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Output Regulation for a Class of Weakly Minimum Phase Systems and Its Application to a Nonlinear Benchmark System

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Abstract— **In this paper, we study the global robust output regulation for a class of weakly minimum phase nonlinear systems. By employing the internal model design technique, we first convert the problem into a global robust stabilization problem of an augmented system whose structure has not been encountered before. Then we develop a methodology to solve this stabilization problem via state feedback, by utilizing the saturation function. Finally, the methodology is applied to solve the disturbance rejection problem of the RTAC system.**

I. Introduction

In this paper, we study the global robust output regulation problem of a class of weakly minimum phase nonlinear systems as follows:

$$\begin{array}{rcl} \dot{z} & = & Az + f_0(x_1, v, w) \\ \dot{x}_i & = & f_i(x_1, x_2, \cdots, x_i, v, w) + x_{i+1} \\ e & = & x_1 \\ \dot{v} & = & A_1 v \end{array} \quad (1)$$

where $z \in \mathbb{R}^{n_z}$ and $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ are the states of the lower triangular plant, $x_{n+1} = u \in \mathbb{R}$ the control input, $v \in \mathbb{R}^q$ the exogenous signal representing the disturbance and/or the reference input, $w \in \mathbb{R}^{n_w}$ the uncertain parameter; the functions f_i are *smooth* and globally defined, satisfying $f_i(0, \cdots, 0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$, $i = 0, 1, \cdots, n$; A and A_1 are constant matrices with suitable dimensions; all the eigenvalues of A_1 are simple with zero real part, while A is critically stable.

The global robust output regulation problem of a nonlinear system has been one of the central nonlinear control problems over the last decade. Most papers are focused on the lower triangular systems [5] and [12]. These papers utilize some sort of high gain feedback control. As a result, these papers invariably assume that the systems under consideration are minimum phase. It can be seen that the zero dynamics of the system (1) is $\dot{z} = Az$ where A is critically stable, hence the high gain feedback control technique cannot handle this system. On the other hand, it is known that the saturated control technique can handle systems with critically stable zero dynamics [2] and [13]. Nevertheless, so far this technique has not been applied to lower triangular systems. In this paper, we will combine the saturated control technique and a recursive robust control design method as can be found in [4] and [5] to deal with our problem.

The problem studied in this paper is motivated by the disturbance rejection problem of the Rotational/Translational ACtuator (RTAC) or Translational Oscillator with a Rotational Actuator (TORA) describe in Section IV. The system was introduced in [1] and various control problem associated with this system have been studied over the last decade, see, for example, [3], [4], [8], [9] and [10]. The problem described in Section IV of this paper has not been studied yet, and the result of this paper will lead to a solution to this problem.

Throughout the paper, we will let $\mathcal{L}_\infty^1$ be the set of all piecewise continuous functions $u : [0, \infty) \to \mathbb{R}$ with a finite supremum norm $\|u\|_\infty = \sup_{t>0} \|u(t)\|$, and let $\|u\|_a = \lim_{t\to 0} \sup \|u(t)\|$, where $\|\cdot\|$ denotes the standard Euclidean norm. A function $\gamma : [0, \infty) \to [0, \infty)$ is called a gain function if it is continuously differentiable and belongs to class $\mathcal{K}_\infty$. Let $*$ denote any constant matrix element, and let $d(t)$ denotes a piecewise continuous time-varying parameter taking values from a given compact set $\mathcal{D} \subset \mathbb{R}^{q+n_w}$.

II. Problem Formulation and Preliminaries

We will consider a dynamic state feedback controller in the following form:

$$u = k_\eta(z, x, \eta), \quad \dot{\eta} = f_\eta(z, x, \eta) \quad (2)$$

where η is the compensator state of dimension n_η to be specified later, and k_η and f_η are sufficiently smooth functions vanishing at the origin.

The global robust output regulation problem (GRORP) for system (1) is described as follows. Given any $\mathcal{V} \subset \mathbb{R}^q$, $\mathcal{W} \subset \mathbb{R}^{n_w}$ with $\mathcal{V}$ and $\mathcal{W}$ compact subsets, design a control law of the form (2) such that the closed-loop system has the two properties as follows.

1) For any $v(0) \in \mathcal{V}$, $w \in \mathcal{W}$, and any initial state of the plant (1) and the controller (2), the trajectory of the closed-loop system exists and is bounded for all $t \geq 0$; and the tracking error $e(t)$ approaches zero; asymptotically, i.e.,

$$\lim_{t\to\infty} e(t) = 0.$$

2) When $v(t) = 0$ for all $t \geq 0$, the closed-loop system is globally asymptotically stable (GAS) and locally exponentially stable (LES),

Remark 2.1: The problem description here is different from what is given in [4] [5], and [12] in that objection 1) is explicitly stated. Without objection 1), the problem can be

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trivially solved by applying the methodology of [5] to the last three equations. In such a case, the state z is left uncontrolled and the steady-state trajectory of z will be dictated by the initial state $z(0)$ which is undesirable. Our approach will drive the solution of the closed-loop system to an invariant manifold parameterized by (v, w) asymptotically. In particular, when $v = 0$, the equilibrium of the closed-loop system at the origin will be globally asymptotically stable and locally exponentially stable. For this purpose, we have to develop a more sophisticated approach utilizing a more complex internal model, and saturation function.

Assumption 2.1: There exists a sufficiently smooth function $\mathbf{z}(v, w)$, with $\mathbf{z}(0, 0) = 0$, satisfying the following equation for all $v \in \mathbb{R}^q$ and $w \in \mathbb{R}^{n_w}$:

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} A_1 v = A\mathbf{z}(v,w) + f_0(0,v,w) \tag{3}$$

Under Assumption 2.1, let $\mathbf{x}_1(v, w) = 0$ and

$$\mathbf{x}_{i+1}(v,w) = \frac{\partial \mathbf{x}_i(v,w)}{\partial v} A_1 v - f_i(\mathbf{x}_1, \cdots, \mathbf{x}_i, v, w),\ i = 1, \cdots, n$$

Also, let $\mathbf{x}(v, w) = (\mathbf{x}_1(v, w), \cdots, \mathbf{x}_n(v, w))$, $\mathbf{u}(v, w) = \mathbf{x}_{n+1}(v, w)$. Then, for all $v \in \mathbb{R}^q$, $w \in \mathbb{R}^{n_w}$, $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$ satisfy

$$\begin{aligned} \mathbf{x}_1(v,w) &= 0 \\ \frac{\partial \mathbf{x}_i(v,w)}{\partial v} A_1 v &= f_i(\mathbf{x}_1, \cdots, \mathbf{x}_i, v, w) + \mathbf{x}_{i+1}(v,w) \end{aligned}$$

where $i = 1, \cdots, n$.

In other words, $\mathbf{z}(v, w)$, $\mathbf{x}(v, w)$, and $\mathbf{u}(v, w)$ are the solution of the regulator equations associated with (1).

Assumption 2.2: There exist sufficiently smooth functions $\pi_i(v, w)$, $i = 1, \cdots, n$, vanishing at $(0, 0)$, such that

$$\begin{aligned} \dot{\pi}_i(v,w) &= \Phi_i \pi_i(v,w) \\ \mathbf{x}_{i+1}(v,w) &= \Psi_i \pi_i(v,w) \\ \mathbf{z}(v,w) &= \Psi_z \pi_n(v,w) \end{aligned} \tag{4}$$

where $\Psi_i \in \mathbb{R}^{1\times r_i}$, $\Phi_i \in \mathbb{R}^{r_i\times r_i}$, $\Psi_z \in \mathbb{R}^{n_z\times r_n}$ and $\pi_i(v, w) : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}^{r_i}$ with the pair (Ψ_i, Φ_i) observable and all the eigenvalues of Φ_i simple with zero real part.

Remark 2.2: Under Assumption 2.2, given a pair of controllable matrices (M_i, N_i) with $M_i \in \mathbb{R}^{r_i\times r_i}$ Hurwitz and $N_i \in \mathbb{R}^{r_i\times 1}$ a column vector, there is a unique and nonsingular matrix $T_i \in \mathbb{R}^{r_i\times r_i}$ satisfying the Sylvester equation [11]

$$T_i \Phi_i - M_i T_i = N_i \Psi_i \tag{5}$$

For $i = 1, \cdots, n$, define

$$\dot{\eta}_i = M_i \eta_i + N_i x_{i+1} - L_i(x_1, \cdots, x_i, \eta_1, \cdots, \eta_{i-1}) \tag{6}$$

where L_i is a linear function satisfying $L_i(\mathbf{x}_1(v, w), \cdots, \mathbf{x}_i(v, w), T_1\pi_1, \cdots, T_{i-1}\pi_{i-1}) = 0$.

It can be verified that (6) is an *internal model* for (1) in the sense of Definition 6.6 in [4].

Remark 2.3: The internal model as defined in (6) is different from the canonical form

$$\dot{\eta}_i = M_i \eta_i + N x_{i+1},$$

which is used in [5], [11], [12]. This is because that we need to render the linearization of the augmented system (16) introduced later certain stabilizability property described in Remark 2.6.

Attaching the internal model (6) to the original system (1) leads to the so called augmented system. Stabilizability of the augmented system in the sense described in Remark 2.4 implies the solvability of the output regulation problem of the original system (1).

Define the coordinate and input transformation

$$\begin{aligned} \bar{z} &= z - \Psi_z T_n^{-1} \eta_n \\ \bar{x}_1 &= x_1 \\ \bar{x}_{i+1} &= x_{i+1} - \Psi_i T_i^{-1} \eta_i \\ \bar{\eta}_i &= \eta_i - T_i \pi_i \end{aligned} \tag{7}$$

where $i = 1, \cdots, n$, and let $\bar{u} = \bar{x}_{n+1}$, $\bar{\eta} = [\bar{\eta}_1^T, \ldots, \bar{\eta}_n^T]^T$, and $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]^T$.

Performing the transformation on (1) and (6) yields

$$\begin{aligned} \dot{\bar{z}} &= \dot{x} - \Psi_z T_n^{-1} \dot{\eta}_n \\ &= Az + f_0 - \Psi_z T_n^{-1}(M_n \eta_n + N_n u - L_n) \\ &= A\bar{z} + (A\Psi_z - \Psi_z \Phi_n) T_n^{-1} \bar{\eta}_n \\ &\quad - \Psi_z T_n^{-1} N_n \bar{u} + \bar{f}_0 + \Psi_z T_n^{-1} L_n \\ \dot{\bar{\eta}}_i &= \dot{\eta}_i - T_i \dot{\pi}_i \\ &= M_i \eta_i + N_i x_{i+1} - L_i - T_i \Phi_i \pi_i \\ &= (M_i + N_i \Psi_i T_i^{-1}) \bar{\eta}_i + N_i \bar{x}_{i+1} - L_i \\ \dot{\bar{x}}_i &= \dot{x}_i - \Psi_{i-1} T_{i-1}^{-1} \dot{\eta}_{i-1} \\ &= f_i + x_{i+1} + \Psi_z T_{i-1}^{-1}(M_{i-1}\eta_{i-1} + N_{i-1} x_i - L_{i-1}) \\ &= \bar{f}_i + \Psi_i T_i^{-1} \bar{\eta}_i + \bar{x}_{i+1} \end{aligned} \tag{8}$$

for $i = 1, \cdots, n$, where

$$\begin{aligned} \bar{f}_0 &= f_0 + (A\Psi_z - \Psi_z \Phi_n)\pi_n \\ \bar{f}_1 &= f_1 + \Psi_1 \pi_1 \\ \bar{f}_i &= f_i + \Psi_i \pi_i - \Psi_{i-1}\Phi_{i-1}(\pi_{i-1} + T_{i-1}^{-1}\bar{\eta}_{i-1}) \\ &\quad - \Psi_{i-1} T_{i-1}^{-1}(N_{i-1}\bar{x}_i - L_{i-1}) \\ i &= 2, \cdots, n. \end{aligned} \tag{9}$$

Remark 2.4: The augmented system (8) has the property that when $\bar{u} = 0$, $(\bar{\eta}, \bar{x}, \bar{z})$ is the equilibrium point for all (v, w) and e is the first component of $\bar{x}$. Therefore, if there exists a controller that make the equilibrium of the augmented system (8) GAS and LES regardless of (v, w), then the GRORP of (1) is solvable.

Next, we want to convert system (8) in a block lower triangular form. For this purpose, performing another coordinate transformation

$$\tilde{z} = \bar{z} + \Psi_z T_n^{-1} N_n \bar{x}_n,\ \tilde{\eta}_i = \bar{\eta}_i - N_i \bar{x}_i,\ i = 1, \cdots, n \tag{10}$$

on (8) yields

$$\begin{aligned}\dot{\tilde{z}}=&\dot{\bar{z}}+\Psi_z T_n^{-1}N_n\dot{\bar{x}}_n\\=&A\tilde{z}+\bar{f}_0-\Psi_z T_n^{-1}(M_nN_n\bar{x}_n-N_n\bar{f}_n-L_n)\\&+(A\Psi_zT_n^{-1}-\Psi_zT_n^{-1}M_n)\tilde{\eta}_n\\\dot{\tilde{\eta}}_i=&\dot{\bar{\eta}}_i-N\dot{\bar{x}}_i\\=&(M_i+N_i\Psi_iT_i^{-1})\bar{\eta}_i+N_i\bar{x}_{i+1}-L_i\\&-N_i(\bar{f}_i+\Psi_iT_i^{-1}\bar{\eta}_i+\bar{x}_{i+1})\\=&M_i\tilde{\eta}_i+M_iN_i\bar{x}_i-N_i\bar{f}_i-L_i\\\dot{\bar{x}}_i=&\bar{f}_i+\Psi_iT_i^{-1}\tilde{\eta}_i+\Psi_iT_i^{-1}N_i\bar{x}_i+\bar{x}_{i+1}\end{aligned}\tag{11}$$

To ensure that the stabilization problem of system (11) can be handled by the stabilization result developed in Section 3, we make two more assumptions.

Assumption 2.3: Let $B=\frac{\partial f_0}{\partial x_1}(0,v,w)$. Then B is a constant column vector for all (v,w) with (A,B) stabilizable.

Assumption 2.4: For $i=1,\cdots,n$, let

$$\begin{aligned}&\hat{f}_i(\bar{x}_1,\tilde{\eta}_1,\cdots,\bar{x}_i,\tilde{\eta}_i,v,w)\\=&f_i(\bar{x}_1,\cdots,\bar{x}_i+\Psi_{i-1}T_{i-1}^{-1}(\tilde{\eta}_{i-1}+\tilde{N}_i\bar{x}_i+T_{i-1}\pi_{i-1}),v,w)\end{aligned}$$

Then, $\partial\hat{f}_i/\partial\bar{x}_j$ and $\partial\hat{f}_i/\partial\tilde{\eta}_j$ at $(\bar{x},\tilde{\eta})=0$ are constant numbers for all (v,w), and $i=1,\cdots,n$ and $j=1,\cdots,i$.

Remark 2.5: Under Assumption 2.4, for $i=1,\ldots,n$, let

$$L_i=M_iN_i\bar{x}_i-N_i\Sigma_{j=1}^i\frac{\partial\bar{f}_j(0,\ldots,0,v,w)}{\partial\bar{x}_j}\bar{x}_i.\tag{12}$$

Then L_i is a linear function of x and η only satisfying $L_i(\mathbf{x}_1(v,w),\cdots,\mathbf{x}_i(v,w),T_1\pi_1,\cdots,T_{i-1}\pi_{i-1})=0$. Further, let

$$\tilde{f}_i=\bar{f}_i-\Sigma_{j=1}^i\frac{\partial\bar{f}_j(0,\ldots,0,v,w)}{\partial\bar{x}_j}\bar{x}_i.$$

Then system (11) is simplified as follows.

$$\dot{\tilde{z}}=A\tilde{z}+\bar{f}_0-\Psi_zT_n^{-1}\tilde{f}_n+(A\Psi_zT_n^{-1}-\Psi_zT_n^{-1}M_n)\tilde{\eta}_n\tag{13}$$
$$\dot{\tilde{\eta}}_i=M_i\tilde{\eta}_i-N_i\tilde{f}_i\tag{14}$$
$$\dot{\bar{x}}_i=\bar{f}_i+\Psi_iT_i^{-1}\tilde{\eta}_i+\Psi_iT_i^{-1}N_i\bar{x}_i+\bar{x}_{i+1}\tag{15}$$

for $i=1,\cdots,n$.

Let $d(t)=(v(t),w)$,

$$g_z=\bar{f}_0-\Psi_zT_n^{-1}\tilde{f}_n+(A\Psi_zT_n^{-1}-\Psi_zT_n^{-1}M_n)\tilde{\eta}_n$$
$$f_z=\begin{pmatrix}\bar{f}_1+\Psi_1T_1^{-1}\tilde{\eta}_1+\Psi_1T_1^{-1}N_1\bar{x}_1+\bar{x}_2\\\cdots\\\bar{f}_n+\Psi_nT_n^{-1}\tilde{\eta}_n+\Psi_nT_n^{-1}N_n\bar{x}_n+\bar{u}\\M_1\tilde{\eta}_1-N_1\tilde{f}_1\\\cdots\\M_n\tilde{\eta}_n-N_n\tilde{f}_n\end{pmatrix}$$

Then (13)-(15) can be put in the following compact form.

$$\begin{aligned}\dot{\tilde{z}}&=A\tilde{z}+g_z(\bar{x},\tilde{\eta},d(t))\\\begin{bmatrix}\dot{\bar{x}}\\\dot{\tilde{\eta}}\end{bmatrix}&=f_z(\bar{x},\tilde{\eta},\bar{u},d(t))\end{aligned}\tag{16}$$

Remark 2.6: In the next section, we will consider to globally stabilize system (16) utilizing the saturation function technique. For this purpose, we require that the linear approximation of the $\tilde{\eta}$ subsystem be independent of $\bar{x}$ so that a controller without relying on $\tilde{\eta}$ is available since $\tilde{\eta}$ is not measurable. This is why the function L_i defined in (12) has to be introduced in the internal model (6).

III. A STABILIZATION RESULT

The global robust stabilization problem of system (16 (13) to (15) is complicated by three factors. First, the zero dynamics (with e being viewed as the output) $\dot{\tilde{z}}=A\tilde{z}$ is not asymptotically stable; second, the system is a time varying system due to the presence of the exogenous signal $v(t)$; and third, the system contains dynamic uncertainty represented by the subsystem governing $\tilde{\eta}$ as the state $\tilde{\eta}$ is not available for feedback. In this section, we will develop a method to stabilize this system by overcoming these three difficulties.

Definition 3.1: [4] Consider the system

$$\dot{x}=f(x,u,d(t))\tag{17}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $d:[0,\ \infty)\to\mathcal{D}$ is a piecewise continuous time function where $\mathcal{D}$ is a compact subset of $\mathbb{R}^{n_d}$, and f is smooth satisfying $f(0,0,d(t))=0$ for all $d(t)\in\mathcal{D}$. The system is called Robust Input-to-State Stable (RISS) with respect to $d(t)$, with input u, and has a gain function $\kappa(\cdot)$, if for all $d(t)\in\mathcal{D}$ and for all $t\geq t_0\geq 0$ the following holds,

$$\|x(t)\|\leq\max\{\beta(\|x(t_0)\|,t-t_0),\kappa(\sup_{t_0\leq\tau\leq t}\|u(\tau)\|)\}$$

where $\beta(\cdot,\cdot)$ belongs to class $\mathcal{KL}$.

Remark 3.1: Since f is smooth, κ can always be made smooth. Thus, the RISS condition implies that, for any given $U>0$, the following holds for all $\|u\|_\infty<U$,

$$\|x\|_a\ \leq\ \gamma_{ux}\|u\|_a\tag{18}$$
$$\|x\|_\infty\ \leq\ \max\{\gamma_{xx}(\|x(0)\|),\gamma_{ux}\|u\|_\infty\}\tag{19}$$

where $\gamma_{xx}(\|x(0)\|)=\beta(\|x(0)\|,0)$, $\gamma_{ux}=\sup_{0<\|u\|\leq U}\frac{\kappa(\|u\|)}{\|u\|}$.

The following definition is from [7] and [13].

Definition 3.2: A piecewise continuous function $\sigma:\mathbb{R}\to\mathbb{R}$ is said to be a saturation function (with level $\bar{k}$), if:
(1) $\sigma(0)=0$ and $s\sigma(s)>0$ for all $s\neq 0$,
(2) there exist $\bar{k}$ and $\underline{k}$ such that $|\sigma(s)|\leq\bar{k}$ for all $s\in\mathbb{R}$ and $\lim_{|s|\to\infty}\inf|\sigma(s)|\geq\underline{k}$,
(3) $\sigma(s)$ is differentiable in a neighborhood of $s=0$ and $\sigma'(0)=1$.

Proposition 3.1: Consider the system

$$\begin{aligned}\dot{z}&=Az+g(\xi,d(t))\\\dot{\xi}&=f(\xi,u,d(t))\end{aligned}\tag{20}$$

where f and g are smooth with $f(0,0,d(t))=0$ and $g(0,d(t))$ for all $d\in\mathcal{D}$. Assume that:

1) there exists a symmetric matrix $P>0$ such that $PA+A^TP\leq 0$,

2) $G_1 = \frac{\partial g}{\partial \xi}(0, d(t))$, $G_2 = \frac{\partial f}{\partial \xi}(0, 0, d(t))$ and $B_1 = \frac{\partial f}{\partial u}(0, 0, d(t))$ are all constant matrices for all $d(t) \in D$ and the pair $\left(\begin{bmatrix} A & G_1 \\ 0 & G_2 \end{bmatrix}, \begin{bmatrix} 0 \\ B_1 \end{bmatrix}\right)$ is stabilizable,
3) the ξ subsystem is RISS with respect to $d(t)$, with u as input and has a gain function.

Then, there exists a feedback law

$$u = \lambda\sigma(\frac{K_z z + K_\xi \xi}{\lambda}) \quad (21)$$

which globally asymptotically stabilizes the equilibrium point $(z, \xi) = 0$.

Proof of the proposition is omitted here due to space limit.

We now turn to the problem of the stabilization of system (13)-(15).

Proposition 3.2: Under Assumptions 2.1-2.4, there exists a control law of the form

$$\bar{u} = k(\bar{x}) + \lambda\sigma(\frac{K_z \tilde{z} + K_x \bar{x}}{\lambda}) \quad (22)$$

where $k(\cdot)$ is a smooth function with $k(0) = 0$, $\sigma(\cdot)$ is a saturation function, and $\lambda > 0$ is a sufficiently small constant, such that the equilibrium of the closed-loop system (16) and (22) is globally asymptotically stable and locally exponentially stable.

Proof: Let $\xi = (\bar{x}, \tilde{\eta})$. Then the ξ subsystem of (16) is in lower triangular form viewing $\tilde{\eta}$ as the dynamic uncertainty. Moreover, this subsystem satisfies the conditions of Theorem 7.6 in [4]. Thus there exists a smooth function $k(\bar{x})$ satisfying $k(0) = 0$, such that, under the controller

$$\bar{u} = k(\bar{x}) + \hat{u}, \quad (23)$$

the closed-loop system composed of the ξ subsystem of (16) and the controller (23) is RISS with respect to (v, w) with $\hat{u}$ as input. Denote the closed-loop system composed of (16) and (23) by the following system

$$\begin{aligned} \dot{\tilde{z}} &= A\tilde{z} + g_z(\bar{x}, \tilde{\eta}, d(t)) \\ \dot{\xi} &= f_z(\bar{x}, \tilde{\eta}, k(\bar{x}) + \hat{u}, d(t)) \end{aligned} \quad (24)$$

which is in the same form as the system (20).

We will show that system (24) satisfies all the three conditions of Proposition 3.1. In fact, condition 1) is obviously satisfied and condition 3) is also satisfied as a result of the application of the control law (23). We only need to verify condition 2). Note that, by Assumption 2.4, the linearization of $\tilde{f}_i$, for $i = 1, \ldots, n$, at $(\bar{x}, \tilde{\eta}) = 0$ is zero. Thus Assumptions 2.3 and 2.4 imply that the three matrices G_1, G_2, and B_1 in condition 2) are all constant for all $d(t)$. More specifically, we have

$$\begin{bmatrix} A & G_1 \\ 0 & G_2 \end{bmatrix} = \begin{bmatrix} A & B & 0 & \ldots & 0 & * \\ 0 & * & 1 & \ldots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \ldots & 1 & * \\ 0 & * & * & \ldots & * & * \\ 0 & 0 & 0 & \ldots & 0 & M \end{bmatrix}, \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where M is a block lower triangular matrix with the diagonal block being given by $(M_1, \cdots, M_n)$. Because (A, B) is stabilizable and M is Hurwitz, the pair

$$\left(\begin{bmatrix} A & G_1 \\ 0 & G_2 \end{bmatrix}, \begin{bmatrix} 0 \\ B_1 \end{bmatrix}\right)$$

is stabilizable by the PBH test.

Having shown that system (24) satisfies all three conditions of Proposition 3.1, we can conclude that there exists a saturated controller in the following form:

$$\hat{u} = \lambda\sigma(\frac{K_z \tilde{z} + K_x \bar{x} + K_\eta \tilde{\eta}}{\lambda}) \quad (25)$$

with K_z, K_x and K_η constant row vectors such that the origin of the closed-loop system (24) and control law (25) is GAS. Moreover, all the eigenvalues of the linear approximation of the closed-loop system have negative-real-part, the equilibrium of the closed-loop system is LES.

Finally, note that the linear approximation of $\tilde{\eta}$ subsystem of (24) at $(\bar{x}, \tilde{\eta}) = 0$ is autonomous and GAS, according to [13], we can always have $K_\eta = 0$. Thus, the controller (25) can be made independent of the unmeasurable state $\tilde{\eta}$.

In summary, we have the following main result.

Theorem 3.1: Under Assumption 2.1-2.4, the GRORP of (1) is solvable by a controller in the following form

$$\begin{aligned} \dot{\eta}_i &= M_i \eta + N_i x_i - L_i \\ u &= \Psi_n T_n^{-1} \eta_n + k(\bar{x}) + \lambda\sigma\left(\frac{K_1 \bar{z} + K_2 \bar{x}}{\lambda}\right) \end{aligned} \quad (26)$$

where $k(\cdot)$ is a smooth function with $k(0) = 0$, $\sigma(\cdot)$ is a saturation function, $i = 1, \cdots, n$.

IV. Application to the RTAC system

The Rotational/Translational Actuator (RTAC) considered in this paper was introduced in [1] and [14]. The system consists of a cart of mass M_0 connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel. The proof-mass actuator attached to the cart has mass m and moment of inertia I about its center of mass, which is located at a distance y from the point about which the proof-mass rotates. Its motion occurs in a horizontal plane so that no gravitational forces need to be considered. The motion of RTAC is described as follows:

$$\begin{aligned} \ddot{y} + y &= \epsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + F_d \\ \ddot{\theta} &= -\epsilon\ddot{y}\cos\theta + u_\theta \end{aligned} \quad (27)$$

where y is the one-dimensional displacement of the cart, θ the angular position of the proof body, F_d the disturbance, and u_θ the control input. The coupling between the translational and rotational motion is captured by the parameter ϵ, which is defined by

$$\epsilon = \frac{m e_0}{\sqrt{(1 + m e_0^2)(M_0 + m)}} \quad (28)$$

where $0 < e_0 < 1$ is the eccentricity of the proof body.

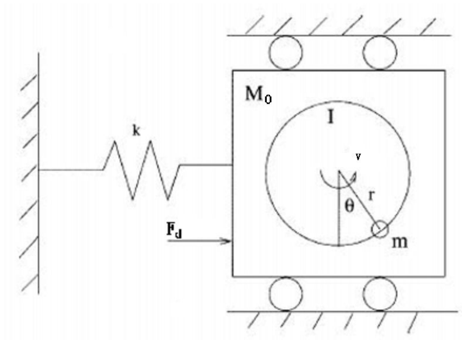

Fig. 1. Rotational/translation actuator [14]

The external disturbance F_d we consider here is a sinusoidal signal, i.e.,

$$F_d(t) \quad = \quad C\sin(\omega t + \phi) \tag{29}$$

where C and ϕ are unknown constants.

Various control problems associated with (27) has been studied in, for example, [3], [4], [8], [9], and [10]. These papers either deal with the stabilization problem or the local disturbance rejection problem. Here we are interested in a global asymptotical disturbance rejection problem described as follows:

Global asymptotical disturbance rejection problem: Design a state feedback control law of the form (2) such that,

1) for any initial state of (31) and any disturbance in the form of (29), with known frequency and range of amplitute, but with unknown amplitude C and initial phase ϕ, the state of the closed-loop system composed of the RTAC and the control law is bounded, and $\lim_{t\to\infty} x_1(t) = 0$.

2) the closed-loop system is GAS and LES, in the absence of disturbances.

As in [14], under the coordinates and input transformation defined by

$$\begin{aligned} z_1 &= y + \epsilon\sin\theta \\ z_2 &= \dot{y} + \epsilon\dot{\theta}\cos\theta \\ x_1 &= \theta \\ x_2 &= \dot{\theta} \\ u &= \frac{u_\theta + \epsilon\cos x_1(z_1 - (1+x_2^2)\epsilon\sin x_1)}{1-\epsilon^2\cos^2 x_1} \end{aligned} \tag{30}$$

system (27) is then converted to

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \epsilon\sin(x_1) + F_d \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{\epsilon\cos x_1}{1-\epsilon^2\cos^2 x_1}F_d + u \end{aligned} \tag{31}$$

Also, we can express $F_d(t)$ as

$$F_d = \begin{bmatrix} 1 & 0 \end{bmatrix} v, \quad \dot{v} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} v \tag{32}$$

We assume that the initial state $v(0)$ belongs to some arbitrarily large known compact set $\mathcal{V} \subset \mathbb{R}^2$. It is clear that the problem is a special case of GRORP.

The equation (3) here is

$$\frac{\partial \mathbf{z}(v,w)}{\partial v}A_1 v = A\mathbf{z}(v,w) + \begin{bmatrix} 0 \\ v_1 \end{bmatrix}$$

which has a solution as follows:

$$\mathbf{z}(v) = \frac{1}{1-\omega^2}\begin{bmatrix} v_1 \\ \omega v_2 \end{bmatrix}$$

Thus, Assumption 2.1 is satisfied. To verify Assumption 2.2, let

$$\pi(v) = \begin{bmatrix} v_1 \\ \omega v_2 \end{bmatrix}, \Phi = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

$$\Psi_z = \frac{1}{1-\omega^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Psi_n = \frac{\epsilon}{1-\epsilon^2}\begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Then it is clear that the following equations hold:

$$\begin{aligned} \dot{\pi} &= \Phi\pi(v) \\ \mathbf{z}(v) &= \Psi_z\pi(v) \\ \mathbf{u}(v) &= \Psi_n\pi(v) \\ \mathbf{x}_1(v) &= \Psi_x\pi(v) \\ \mathbf{x}_2(v) &= \Psi_x\pi(v) \end{aligned}$$

where Ψ_x is a zero row vector with suitable dimension. Thus Assumption 2.2 is also satisfied.

Next, noting that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \epsilon \end{bmatrix}$$

shows Assumption 2.3 is satisfied. Finally, it is clear that Assumption 2.4 is also satisfied.

For the RTAC system, the augmented system (13)-(15) now becomes,

$$\begin{aligned} \dot{\tilde{z}} &= A\tilde{z} + B\sin(x_1) + (A\Psi_z T^{-1} - \Psi_z T^{-1}M)\tilde{\eta} \\ &\quad + \Psi_z T^{-1}Nh(x_1)g(v) \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= h(x_1)g(v) + \Psi T^{-1}\tilde{\eta} + \Psi T^{-1}Nx_2 + \bar{u} \\ \dot{\tilde{\eta}} &= M\tilde{\eta} - Nh(x_1)g(v) \end{aligned}$$

where $h(x_1) = \frac{\epsilon}{1-\epsilon^2} - \frac{\epsilon\cos x_1}{1-\epsilon^2\cos^2 x_1}$, and $g(v) = F_d$.

It is noted that since both $\mathbf{x}_1(v)$ and $\mathbf{x}_2(v)$ are identically 0, hence we can obtain a simplified internal model in the following form

$$\dot{\eta} \quad = \quad M\eta + Nu - MNx_2 \tag{33}$$

where $\eta \in \mathbb{R}^2$.

Applying the design procedure detailed in Section 3 gives the following specific controller:

$$\dot{\eta} = M\eta + Nu - MNx_2 \tag{34}$$

$$u = \Psi T^{-1}\eta - Kx + \lambda\sigma\left(\frac{K_z(z - \Psi_z T^{-1}\eta) + K_x x}{\lambda}\right) \tag{35}$$

where $\eta \in \mathbb{R}^2$. Letting

$$M = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \; N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and solving the Sylvester equation (5) gives

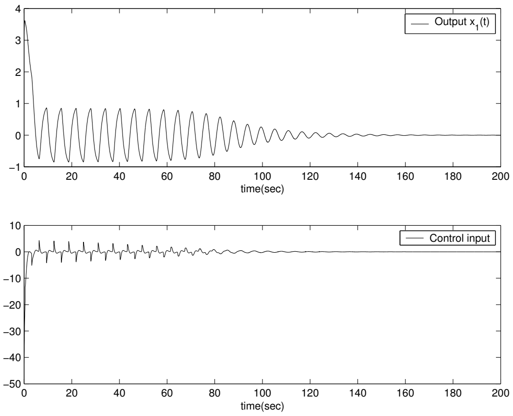

Fig. 2. Profile of the output error and control input ($v(0) = 0$)

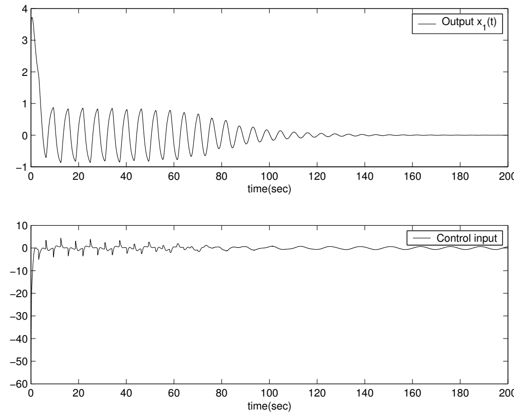

Fig. 4. Profile of the output error and control input ($v(0) \neq 0$)

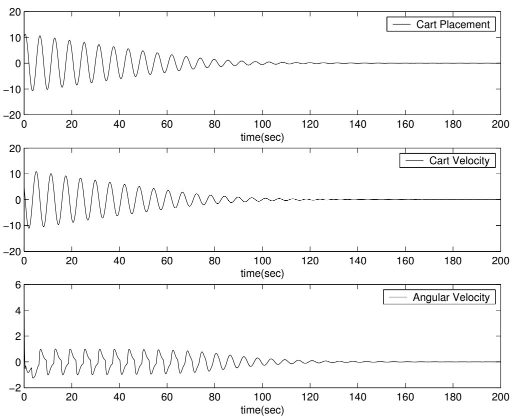

Fig. 3. Profile of the other state ($v(0) = 0$)

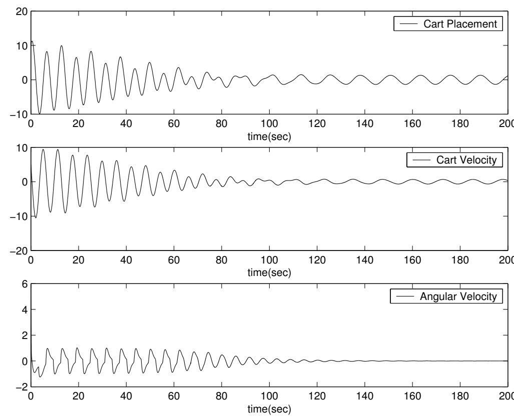

Fig. 5. Profile of the other state ($v(0) \neq 0$)

$$T = \begin{bmatrix} 0.2141 & -0.1557 \\ 0.0389 & 0.2141 \end{bmatrix}.$$

Also noted that, for this particular system, we can actually find $k(x)$ which is in linear form Kx. The details of the design are omitted due to the space limit.

Finally we set $\lambda = 1$, $K = [3\ 6]$, $K_x = [-11\ \ -100]$ and $K_z = [8\ \ -24]$. The initial condition is arbitrary chosen to be $[z_1(0), z_2(0), x_1(0), x_2(0), \eta(0)^T] = [10, 5, 3, 6, 0.2, 8]$. Figures 2-5 show the simulation results for the two scenarios with $v(0) = [0, 0]^T$ and $v(0) = [1, 0]^T$, respectively.

The simulation results show that the angular position $x_1(t)$ can be driven to the origin asymptotically in the presence or absence of the disturbance, and all other quantities are bounded.

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