loop) dynamical system described by

$$x_{cl}(k+1) = f_{\kappa(x_{cl}(k))}(x_{cl}(k)), \ k \in \mathbb{Z}^+,$$
$$x_{cl}(0) = x_0 \in \mathbb{R}^n, \quad (3)$$

then, it follows that $x(\cdot\,; x_0, \mathcal{K}_\kappa(x)) = x_{cl}(\cdot\,; x_0)$.

In this work we will adopt the following definitions.

Definition 1: The switched system (2) is state-feedback exponentially stabilizable whenever there exist a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ and scalars $\alpha \geq 1$ and $0 < \beta < 1$ such that the motions of the associated (closed-loop) dynamical system (3) satisfy

$$\|x_{cl}(k; x_0)\| \leq \alpha\beta^k \|x_0\|, \ k \in \mathbb{Z}^+, \ x_0 \in \mathbb{R}^n.$$

Definition 2: The switched system (2) is uniformly exponentially convergent whenever there exist scalars $\alpha \geq 1$ and $0 < \beta < 1$ that obey the following property:
For each $x_0 \in \mathbb{R}^n$ there exists $q_{x_0} \in \mathcal{Q}_\infty$ such that the corresponding motion of (2) satisfies

$$\|x(k; x_0, q_{x_0})\| \leq \alpha\beta^k \|x_0\|, \ k \in \mathbb{Z}^+.$$

III. Feedback Stabilizability of the Switched Homogeneous System

It is convenient to associate to the sets $\mathcal{Q}_k$, $k \in \mathbb{Z}^+$, $k > 0$, of control functions the following sets $\mathcal{F}_k$, $k \in \mathbb{Z}^+$, $k > 0$, of continuous homogeneous functions:

$$\mathcal{F}_k = \{F \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n) : F = f_{q(k-1)} \circ \ldots \circ f_{q(0)}, q \in \mathcal{Q}_k\}.$$

We associate, also, to the switched homogeneous system (2), the sequence of functions $\{V_k\}$, where $V_k : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, is defined by

$$V_k(x_0) = \min_{q \in \mathcal{Q}_k} \|x(k; x_0, q)\|^2 = \min_{F \in \mathcal{F}_k} \|F(x_0)\|^2, \quad (4)$$

and the sequence $\{\mu_k\}$, where $\mu_k \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, is defined by

$$\mu_k = \max_{\|x_0\| \leq 1} V_k(x_0) = \max_{\|x_0\| \leq 1} \min_{q \in \mathcal{Q}_k} \|x(k; x_0, q)\|^2. \quad (5)$$

Some simple properties of $\{V_k\}$ and $\{\mu_k\}$ are collected in the next Fact.

Fact 1: For each given $k \in \mathbb{Z}^+$, $k > 0$, it follows that

(1) V_k is continuous.
(2) $V_k(\lambda x_0) = \lambda^2 V_k(x_0), \ \lambda \in \mathbb{R}^+, \ x_0 \in \mathbb{R}^n.$
(3) $\mu_k = \max_{\|x_0\|=1} V_k(x_0).$
(4) $\mu_k \leq \min_{F \in \mathcal{F}_k} \max_{\|x_0\| \leq 1} \|F(x_0)\|^2.$
(5) For each given $h \in \mathbb{Z}^+$, $h > 0$, it follows that

$$\mu_{hk} \leq \left(\mu_k\right)^h.$$

Next, we present the main result of this work.

Theorem 1: The switched homogeneous system (2), with associated sequence $\{\mu_k\}$, is state-feedback exponentially stabilizable if and only if any (and then all) of the following equivalent conditions are satisfied:

(i) There exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$.
(ii) $\lim_{k \to +\infty} \mu_k = 0.$
(iii) There exists a function $W : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ satisfying
- $W(\lambda x_0) = \lambda^2 W(x_0), \ \lambda \in \mathbb{R}^+, \ x_0 \in \mathbb{R}^n$,
- $\|x_0\|^2 \leq W(x_0) \leq \gamma \|x_0\|^2, \ x_0 \in \mathbb{R}^n$, for some $\gamma > 1$,

which solves the following associated dynamic programming equation:

$$W(x_0) = \|x_0\|^2 + \min_{q \in \mathcal{Q}} W(f_q(x_0)), \ x_0 \in \mathbb{R}^n. \quad (6)$$

Moreover, a function W as in (iii) defines a state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ via

$$\kappa(x_0) \in \arg\min_{q \in \mathcal{Q}} W(f_q(x_0)), \ x_0 \in \mathbb{R}^n. \quad (7)$$

Any such a state-feedback (7) exponentially stabilizes the switched system (2). And the function W, is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system (3).

Proof: This constructive proof is organized as follows. In the Necessity part, we prove that the existence of an exponentially stabilizing feedback for the switched system (2) implies the satisfaction of condition (i). In the Sufficiency part, we prove that the satisfaction of condition (i) implies the satisfaction of conditions (ii) and (iii), and the last one implies the existence of an exponentially stabilizing feedback for the switched system (2). (Trivially, (ii) $\Longrightarrow$ (i).)
(Necessity.) By assumption there exist a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ and scalars $\alpha \geq 1$ and $0 < \beta < 1$ such that the motions of the associated system (3) satisfy

$$\|x_{cl}(k; x_0)\| \leq \alpha\beta^k \|x_0\|, \ k \in \mathbb{Z}^+, \ x_0 \in \mathbb{R}^n.$$

Choose $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\alpha^2\beta^{2k_0} < 1$. And define the following family of control functions:

$$q_{x_0} = \mathcal{K}_\kappa(x_{cl}(\cdot\,; x_0)), \ x_0 \in \mathbb{R}^n, \ \|x_0\| \leq 1.$$

Then, using the definition of V_{k_0}, we have that

$$V_{k_0}(x_0) = \min_{q\in\mathcal{Q}_{k_0}} \|x(k_0; x_0, q)\|^2 \leq \|x(k_0; x_0, q_{x_0})\|^2 = \|x_{cl}(k_0; x_0)\|^2 \leq \alpha^2\beta^{2k_0}\,,\ x_0 \in \mathbb{R}^n\,,\ \|x_0\| \leq 1\,.$$

Therefore,

$$\mu_{k_0} = \max_{\|x_0\|\leq 1} V_{k_0}(x_0) \leq \alpha^2\beta^{2k_0} < 1\,.$$

(Sufficiency.) By assumption there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$. It will be assumed, without loss of generality, that $k_0 > 1$. (Notice that in case that $k_0 = 1$ we can appeal to Fact 1 (property (5)) to define a new $k_0^{\text{new}} = hk_0$ with $h \in \mathbb{Z}^+$, $h > 1$. Thus, $k_0^{\text{new}} > 1$, and moreover $\mu_{k_0^{\text{new}}} \leq (\mu_{k_0})^h < 1$.) Now, for a given $x_0 \in \mathbb{R}^n$, we will consider the optimal control problem

$$\min_{q\in\mathcal{Q}_{k_0}} \|x(k_0; x_0, q)\|^2\,, \tag{8}$$

and we will use $\hat{q}_{k_0,x_0}$ to denote a solution for that problem. Therefore, for any given $x_0 \in \mathbb{R}^n$

$$\|x(k_0; x_0, \hat{q}_{k_0,x_0})\|^2 = \min_{q\in\mathcal{Q}_{k_0}} \|x(k_0; x_0, q)\|^2 = V_{k_0}(x_0) \leq \mu_{k_0}\|x_0\|^2\,.$$

Let us define

$$M_{k_0} = \max\{1\,,\ \max_{F\in\bigcup_{j=1}^{k_0-1}\mathcal{F}_j}\ \max_{\|z\|\leq 1} \|F(z)\|^2\}\,.$$

Let $x_0 \in \mathbb{R}^n$ be given, and let $h \in \mathbb{Z}^+$, $h > 0$, be given. Let $\tilde{q}_{hk_0,x_0} \in \mathcal{Q}_{hk_0}$ be a control function made up by concatenating solutions of the optimal control problem (8) with the following initial conditions:

$$\hat{x}_0 = x_0\,,\ \hat{x}_1 = x(k_0; \hat{x}_0, \hat{q}_{k_0,\hat{x}_0})\,, \ldots\,, \hat{x}_{h-1} = x(k_0; \hat{x}_{h-2}, \hat{q}_{k_0,\hat{x}_{h-2}})\,.$$

That is, using the above notation, the control function $\tilde{q}_{hk_0,x_0}$ is defined by

$$\tilde{q}_{hk_0,x_0}(jk_0 + i) = \hat{q}_{k_0,\hat{x}_j}(i)\,,\quad i \in \{0, \ldots, (k_0 - 1)\}\,,\ j \in \{0, \ldots, (h-1)\}\,.$$

Now, it is easy to see that, with the above defined control function $\tilde{q}_{hk_0,x_0}$ the following inequalities are satisfied:

$$\|x(k; x_0, \tilde{q}_{hk_0,x_0})\|^2 \leq M_{k_0}\mu_{k_0}^j\|x_0\|^2\,,\quad k \in \mathbb{Z}^+\,,\ k \in \{jk_0, \ldots, jk_0 + (k_0-1)\}\,,\quad j \in \{0, \ldots, (h-1)\}\,,\ \text{and}$$

$$\|x(k; x_0, \tilde{q}_{hk_0,x_0})\|^2 \leq M_{k_0}\mu_{k_0}^h\|x_0\|^2\,,\ k = hk_0\,.$$

It is then clear that the above expression implies that $\lim_{k\to+\infty}\mu_k = 0$. In effect, given $\epsilon > 0$ arbitrary, we choose $j_0 \in \mathbb{Z}^+$, $j_0 > 0$, such that $M_{k_0}\mu_{k_0}^{j_0} < \epsilon$. Then, for any $k \in \mathbb{Z}^+$, $k \geq j_0k_0$, it is verified that (where we have chosen $h \in \mathbb{Z}^+$, $h > 0$, such that $k \leq hk_0$; thus $j \geq j_0$)

$$\mu_k = \max_{\|z\|\leq 1} V_k(z) = V_k(x_0) = \min_{q\in\mathcal{Q}_k} \|x(k; x_0, q)\|^2 \leq \|x(k; x_0, \tilde{q}_{hk_0,x_0})\|^2 \leq M_{k_0}\mu_{k_0}^j \leq M_{k_0}\mu_{k_0}^{j_0} < \epsilon\,,$$

where x_0 denotes an optimal solution of the problem $\max_{\|z\|\leq 1} V_k(z)$. For each $k \in \mathbb{Z}^+$, $k > 0$, we now define the cost functional $J_k : \mathbb{R}^n \times \mathcal{Q}_k \longrightarrow \mathbb{R}^+$ by

$$J_k(x_0, q) = \sum_{i=0}^{k} \|x(i; x_0, q)\|^2\,. \tag{9}$$

For any given $x_0 \in \mathbb{R}^n$ we will consider the following family of optimal control problems (where $k \in \mathbb{Z}^+$, $k > 0$):

$$\min_{q\in\mathcal{Q}_k} J_k(x_0, q)\,, \tag{10}$$

and we will denote by $U_k(x_0)$ the optimal values of those problems. It immediately follows that the functions $U_k : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ are continuous and also verify

$$U_k(\lambda x_0) = \lambda^2 U_k(x_0)\ ,\quad \lambda \in \mathbb{R}^+\ ,\ x_0 \in \mathbb{R}^n\ ,\ k \in \mathbb{Z}^+\ ,\ k > 0\,. \tag{11}$$

Moreover, for any given $x_0 \in \mathbb{R}^n$, we have that (choosing $h \in \mathbb{Z}^+$, $h > 0$, such that $k \leq hk_0$)

$$U_k(x_0) = \min_{q\in\mathcal{Q}_k} J_k(x_0, q) \leq \min_{q\in\mathcal{Q}_{hk_0}} J_{hk_0}(x_0, q) \leq J_{hk_0}(x_0, \tilde{q}_{hk_0,x_0}) = \sum_{i=0}^{hk_0} \|x(i; x_0, \tilde{q}_{hk_0,x_0})\|^2 \leq \sum_{j=0}^{h} k_0 M_{k_0}\mu_{k_0}^j\|x_0\|^2 \leq \frac{k_0 M_{k_0}}{(1-\mu_{k_0})}\|x_0\|^2\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,.$$

It was therefore proved that

$$\|x_0\|^2 \leq U_k(x_0) \leq k_0 M_{k_0}\frac{1}{(1-\mu_{k_0})}\|x_0\|^2\,,\quad x_0 \in \mathbb{R}^n\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,. \tag{12}$$

It is also easy to see that the following property is verified:

$$U_{k+1}(x_0) \geq U_k(x_0)\,,\ x_0 \in \mathbb{R}^n\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,. \tag{13}$$

It then follows that, for each given $x_0 \in \mathbb{R}^n$, the limite $\lim_{k\to+\infty} U_k(x_0)$ exists. That fact lead us to the introduction of the function $W : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ defined by

$$W(x_0) = \lim_{k\to+\infty} U_k(x_0)$$

which immediately satisfies

$$\|x_0\|^2 \leq W(x_0) \leq k_0 M_{k_0}\frac{1}{(1-\mu_{k_0})}\|x_0\|^2\,,\ x_0 \in \mathbb{R}^n\,,$$

and also

$$W(\lambda x_0) = \lambda^2 W(x_0)\ ,\ \lambda \in \mathbb{R}^+\ ,\ x_0 \in \mathbb{R}^n\,.$$

Furthermore, since

$$\begin{aligned} U_{k+1}(x_0) &= \left(\|x_0\|^2 + \min_{q\in\mathcal{Q}} U_k(x(1; x_0, q))\right) \\ &= \left(\|x_0\|^2 + \min_{q\in\mathcal{Q}} U_k(f_q(x_0))\right)\,, \\ &\qquad x_0 \in \mathbb{R}^n\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,, \end{aligned}$$

it then follows that W is a solution of the following dynamic programming equation:

$$W(x_0) = \|x_0\|^2 + \min_{q\in\mathcal{Q}} W(f_q(x_0))\,,\ x_0 \in \mathbb{R}^n\,.$$

Hence, there exists a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ satisfying

$$\kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(f_q(x_0))\,,\ x_0 \in \mathbb{R}^n\,.$$

Thus, it is verified that

$$W(f_{\kappa(x_0)}(x_0)) - W(x_0) = -\|x_0\|^2\,,\ x_0 \in \mathbb{R}^n\,,$$

which means that W is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system (3). In effect, it is easy to verify that (with that state-feedback mapping) the motions of the associated closed-loop dynamical system (3) satisfy

$$\|x_{cl}(k; x_0)\| \le \alpha\beta^k\|x_0\|\,,\ k \in \mathbb{Z}^+\,,\ x_0 \in \mathbb{R}^n\,,$$

with

$$\alpha = \sqrt{\frac{k_0 M_{k_0}}{(1-\mu_{k_0})}}\,,\ \beta = \sqrt{\frac{k_0 M_{k_0} - (1-\mu_{k_0})}{k_0 M_{k_0}}}\,,$$

which completes the proof of the Theorem. ■

Remark 1: It follows from the previous result that (in case the switched homogeneous system (2) is state-feedback exponentially stabilizable) a exponentially stabilizing state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$, given by (7), can always be chosen with the following property:

$$\kappa(\lambda x_0) = \kappa(x_0)\,,\ \lambda \in (\mathbb{R}^+ \setminus \{0\})\,,\ x_0 \in \mathbb{R}^n\,.$$

Notice also, that in such a case

$$f_{\kappa(\lambda x_0)}(\lambda x_0) = \lambda f_{\kappa(x_0)}(x_0)\,,\ \lambda \in \mathbb{R}^+\,,\ x_0 \in \mathbb{R}^n\,,$$

therefore, the resulting closed-loop dynamical system (3) is also homogeneous.

Remark 2: We remark that in case the homogeneity property in (1), for the functions f_i, $i \in \mathcal{Q}$, is further held for $\lambda \in \mathbb{R}$ (no just for $\lambda \in \mathbb{R}^+$), then, this property is also inherited by V_k, U_k, ($k \in \mathbb{Z}^+$, $k > 0$), W, κ, and f_κ.

We observe that our proof of Theorem 1 also proves the following result: The switched homogeneous system (2) is uniformly exponentially convergent if and only if there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\mu_{k_0} < 1$. We have therefore the following conclusion from Theorem 1.

Corollary 1: The switched homogeneous system (2) is state-feedback exponentially stabilizable if and only if it is uniformly exponentially convergent.

IV. On the Solvability of the Associated Dynamic Programming Equation

Given the relevance of the associated dynamic programming equation (6), this section is devoted to present results, which are related with Theorem 1, and which are concerned with its solvability, number of solutions, and properties of the solutions. Let us begin by introducing the following sets, $\mathcal{W}^+$, and $\mathcal{W}^{++}$, of (positive semi-definite and positive definite decrescent) functions as

$$\mathcal{W}^+(\mathcal{W}^{++}) = \{\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^+ : \exists\,(\delta_1 > 0\,,)\,\delta_2 > 0 : (\delta_1\|x_0\|^2 \le)\,\Phi(x_0) \le \delta_2\|x_0\|^2\,,\ \forall x_0 \in \mathbb{R}^n\}\,.$$

Theorem 2: The dynamic programming equation (6) associated to the switched homogeneous system (2) has a solution W inside the convex cone $\mathcal{W}^+$, if and only if, the switched homogeneous system (2) is state-feedback exponentially stabilizable. Moreover:

(1) The convex cone $\mathcal{W}^+$ admits at most one solution of the dynamic programming equation (6).

(2) If $W \in \mathcal{W}^+$ is the solution of the dynamic programming equation (6), then, $W \in \mathcal{W}^{++}$ and it has the following properties:

- W is continuous.
- It is verified that

$$W(\lambda x_0) = \lambda^2 W(x_0)\,,\ \lambda \in \mathbb{R}^+\,,\ x_0 \in \mathbb{R}^n\,.$$

- It is also verified that

$$0 \le W(x_0) - U_k(x_0) = W(x_0) - \min_{\Phi\in\mathcal{G}_k} \Phi(x_0) \le \Big(\frac{k_0 M_{k_0}}{(1-\mu_{k_0})} - 1\Big)\frac{1}{\big(1 + \frac{(1-\mu_{k_0})}{k_0 M_{k_0} M_2}\big)^k}\,\|x_0\|^2\,,$$
$$x_0 \in \mathbb{R}^n\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,,$$

where $k_0 \in \mathbb{Z}^+$, $k_0 > 1$, is such that $\mu_{k_0} < 1$,

$$M_l = \max\{1\,,\ \max_{F\in\bigcup_{j=1}^{l-1}\mathcal{F}_j}\ \max_{\|z\|\le 1} \|F(z)\|^2\}\,,$$

$l \in \mathbb{Z}^+$, $l > 1$, and where the sets of functions $\mathcal{G}_k$, $k \in \mathbb{Z}^+$, $k > 0$, are defined as follows:

$$\mathcal{G}_k = \{\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^+ : \Phi(x_0) = \|x_0\|^2 + \sum_{l=1}^{k} \|(f_{q(l-1)} \circ \ldots \circ f_{q(0)})(x_0)\|^2\,,\ q \in \mathcal{Q}_k\}\,.$$

Proof: In Part 1 we prove the necessary and sufficient condition for existence of solution in $\mathcal{W}^+$. And then, in Part 2, the rest of the statement is proved.

Part 1.- (Sufficiency.) It was already proved, in Theorem 1, that if the switched homogeneous system (2) is state-feedback exponentially stabilizable, then, there exists $W \in \mathcal{W}^{++} \subset \mathcal{W}^+$ that solves the dynamic programming equation. (Necessity.) If $W \in \mathcal{W}^+$ and solves the dynamic programming equation (6), then, it is straightforward that $W \in \mathcal{W}^{++}$, with $\delta_1 = 1$, and any state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ obeying $\kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(f_q(x_0))\,,\ x_0 \in \mathbb{R}^n$, exponentially stabilizes the switched homogeneous system (2); since in that case we have

$$W(f_{\kappa(x_0)}(x_0)) - W(x_0) = -\|x_0\|^2\,,\ \forall x_0 \in \mathbb{R}^n\,.$$

Part 2.- Assume $W \in \mathcal{W}^+$ is solution of the dynamic programming equation (6). Then, since the switched homogeneous system (2) is state-feedback stabilizable, it follows from Theorem 1 that there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 1$, such that

$\mu_{k_0} < 1$. As in the sufficiency part of the proof of Theorem 1, we consider the costs functionals $J_k : \mathbb{R}^n \times \mathcal{Q}_k \longrightarrow \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, defined by (9), and for given $x_0 \in \mathbb{R}^n$ we consider the family of optimal control problems defined in (10) and we will denote by $U_k(x_0)$ the optimal values of these problems. It follows (as it was shown in the sufficiency part of the proof of Theorem 1) that the functions $U_k : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ are continuous, and they also obey properties (11), (13), and (12). As a result of all that, we can define a function $U_\infty : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ by

$$U_\infty(x_0) = \lim_{k\to+\infty} U_k(x_0)$$

which therefore satisfies

$$U_\infty(\lambda x_0) = \lambda^2 U_\infty(x_0)\ ,\ \lambda \in \mathbb{R}^+\ ,\ x_0 \in \mathbb{R}^n\ ,$$
$$U_\infty(x_0) \geq U_k(x_0)\ ,\ x_0 \in \mathbb{R}^n\ ,\ k \in \mathbb{Z}^+\ ,\ k > 0\ ,$$

and

$$\|x_0\|^2 \leq U_\infty(x_0) \leq k_0 M_{k_0} \frac{1}{(1-\mu_{k_0})} \|x_0\|^2\ ,\ x_0 \in \mathbb{R}^n\ .$$

Further, since

$$\begin{aligned} U_{k+1}(x_0) &= \left(\|x_0\|^2 + \min_{q\in\mathcal{Q}} U_k(f_q(x_0))\right)\ , \\ &\quad x_0 \in \mathbb{R}^n\ ,\ k \in \mathbb{Z}^+\ , \end{aligned} \tag{14}$$

it then follows that

$$U_\infty(x_0) = \|x_0\|^2 + \min_{q\in\mathcal{Q}} U_\infty(f_q(x_0))\ ,\ x_0 \in \mathbb{R}^n\ .$$

We now claim that $U_\infty = W$. In effect, by assumption $W \in \mathcal{W}^{++}$, that is, there are $\delta_1 > 0$, $\delta_2 > 0$, such that

$$\delta_1\|x_0\|^2 \leq W(x_0) \leq \delta_2\|x_0\|^2\ ,\ x_0 \in \mathbb{R}^n\ . \tag{15}$$

Without loss of generality, we will assume that $\delta_1 \leq 1$, and $\delta_2 > 1$. We now use (14) and (15) to invoke Lemma 1 (stated after this proof) from which it is concluded that

$$\frac{-(\delta_1^{-1}-1)\delta_2}{(1+(\delta_2 M_2)^{-1})^k}\|x_0\|^2 \leq W(x_0) - U_k(x_0) \leq$$
$$\frac{(1-\delta_2^{-1})\delta_2}{(1+(\delta_2 M_2)^{-1})^k}\|x_0\|^2\ ,\ x_0 \in \mathbb{R}^n\ ,\ k \in \mathbb{Z}^+\ ,\ k > 0\ .$$

The above bounds imply that for each given $x_0 \in \mathbb{R}^n$

$$\lim_{k\to+\infty} U_k(x_0) = W(x_0)\ .$$

Hence, $U_\infty = W$, and the claim was proved. The above bounds also imply that

$$\lim_{k\to+\infty} \sup_{\|x_0\|\leq 1} |W(x_0) - U_k(x_0)| = 0\ ,$$

and therefore, the continuity of W follows from the continuity of the functions U_k, $k \in \mathbb{Z}^+$, $k > 0$. The final properties of the solution $W \in \mathcal{W}^+$ follow now from the properties we have already established on U_∞, and (then) by setting $\delta_2 = \frac{k_0 M_{k_0}}{(1-\mu_{k_0})}$ on the above upper bound. ■

In the proof of Theorem 2 we have used the next result that we have adapted from [8] to fit in our setting.

Lemma 1: Consider the switched homogeneous system (2). Let $W : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying

$$\delta_1\|x_0\|^2 \leq W(x_0) \leq \delta_2\|x_0\|^2\ ,\ \forall x_0 \in \mathbb{R}^n\ ,$$

for some given $1 \geq \delta_1 > 0$, $\delta_2 > 0$, be a solution of the associated dynamic programming equation (6). Let $\{U_k\}$, $k \in \mathbb{Z}^+$, be the sequence of functions generated by

$$\begin{aligned} U_{k+1}(x_0) &= \left(\|x_0\|^2 + \min_{q\in\mathcal{Q}} U_k(f_q(x_0))\right)\ , \\ &\quad U_0(x_0) = \|x_0\|^2\ ,\ x_0 \in \mathbb{R}^n\ ,\ k \in \mathbb{Z}^+\ . \end{aligned}$$

Then, under these conditions,

$$\frac{-(\delta_1^{-1}-1)}{(1+(\delta_2 M_2)^{-1})^k} W(x_0) \leq W(x_0) - U_k(x_0) \leq$$
$$\frac{(1-\delta_2^{-1})}{(1+(\delta_2 M_2)^{-1})^k} W(x_0)\ ,\ x_0 \in \mathbb{R}^n\ ,\ k \in \mathbb{Z}^+\ ,$$

where $M_2 = \max\{1\,, \max_{q\in\mathcal{Q}} \max_{\|z\|\leq 1} \|f_q(z)\|^2\}$.

V. Some Remarks on Lyapunov Functions

A simple consequence of Theorem 2, which deserves some comments, is the next result.

Corollary 2: If the switched homogeneous system (2) is state-feedback exponentially stabilizable, then, there exists $k_1 \in \mathbb{Z}^+$, $k_1 > 0$, having the following property:

- Every state-feedback mapping $\kappa_k : \mathbb{R}^n \longrightarrow \mathcal{Q}$, $k \in \mathbb{Z}^+$, $k \geq k_1$, satisfying
$$\kappa_k(x_0) \in \arg\min_{q\in\mathcal{Q}} \min_{\Phi\in\mathcal{G}_k} \Phi(f_q(x_0))\ ,\ x_0 \in \mathbb{R}^n\ , \tag{16}$$
exponentially stabilizes the switched system (2).
- Moreover, the corresponding function U_k, $k \in \mathbb{Z}^+$, $k \geq k_1$, which can be expressed as
$$U_k(x_0) = \min_{\Phi\in\mathcal{G}_k} \Phi(x_0)\ ,\ x_0 \in \mathbb{R}^n\ , \tag{17}$$
is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop system (3) (for the corresponding feedback mapping κ_k).

Proof: By Theorem 2 there exists $W \in \mathcal{W}^{++}$ solution of the associated dynamic programming equation (6). Choose ϵ_1 such that $0 < \epsilon_1 < 1$. It is a consequence of Theorem 2 that there is $k_1 \in \mathbb{Z}^+$, $k_1 > 0$, such that for every $k \in \mathbb{Z}^+$, $k \geq k_1$, we have

$$0 \leq W(x_0) - U_k(x_0) \leq (1-\epsilon_1)\|x_0\|^2\ ,\ \forall x_0 \in \mathbb{R}^n\ .$$

Hence, for every $k \in \mathbb{Z}^+$, $k \geq k_1$,

$$\min_{q\in\mathcal{Q}} U_k(f_q(x_0)) - U_k(x_0) \leq -\epsilon_1\|x_0\|^2\ ,\ \forall x_0 \in \mathbb{R}^n\ .$$

Noticing that U_k, $k \in \mathbb{Z}^+$, $k > 0$, obeys property (12) ($\Longrightarrow U_k \in \mathcal{W}^{++}$) completes the proof. ■

Remark 3: Under the hypothesis of Corollary 2, exponentially stabilizing state-feedback mappings $\kappa_k : \mathbb{R}^n \longrightarrow \mathcal{Q}$, $k \in \mathbb{Z}^+$, $k \geq k_1$, given by (16), can always be chosen having the following property:

$$\kappa_k(\lambda x_0) = \kappa_k(x_0)\ ,\ \lambda \in (\mathbb{R}^+ \setminus \{0\})\ ,\ x_0 \in \mathbb{R}^n\ .$$

Remark 4: The above result says that, under the present hypothesis, it can always be used (for the exponential stability property of the trivial solution of a κ_k-stabilized associated closed-loop system (3)) Lyapunov functions which are defined as in (17). That is, they are defined as the point-wise minimum over the finite set of functions $\mathcal{G}_k$. Notice that, in case that the switched system is linear; that is, when for given $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{Q}$, the switched homogeneous system (2) is defined by the functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying $f_i(x_0) = A_i x_0$, $\forall x_0 \in \mathbb{R}^n$, $i \in \mathcal{Q}$, then, the aforementioned Lyapunov functions U_k, $k \in \mathbb{Z}^+$, $k \geq k_1$, can be written as

$$U_k(x_0) = \min_{P\in\mathcal{P}_k} x_0^* P x_0 \, , \ x_0 \in \mathbb{R}^n \, ,$$

where the finite sets $\mathcal{P}_k \subset \mathbb{R}^{n\times n}$, $k \in \mathbb{Z}^+$, $k > 0$, are

$$\mathcal{P}_k = \{P \in \mathbb{R}^{n\times n} \, : \, P = I + \sum_{l=1}^{k} (A_{q(l-1)} \dots A_{q(0)})^* (A_{q(l-1)} \dots A_{q(0)}) \, , \, q \in \mathcal{Q}_k\} \, .$$

Remark 5: Notice that, from Corollary 2 and Theorem 2, by setting $N = 1$, we recover a well-known Lyapunov converse result. In effect, we have also proved that if the trivial solution of the dynamical homogenous system described by

$$x(k+1) = f(x(k)) \, , \ k \in \mathbb{Z}^+ \, , \ x(0) = x_0 \in \mathbb{R}^n \, ,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous, $f(0) = 0$, and $f(\lambda x_0) = \lambda f(x_0)$, $\lambda \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^n$, is exponentially stable, then, there exists $k_1 \in \mathbb{Z}^+$, $k_1 > 0$, such that for every $k \in \mathbb{Z}^+$, $k \geq k_1$, the function U_k defined by

$$U_k(x_0) = \|x_0\|^2 + \sum_{l=1}^{k} \|(\underbrace{f \circ \ldots \circ f}_{l})(x_0)\|^2 \, , \ x_0 \in \mathbb{R}^n$$

is a Lyapunov function for that stability property. Moreover, the above defined sequence $\{U_k\}$, $k \in \mathbb{Z}^+$, $k > 0$, converges uniformly on compacts to the Lyapunov function W, where

$$W(x_0) = \lim_{k\to+\infty} \big(\|x_0\|^2 + \sum_{l=1}^{k} \|(\underbrace{f \circ \ldots \circ f}_{l})(x_0)\|^2\big) \, , \ x_0 \in \mathbb{R}^n$$

satisfies $W(f(x_0)) - W(x_0) = -\|x_0\|^2$, $x_0 \in \mathbb{R}^n$.

VI. On a Related Quadratic Regulator Problem

In this section we present a complete solution for a related optimal control problem involving the switched homogeneous system (2): A related infinite-horizon quadratic regulator problem. We begin with the following definition.

Definition 3: The switched system (2) is said to be uniformly quadratically bounded whenever there exists a scalar $\gamma \geq 1$ that obeys the following property:
For each $x_0 \in \mathbb{R}^n$ there exists $q_{x_0} \in \mathcal{Q}_\infty$ such that the corresponding motion of (2) satisfies

$$\sum_{i=0}^{k} \|x(i; x_0, q_{x_0})\|^2 \leq \gamma \|x_0\|^2 \, , \ k \in \mathbb{Z}^+ \, .$$

Let us introduce the following cost functional $J_\infty : \mathbb{R}^n \times \mathcal{Q}_\infty \longrightarrow (\mathbb{R}^{(\mathrm{ext})})^+$ defined as

$$J_\infty(x_0, q) = \lim_{k\to+\infty} \sum_{i=0}^{k} \|x(i; x_0, q)\|^2 \, . \tag{18}$$

For each given $x_0 \in \mathbb{R}^n$, we will consider (and solve) in this section, the following optimal control problem:

$$\inf_{q\in\mathcal{Q}_\infty} J_\infty(x_0, q) \, , \tag{19}$$

for which it will be denoted by $U : \mathbb{R}^n \longrightarrow (\mathbb{R}^{(\mathrm{ext})})^+$ the optimal cost functional

$$U(x_0) = \inf_{q\in\mathcal{Q}_\infty} J_\infty(x_0, q) \, . \tag{20}$$

We further associate, to the switched homogeneous system (2), the sequence $\{\upsilon_k\}$, $k \in \mathbb{Z}^+$, $k > 0$, defined as

$$\upsilon_k = \max_{\|x_0\|\leq 1} U_k(x_0) = \max_{\|x_0\|\leq 1} \min_{q\in\mathcal{Q}_k} J_k(x_0, q) \, . \tag{21}$$

Now, regarding the above posed quadratic optimal control problem (19) we have the following result.

Theorem 3: The optimal cost functional U, defined in (20), is continuous at $x_0 = 0$, if and only if, any (and then all) of the following equivalent conditions are satisfied:

(i) The switched homogeneous system (2) is uniformly quadratically bounded.
(ii) The switched homogeneous system (2) is state-feedback exponentially stabilizable.
(iii) The associated sequence $\{\upsilon_k\}$, $k \in \mathbb{Z}^+$, $k > 0$, is bounded.

Further, in case the above conditions are satisfied, we have that

$$U(x_0) = \min_{q\in\mathcal{Q}_\infty} J_\infty(x_0, q) = W(x_0) \, , \ x_0 \in \mathbb{R}^n \, ,$$

where $W \in \mathcal{W}^{++}$ is the solution of the associated dynamic programming equation (6). Moreover,

$$\hat{q}_{x_0} = \mathcal{K}_\kappa(x(\cdot; x_0, \mathcal{K}_\kappa(x))) \, , \ x_0 \in \mathbb{R}^n$$

with

$$\kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(f_q(x_0)) \, , \ x_0 \in \mathbb{R}^n$$

is an optimal solution for the optimal control problem under consideration.

Proof: (Sufficiency.) Since the switched homogeneous system (2) is uniformly quadratically bounded, then

$$0 \leq U(x_0) \leq J_\infty(x_0, q_{x_0}) \leq \gamma \|x_0\|^2 \, , \ x_0 \in \mathbb{R}^n \, ,$$

which implies the continuity of U at $x_0 = 0$.
(Necessity.) By the hypothesis of continuity, there is $\delta > 0$ such that $U(x_0) \leq 1$, $\forall x_0 \in \mathbb{R}^n : \|x_0\| \leq \delta$. Since U is homogeneous degree-two, we can write

$$U(x_0) \leq \frac{1}{\delta^2} \|x_0\|^2 \, , \ \forall x_0 \in \mathbb{R}^n \, ,$$

from which it clearly follows, after choosing $\gamma = \frac{2}{\delta^2}$, the uniformly quadratically boundness of the switched homogeneous system (2).

((i) $\Longrightarrow$ (iii)) If the switched system (2) is uniformly quadratically bounded, we can write

$$U_k(x_0) \leq J_k(x_0, q_{x_0}) \leq \gamma \|x_0\|^2 \,,\; x_0 \in \mathbb{R}^n \,,\; k \in \mathbb{Z}^+ \,,\; k > 0$$

implying that $\upsilon_k \leq \gamma$, $k \in \mathbb{Z}^+$, $k > 0$.

((iii) $\Longrightarrow$ (ii)) Invoking Lemma 2 (stated after this proof), the above upper bound gives

$$\mu_k \leq \upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j}) \leq \gamma (1 - \frac{1}{\gamma})^k \,,\; k \in \mathbb{Z}^+ \,,\; k > 0 \,,$$

hence there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, obeying $\mu_{k_0} < 1$, and from Theorem 1, this implies the switched system (2) is state-feedback exponentially stabilizable.

((ii) $\Longrightarrow$ (i)) If there exists a state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ which exponentially stabilizes the switched system (2), then, such a mapping generates control functions $q_{x_0} \in \mathcal{Q}_\infty$ via $q_{x_0} = \mathcal{K}_\kappa(x(\cdot; x_0, \mathcal{K}_\kappa(x)))$, $x_0 \in \mathbb{R}^n$, and also, there exist scalars $\alpha \geq 1$ and $0 < \beta < 1$ for which the motions of (2) hold

$$\|x(k; x_0, q_{x_0})\| \leq \alpha \beta^k \|x_0\| \,,\; k \in \mathbb{Z}^+ \,,\; x_0 \in \mathbb{R}^n \,.$$

Therefore, the conditions in Definition 3 are clearly met with $\gamma = \frac{\alpha^2}{(1-\beta^2)}$.

Now, we prove the last part of the Theorem. By assumption there is $W \in \mathcal{W}^{++}$, the unique solution of the associated dynamic programming equation (6). Consider a state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ defined as in (7), and consider the family of control functions $\hat{q}_{x_0} \in \mathcal{Q}_\infty$, $x_0 \in \mathbb{R}^n$, generated when closing the loop with the above feedback, that is

$$\hat{q}_{x_0} = \mathcal{K}_\kappa(x(\cdot; x_0, \mathcal{K}_\kappa(x))) \,,\; x_0 \in \mathbb{R}^n \,.$$

Then, for each given $x_0 \in \mathbb{R}^n$, we have that

$$J_\infty(x_0, \hat{q}_{x_0}) = \lim_{k \to +\infty} \sum_{i=0}^{k} \|x(i; x_0, \hat{q}_{x_0})\|^2 = \lim_{k \to +\infty} \big(W(x_0) - W(x(k+1; x_0, \hat{q}_{x_0})) \big) = W(x_0) \,.$$

Now, for each given $x_0 \in \mathbb{R}^n$, let $q_{x_0} \in \mathcal{Q}_\infty$ be any control function for which $J_\infty(x_0, q_{x_0})$ is finite. It then follows that $\lim_{i \to +\infty} \|x(i; x_0, q_{x_0})\|^2 = 0$, implying that $\lim_{i \to +\infty} W(x(i; x_0, q_{x_0})) = 0$. It also follows that

$$J_\infty(x_0, q_{x_0}) = \lim_{k \to +\infty} \sum_{i=0}^{k} \|x(i; x_0, q_{x_0})\|^2 \geq \lim_{k \to +\infty} \sum_{i=0}^{k} \big(W(x(i; x_0, q_{x_0})) - W(x(i+1; x_0, q_{x_0})) \big) = \lim_{k \to +\infty} \big(W(x_0) - W(x(k+1; x_0, q_{x_0})) \big) = W(x_0) \,,$$

which completes the proof of the Theorem. ■

In the proof of Theorem 3 we had to use the following result that provides with a relation between the associated sequences $\{\upsilon_k\}$ and $\{\mu_k\}$.

Lemma 2: Let $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}^+$, $k > 0$, be given. Let $\hat{q}_{k,x_0} \in \mathcal{Q}_k$ be such that

$$U_k(x_0) = J_k(x_0, \hat{q}_{k,x_0}) = \min_{q \in \mathcal{Q}_k} J_k(x_0, q) \,.$$

Then, under these conditions,

$$\|x(k; x_0, \hat{q}_{k,x_0})\|^2 \leq \upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j}) \|x_0\|^2 \,.$$

Therefore,

$$V_k(x_0) \leq \upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j}) \|x_0\|^2 \,, \text{ and}$$

$$\mu_k \leq \upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j}) \,.$$

Proof: Let us shorten the notation by using $x_i = x(i; x_0, \hat{q}_{k,x_0})$, $i = 0, \ldots, k$, and $z_i = \|x(i; x_0, \hat{q}_{k,x_0})\|^2$, $i = 0, \ldots, k$. By the optimality of $\hat{q}_{k,x_0} \in \mathcal{Q}_k$ it follows that

$$\sum_{i=j}^{k} z_i = U_{k-j}(x_j) \leq \upsilon_{k-j} \|x_j\|^2 = \upsilon_{k-j}\, z_j \,, \quad j \in \{0, \ldots, k-1\} \,.$$

Therefore, in order to compute an upper bound for z_k, we use the above k linear inequalities to pose and solve the following linear programming problem (in the variables $z_1, \ldots, z_k$):

$$\begin{aligned} &\max \; z_k \\ &\text{s.t.} \\ &X_k \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \leq b_k \,, \\ &z_1 \geq 0 \,, \ldots \,, z_k \geq 0 \,, \end{aligned}$$

where $X_k \in \mathbb{R}^{k \times k}$, $b_k \in \mathbb{R}^k$ are

$$X_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ (1 - \upsilon_{k-1}) & 1 & 1 & \cdots & 1 & 1 \\ 0 & (1 - \upsilon_{k-2}) & 1 & \cdots & 1 & 1 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & (1 - \upsilon_1) & 1 \end{pmatrix} ,$$

$$b_k = \big((\upsilon_k - 1) z_0 \quad 0 \quad 0 \quad \cdots \quad 0 \big)^* \,.$$

The above linear programming problem can easily be explicitly solved as a function of the above data. Therefore, the proof of the Lemma is completed from the fact that the optimal value for that linear programming problem is given by $\upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j})\, z_0$. ■

VII. A Numerical Example

The next example illustrates on results here reported.

Example 1: Consider the switched system (2) defined by $N = 2$ and $f_i(x_0) = A_i x_0$, $i \in \{1, 2\}$, where

$$A_1 = \begin{pmatrix} 1.025 & 1.5 \\ 0 & 1.005 \end{pmatrix} ,\; A_2 = \begin{pmatrix} 0.825 & 0.6 \\ -0.6 & 0.825 \end{pmatrix} .$$

For these matrices, we have that, $|\lambda_1(A_1)| = 1.005$, $|\lambda_2(A_1)| = 1.025$, and $|\lambda_1(A_2)| = |\lambda_2(A_2)| \approx 1.02$. Thus, neither of these matrices have (Schur) stable invariant subspaces. Numerical computations were performed

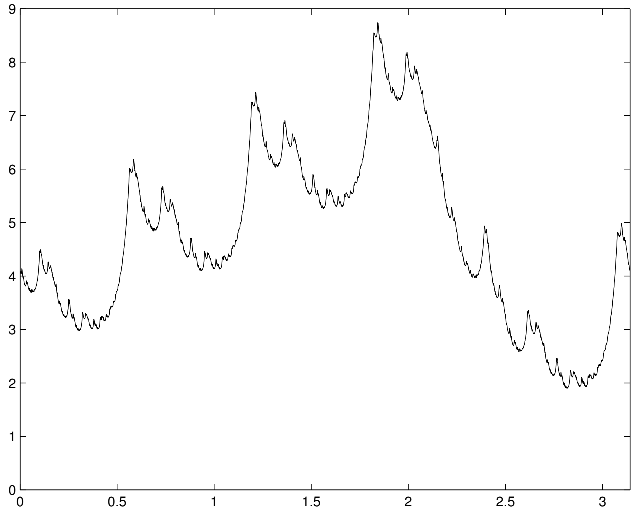

Fig. 1. Graphical representation of W for the system in Example 1.

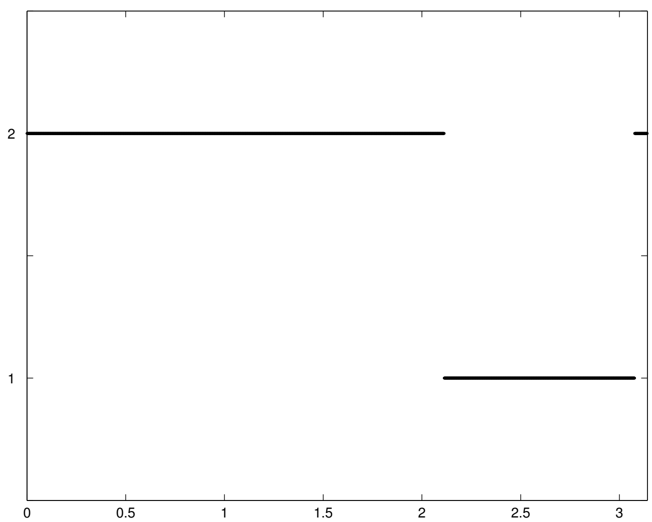

Fig. 2. Graphical representation of a stabilizing state-feedback mapping κ for the system in Example 1.

to evaluate some of the elements of the sequence $\{\mu_k\}$. These computations give that $\mu_8 \approx 0.3453$. Then, using Theorem 1, we conclude that the switched system is state-feedback exponentially stabilizable. Using the recursion (14) we computed the solution W for the dynamic programming equation (6). The graphical representation of the computed $W(\begin{pmatrix}\cos(\cdot)\\ \sin(\cdot)\end{pmatrix}) : [0, \pi) \longrightarrow \mathbb{R}^+$ is shown in Figure 1. A stabilizing state-feedback mapping, $\kappa : \mathbb{R}^2 \longrightarrow \{1, 2\}$, defined via $\kappa(x) \in \arg\min_{q\in\{1,2\}} W(A_q x)$ and satisfying the property discussed in Remark 1 was also computed. Figure 2 depicts the graphical representation of such a computed mapping $\kappa(\begin{pmatrix}\cos(\cdot)\\ \sin(\cdot)\end{pmatrix}) : [0, \pi) \longrightarrow \{1, 2\}$. Notice that, according with Remark 2, the above functions only needed to be computed on $[0, \pi)$.

VIII. Summary and Concluding Remarks

We have proved that a discrete-time switched homogeneous system is state-feedback exponentially stabilizable if and only if its associated sequence $\{\mu_k\}$ converges to zero; or equivalently, if and only if there is $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\mu_{k_0} < 1$. It was also shown that a switched homogeneous system is state-feedback exponentially stabilizable if and only if an associated dynamic programming equation has a solution W on a convex cone. Such a solution, W, which was shown to be unique, defines (that is, provides us with) a stabilizing state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ via $\kappa(x) \in \arg\min_{q\in\mathcal{Q}} W(f_q(x))$. Such a mapping κ can always be chosen to be degree-zero homogeneous, or in other words, conic-wise constant, resulting in a closed-loop dynamical system having the same homogeneity property as that of the original switched system. That function W, is a degree-two homogeneous Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system. It was also proved in this work, that a switched homogeneous system is state-feedback exponentially stabilizable, if and only if, it is uniformly exponentially convergent, if and only if, it is uniformly quadratically bounded. Further, the state-feedback exponential stabilizability of the switched homogeneous system was related to an infinite-horizon quadratic regulator problem whose solution was also presented. The optimal cost functional for that quadratic regulator problem was shown to be the aforementioned function W, and a state-feedback mapping κ, as above, was shown to generate an optimal control. A numerical example provided illustration on the results reported.

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