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Generalized Real Perturbation Values with Applications to the Structured Real Controllability Radius of LTI Systems

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Abstract— **This paper generalizes the notion of real perturbation values of a complex matrix to account for a more general perturbation structure. Formulas for computing the so-called *generalized real perturbation values* of a matrix are derived and presented. Using these results, we revisit the computation of the structured real controllability radius that was previously used to evaluate the robustness of the multi-link inverted pendulum system, and we also study a new normalized version of the transmission zero at s radius.**

I. Introduction

In the current literature, various continuous measures have been proposed to measure the robustness of various linear time-invariant (LTI) system properties in the presence of parametric perturbations. This class of robustness radii includes, for instance, the stability radius ([1]), the controllability/observability radius ([2], [3]), the decentralized fixed-mode radius ([4], [5]), the transmission zero at s radius ([6]), and the minimum-phase radius ([6]).

Consider the following LTI multivariable system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^r$ are respectively the state, input, and output vectors, and A, B, C, and D are constant matrices with the appropriate dimensions for $n \geq 1$, $m \geq 1$, $r \geq 1$, and $\max(r, m) \leq n$. Suppose the system is controllable and observable, then it is well known that there exists a LTI controller that can assign the eigenvalues of the closed-loop system to any arbitrary spectrum. However, if the system is subjected to parametric perturbations (i.e. $A \to A + \Delta_A$ and $B \to B + \Delta_B$), which may result from numerical errors, modeling errors, etc., then the system may become uncontrollable. Hence, a continuous *controllability radius* to measure how "close" a controllable system is to becoming uncontrollable is more informative than the traditional 'yes/no' controllability metric, which simply determines whether a system is controllable or not.

The same can be said about other system properties, and various measures have been introduced. The *stability radius* in [1] measures how close a stable system is to an unstable one. The *decentralized fixed-mode (DFM) radius* ([4], [5]) measures how close a system with no DFMs is to having one. More recently, such robustness measures have been extended to characterize the robustness of a system's transmission zero properties. The *minimum-phase radius* in [6], for instance, measures how close a minimum-phase system is to becoming nonminimum-phase, and the *transmission zero at s radius* measures how close a system is to having a transmission zero at a given point $s \in \mathbb{C}$.

The formulas for computing the radii mentioned above are all based on the singular values and real perturbation values [7] of complex matrices. Similar to the generalization of the singular values in [8], [9], one of the main contributions of this paper is the generalization of the real perturbation values to account for a more general perturbation structure. Another contribution is the development of formulas for computing the subsequent so-called *generalized real perturbation values* of a matrix. Also, we will apply the generalized real perturbation values to two examples found in systems control that are related to two of the robustness radii mentioned above. In particular, in the first example, we will revisit the pendulum problem in [10], and using the generalized real perturbation values, compute a more accurate estimate of the structured real controllability radius of the multi-link inverted pendulum. In the second example, we will propose a new normalized version of the transmission zero at s radius, and using the generalized real perturbation values, compute the structured transmission zero at $s = 0$ radius of a number of industrial systems that were studied in [6].

The paper is organized as follows. In Section II, the real perturbation values and the generalized singular values of a matrix are reviewed. Definitions of the generalized real perturbation values and formulas for computing these values are then presented in Section III. Finally in Section IV, we will apply the results of Section III to the two examples mentioned above.

II. Preliminaries and Review

A. Notations

In this paper, the field of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The i-th singular value of a matrix $M \in \mathbb{C}^{p \times q}$ is denoted by $\sigma_i(M)$, where $\sigma_1(M) \geq \cdots \geq \sigma_{\min(p,q)}(M)$. $\|M\|_2$ denotes the spectral norm of a matrix M and is equal to $\sigma_1(M)$. Also, $\overline{M}$, M^T, and M^H denote respectively the complex conjugate, transpose, and complex conjugate transpose of M. Furthermore, M^{-H} denotes $\left(M^H\right)^{-1}$. The real and imaginary components of the matrix M are given by $\operatorname{Re} M$ and $\operatorname{Im} M$ respectively. The set of eigenvalues of a square matrix $M \in \mathbb{C}^{p \times p}$ is denoted by $\lambda(M)$. If M is Hermitian (i.e. $M = M^*$), then the i-th

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eigenvalue of M is denoted by $\lambda_i(M)$, where $\lambda_1(M) \geq \cdots \geq \lambda_p(M)$. The set of generalized eigenvalues of the matrix pair (M, N), where $M \in \mathbb{C}^{p\times p}$ and $N \in \mathbb{C}^{p\times p}$, is denoted by $\lambda(M, N)$. Recall that the generalized eigenvalues of (M, N) are the set of (complex) values such that for $\lambda \in \lambda(M, N)$, $Mx = \lambda Nx$ is satisfied for some non-zero $x \in \mathbb{C}^p$ (x is a generalized eigenvector of (M, N)). Finally, it will be understood that $\lambda(M) = \lambda(M, I)$.

B. Review of Real Perturbation Values

The real perturbation values of a complex matrix are a set of numbers that are analogous to the singular values of the matrix with the exception that the former takes into account only real perturbations. The real perturbation values as introduced in [7] are defined as follows.

Definition 2.1 (Real perturbation values [7]): Given $M \in \mathbb{C}^{q\times l}$, the i-th *real perturbation value* of the *first kind* of M are defined as:

$$\tau_i(M) := \frac{1}{\inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{R}^{l\times q}, \text{nullity}(I_l - \Delta M) \geq i\}}$$

and the i-th *real perturbation value* of the *second kind* are defined as:

$$\tilde{\tau}_i(M) := \inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{R}^{q\times l}, \text{rank}\,(M - \Delta) < i\}$$

where $i = 1, \ldots, \min(q, l)$.

Remark 2.1: It can be shown that given a matrix $M \in \mathbb{C}^{q\times l}$, the singular values of M satisfy (e.g. see [7]):

$$\sigma_i(M) = \inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{C}^{q\times l}, \text{rank}(M - \Delta) < i\} = \frac{1}{\inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{C}^{l\times q}, \text{nullity}(I_l - \Delta M) \geq i\}}$$

for $i = 1, \ldots, \min(q, l)$. Therefore, as mentioned earlier, the real perturbation values are closely related to the singular values with the difference that the former takes into account only real perturbations, whereas the latter considers all complex perturbations.

The real perturbation values can be computed by the following formulas [7].

Theorem 2.1 ([7]): Given $M \in \mathbb{C}^{q\times l}$ and $i = 1, \ldots, \min(q, l)$,

$$\tau_i(M) = \inf_{\gamma\in(0,1]} \sigma_{2i}(P(\gamma, M)) \quad (2)$$

$$\tilde{\tau}_i(M) = \sup_{\gamma\in(0,1]} \sigma_{2i-1}(P(\gamma, M)) \quad (3)$$

where

$$P(\gamma, M) := \begin{bmatrix} \text{Re}\, M & -\gamma\, \text{Im}\, M \\ \gamma^{-1}\, \text{Im}\, M & \text{Re}\, M \end{bmatrix} \quad (4)$$

C. Review of Generalized Singular Values

The generalized singular values of a matrix pair arises from the generalized singular value decomposition given as follows.

Theorem 2.2 ([8], [9]): Given $M \in \mathbb{C}^{q\times l}$ and $N \in \mathbb{C}^{p\times l}$, the generalized singular value decomposition is given by

$$M = U \begin{bmatrix} \Sigma_M & 0 \end{bmatrix} Q$$
$$N = V \begin{bmatrix} \Sigma_N & 0 \end{bmatrix} Q$$

where $U \in \mathbb{C}^{q\times q}$ and $V \in \mathbb{C}^{p\times p}$ are unitary matrices, $Q \in \mathbb{C}^{l\times l}$ is nonsingular, and

$$\Sigma_M = \begin{bmatrix} I_r & & \\ & S_M & \\ & & 0 \end{bmatrix}, \quad \Sigma_N = \begin{bmatrix} 0 & & \\ & S_N & \\ & & I_{k-r-s} \end{bmatrix}$$

where $S_M = \text{diag}\,(\alpha_{r+1}, \cdots, \alpha_{r+s})$, $S_N = \text{diag}\,(\beta_{r+1}, \cdots, \beta_{r+s})$, and α_{r+i} and β_{r+i} are real numbers satisfying $0 \leq \alpha_{r+i}, \beta_{r+i} \leq 1$ and $\alpha_{r+i}^2 + \beta_{r+i}^2 = 1$, for $i = 1, \ldots, s$. Here the dimensions k, r, and s satisfy $k = \text{rank}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right)$, $r = \text{rank}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right) - \text{rank}(N)$, and $s = \text{rank}(M) + \text{rank}(N) - \text{rank}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right)$.

Taking the diagonal elements of Σ_M and Σ_N, there are four kinds of pairs, which we will number as follows:

- For $i = 1, \ldots, r$: $(\alpha_i, \beta_i) = (1, 0)$
- For $i = r + 1, \ldots, r + s$: $\alpha_i \neq 0$, $\beta_i \neq 0$
- For $i = r + s + 1, \ldots, k$: $(\alpha_i, \beta_i) = (0, 1)$
- For $i = k + 1, \ldots, l$: $(\alpha_i, \beta_i) = (0, 0)$

The ratios $\sigma_i = \frac{\alpha_i}{\beta_i}$, for $i = 1, \ldots, k$, are called the *nontrivial* generalized singular values of the matrix pair (M, N), and can be infinite, non-zero finite, or zero. σ_i for $i = k+1, \ldots, l$ are called the *trivial* generalized singular values and have no particular numbers assigned to them.

From here on, the i-th (nontrivial) generalized singular value of a matrix pair (M, N) will be denoted by $\sigma_i(M, N)$. To avoid confusion in notations, it will be understood that $\sigma_i(M) = \sigma_i(M, I)$. Also, without loss of generality, the non-trivial generalized singular values are assumed to be arranged in nonincreasing order; i.e. $\sigma_1(M, N) \geq \sigma_2(M, N) \geq \cdots$.

Remark 2.2: Recall that for a given matrix $M \in \mathbb{C}^{q\times l}$, where $q \geq l$, the eigenvalues of $M^H M$ are the squares of the singular values of M. This extends to the matrix pair (M, N) by the fact that

$$M^H M = Q^H \begin{bmatrix} \Sigma_M^H \Sigma_M & 0 \\ 0 & 0 \end{bmatrix} Q \quad (5)$$

and

$$N^H N = Q^H \begin{bmatrix} \Sigma_N^H \Sigma_N & 0 \\ 0 & 0 \end{bmatrix} Q \quad (6)$$

Therefore, the nontrivial generalized eigenvalues[1] of the matrix pair $\left(M^H M, N^H N\right)$ are the squares of the nontrivial generalized singular values of (M, N).

Remark 2.3: It can be shown that the i-th generalized singular value of (M, N), for $i = 1, \ldots, \min(q, l)$, satisfy

$$\sigma_i(M, N) = \inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{C}^{q\times p}, \text{rank}(M - \Delta N) < i\} = \frac{1}{\inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{C}^{p\times q}, \text{nullity}(N - \Delta M) \geq i\}}$$

[1]The trivial generalized eigenvalues are those corresponding to the last $l - k$ columns of Q in (5) and (6), which span the common null space of $M^H M$ and $N^H N$.

III. MAIN RESULT

The following generalization is made.

Definition 3.1 (Generalized real perturbation values): Given $M \in \mathbb{C}^{q\times l}$ and $N \in \mathbb{C}^{p\times l}$, the i-th *generalized real perturbation value* of the *first kind* of the matrix pair (M, N) is defined as:

$$\tau_i(M,N) := \frac{1}{\inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{R}^{p\times q}, \mathrm{nullity}(N-\Delta M) \geq i\}}$$

and the i-th *generalized real perturbation value* of the *second kind* is defined as:

$$\tilde{\tau}_i(M,N) := \inf\{\|\Delta\|_2 \,|\Delta \in \mathbb{R}^{q\times p}, \mathrm{rank}\,(M-\Delta N) < i\}$$

where $i = 1,\ldots,\min(q,l)$. If there exists no real matrix Δ such that $\mathrm{nullity}(N-\Delta M) \geq i$, then $\tau_i(M,N) = 0$. Likewise if there exists no real matrix Δ such that $\mathrm{rank}\,(M-\Delta N) < i$, then $\tilde{\tau}_i(M,N) = \infty$.

Remark 3.1: It can be shown that if N is nonsingular, then the generalized real perturbation value problem reduces to the original real perturbation value problem of a single matrix (i.e. see Definition 2.1). In particular, since

$$\mathrm{nullity}(N-\Delta M) = \mathrm{nullity}\big(I - \Delta M N^{-1}\big)$$

and

$$\mathrm{rank}(M-\Delta N) = \mathrm{rank}\big(MN^{-1} - \Delta\big)$$

then it can be shown that $\tau_k(M,N) = \tau_k\big(MN^{-1}\big)$ and $\tilde{\tau}_k(M,N) = \tilde{\tau}_k\big(MN^{-1}\big)$. In this paper, we will assume that N is not necessarily invertible (or even square).

One of the main results of this paper is the following theorem, which provides formulas for computing the generalized real perturbation values of a matrix pair.

Theorem 3.1: Given matrices $M \in \mathbb{C}^{q\times l}$ and $N \in \mathbb{C}^{p\times l}$, and $i = 1,\ldots,\min(q,l)$, then

$$\tau_i(M,N) = \inf_{\gamma\in(0,1]} \sigma_{2i}(P(\gamma,M)\,,P(\gamma,N)) \quad (7)$$

$$\tilde{\tau}_i(M,N) = \sup_{\gamma\in(0,1]} \sigma_{2i-1}(P(\gamma,M)\,,P(\gamma,N)) \quad (8)$$

where $P(\gamma,M)$ and $P(\gamma,N)$ are defined by (4).

Proof: The proof of Theorem 3.1 is similar to the proof of Theorem 2.1 found in [7], which is based on the following results, namely Lemma 3.1 and Theorem 3.2.

Lemma 3.1 ([7]): Given matrices $T_1 \in \mathbb{C}^{q\times l}$ and $T_2 \in \mathbb{C}^{p\times l}$, there exists a real contraction $\Delta \in \mathbb{R}^{p\times q}$ (with $\|\Delta\|_2 \leq 1$) such that $\Delta T_1 = T_2$ if and only if

$$\begin{bmatrix} T_2 & \overline{T_2} \end{bmatrix}^H \begin{bmatrix} T_2 & \overline{T_2} \end{bmatrix} \leq \begin{bmatrix} T_1 & \overline{T_1} \end{bmatrix}^H \begin{bmatrix} T_1 & \overline{T_1} \end{bmatrix}$$

Theorem 3.2 ([7]): Given $A = A^H \in \mathbb{C}^{n\times n}$ and $B = B^T \in \mathbb{C}^{n\times n}$, the following conditions are equivalent for $k = 1,\ldots,n$:

1) There exists a complex matrix S_k of rank $\geq k$ such that

$$\begin{bmatrix} S_k & 0 \\ 0 & \overline{S_k} \end{bmatrix}^H \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix} \begin{bmatrix} S_k & 0 \\ 0 & \overline{S_k} \end{bmatrix} \geq 0$$

2) The matrix $\begin{bmatrix} A & \alpha\overline{B} \\ \alpha B & \overline{A} \end{bmatrix}$ has at least $2k$ nonnegative eigenvalues for every real $|\alpha| \leq 1$; i.e.

$$\inf_{|\alpha|\leq 1} \lambda_{2k}\left(\begin{bmatrix} A & \alpha\overline{B} \\ \alpha B & \overline{A} \end{bmatrix}\right) \geq 0 \qquad \alpha \in \mathbb{R}$$

In the remainder of this section, we will provide a proof of (8). The proof for (7) is similar and is omitted.

Consider a real $\tau \geq 0$. From Definition 3.1, $\tau \geq \tilde{\tau}_i(M,N)$ if and only if there exists a real Δ such that $\mathrm{rank}(M-\Delta N) < i$ and $\|\Delta\|_2 \leq \tau$. This implies that there exists some complex matrix S with $\mathrm{rank}(S) \geq l-(i-1)$ such that $(M-\Delta N)\,S = 0$, or equivalently, $(\Delta/\tau)\,NS = MS/\tau$. By Lemma 3.1, this is equivalent to

$$\tau^2 \begin{bmatrix} NS & \overline{NS} \end{bmatrix}^H \begin{bmatrix} NS & \overline{NS} \end{bmatrix} \geq \begin{bmatrix} MS & \overline{MS} \end{bmatrix}^H \begin{bmatrix} MS & \overline{MS} \end{bmatrix} \quad (9)$$

Let $A_r = \tau^2 N^H N - M^H M$ and $B_r = \tau^2 N^T N - M^T M$, then (9) is equivalent to

$$\begin{bmatrix} S & 0 \\ 0 & \overline{S} \end{bmatrix}^H \begin{bmatrix} A_r & \overline{B_r} \\ B_r & \overline{A_r} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & \overline{S} \end{bmatrix} \geq 0 \quad (10)$$

Therefore by Theorem 3.2, $\tau \geq \tilde{\tau}_i(M,N)$ if and only if $\begin{bmatrix} A_r & \alpha\overline{B_r} \\ \alpha B_r & \overline{A_r} \end{bmatrix}$ has at least $2\,(l-(i-1))$ nonnegative eigenvalues for $\alpha \in (-1,0]$[2]; i.e. for $\alpha \in (-1,0]$,

$$\lambda_{2(l-(i-1))}\left(\begin{bmatrix} A_r & \alpha\overline{B_r} \\ \alpha B_r & \overline{A_r} \end{bmatrix}\right) \geq 0$$

or equivalently,

$$\lambda_{2i-1}\left(\begin{bmatrix} \left(M^H M - \tau^2 N^H N\right) & \alpha\left(\overline{M^T M - \tau^2 N^T N}\right) \\ \alpha\left(M^T M - \tau^2 N^T N\right) & \overline{M^H M - \tau^2 N^H N} \end{bmatrix}\right) \leq 0 \quad (11)$$

since for Hermitian matrix $H = H^H \in \mathbb{C}^{n\times n}$, $\lambda_k(H) = -\lambda_{n-k+1}(-H)$. Define

$$T_{\alpha,n} := \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1+\alpha}} I_n & \frac{j}{\sqrt{1-\alpha}} I_n \\ \frac{1}{\sqrt{1+\alpha}} I_n & -\frac{j}{\sqrt{1-\alpha}} I_n \end{bmatrix} \quad (12)$$

which satisfies $T_{\alpha,n}^{-H} T_{\alpha,n}^{-1} = \begin{bmatrix} I_n & \alpha I_n \\ \alpha I_n & I_n \end{bmatrix}$. Furthermore,

$$T_{\alpha,q}^{-1} \begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix} T_{\alpha,l} = P\left(\sqrt{\frac{1+\alpha}{1-\alpha}}, M\right) \quad (13)$$

where $P(\gamma,M)$ is defined in (4).

[2]Due to symmetry, it is sufficient to take only $\alpha \in (-1,0]$ instead of all values $|\alpha| \leq 1$ (e.g. see [7]).

Finally, the following equivalences are true for all $\tau \geq 0$.

$$\begin{aligned}
& \tau \geq \tilde{\tau}_i(M,N) \\
\Leftrightarrow\ & \lambda_{2i-1}\left(\begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix}^H \begin{bmatrix} I_q & \alpha I_q \\ \alpha I_q & I_q \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix} \right. \\
& \left. -\tau^2 \begin{bmatrix} N & 0 \\ 0 & \overline{N} \end{bmatrix}^H \begin{bmatrix} I_p & \alpha I_p \\ \alpha I_p & I_p \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & \overline{N} \end{bmatrix}\right) \leq 0 \\
& \text{for } \alpha \in (-1,0] \ \ \text{(from (11))} \\
\Leftrightarrow\ & \lambda_{2i-1}\left(T_{\alpha,l}^H \left(\begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix}^H T_{\alpha,q}^{-H} T_{\alpha,q}^{-1} \begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix} \right.\right. \\
& \left.\left. -\tau^2 \begin{bmatrix} N & 0 \\ 0 & \overline{N} \end{bmatrix}^H T_{\alpha,p}^{-H} T_{\alpha,p}^{-1} \begin{bmatrix} N & 0 \\ 0 & \overline{N} \end{bmatrix}\right) T_{\alpha,l}\right) \leq 0 \\
& \text{for } \alpha \in (-1,0] \\
\Leftrightarrow\ & \lambda_{2i-1}\left(P\left(\sqrt{\frac{1+\alpha}{1-\alpha}}, M\right)^H P\left(\sqrt{\frac{1+\alpha}{1-\alpha}}, M\right)\right. \\
& \left. -\tau^2 P\left(\sqrt{\frac{1+\alpha}{1-\alpha}}, N\right)^H P\left(\sqrt{\frac{1+\alpha}{1-\alpha}}, N\right)\right) \leq 0 \\
& \text{for } \alpha \in (-1,0] \\
\Leftrightarrow\ & \lambda_{2i-1}\Big(P(\gamma,M)^H P(\gamma,M)\,, \\
& \qquad P(\gamma,N)^H P(\gamma,N)\Big) \leq \tau^2 \qquad \text{for } \gamma \in (0,1] \\
\Leftrightarrow\ & \sigma_{2i-1}(P(\gamma,M)\,, P(\gamma,N)) \leq \tau \qquad \text{for } \gamma \in (0,1] \\
\Leftrightarrow\ & \sup_{\gamma \in (0,1]} \sigma_{2i-1}(P(\gamma,M)\,, P(\gamma,N)) \leq \tau
\end{aligned}$$

So it is concluded that (8) must be true. ∎

IV. Applications

In this section, we apply the results on the generalized real perturbation values to two examples found in systems control. The first example looks at obtaining a better estimate of the structured real controllability radius of the multi-link inverted pendulum system that was studied in [10], and the second example studies a new normalized version of the transmission zero at s radius.

A. Structured Real Controllability Radius of the Pendulum

In [10], the controllability radius was used to study the difficulty of balancing a multi-link inverted pendulum system. In particular, it was claimed that the difficulty is related to the pendulum's controllability robustness, and it was shown that as the number of pendulum links increases, the real controllability radius becomes smaller, indicating that the system gets closer to becoming uncontrollable. In [10], perturbations with a particular structure were considered, which led to a controllability radius problem that was difficult to solve. Subsequently, a somewhat ad-hoc normalization method involving a random matrix was used in [10] to compute an estimate of the corresponding radius. In this section, we will use the results on generalized real perturbation values to obtain a more accurate estimate of the structured real controllability radius of the pendulum system.

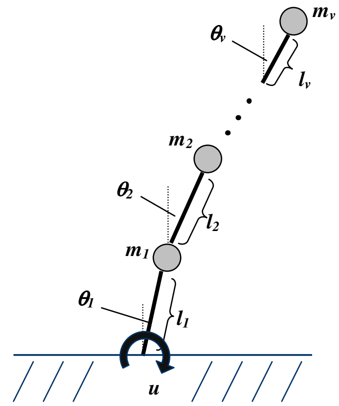

Fig. 1. Model of a multi-link inverted pendulum with v links [10].

Fig. 1 illustrates the single-input single-output multi-link inverted pendulum system with v links that was considered in [10], where the i-th link is modeled as a point-mass, m_i, attached via a massless rigid rod of length, l_i, for $i = 1, \ldots, v$. The control input, u, is a single torque applied at the pivot of the bottom link. All angles are measured with respect to the vertical. Defining $x = \begin{bmatrix} \theta_1 & \cdots & \theta_v & \dot{\theta}_1 & \cdots & \dot{\theta}_v \end{bmatrix}^T$ as the state vector, and θ_1 as the output (i.e. the bottom link's angle is measured), the linear state space model (linearized about the vertical zero equilibrium point, $(x,u) = (0,0)$) is given by [10]:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & I \\ (M_v L_v)^{-1} M_a & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ (M_v L_v)^{-1} M_b \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \qquad\quad , D = 0 \qquad (14)
\end{aligned}$$

where $M_a = \operatorname{diag}\left(g\sum_{i=1}^{v} m_i, g\sum_{i=2}^{v} m_i, \cdots, m_v g\right)$, $M_b = \begin{bmatrix} \frac{1}{l_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $M_v = \begin{bmatrix} m_1 & m_2 & \cdots & m_v \\ 0 & m_2 & \cdots & m_v \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_v \end{bmatrix}$, and $L_v = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ l_1 & l_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ l_1 & l_2 & \cdots & l_v \end{bmatrix}$.

As mentioned earlier, the real controllability radius of a LTI system measures how "close", with respect to real parametric perturbations (i.e. $A \to A + \Delta_A$ and $B \to B + \Delta_B$, where Δ_A and Δ_B are real), the system is to being an uncontrollable system. More precisely, the real controllability radius is defined as follows [3]:

Definition 4.1: Given a LTI system (1), the *real controllability radius*, $r_{\mathbb{R}}^c$, is defined to be:

$$\begin{aligned}
r_{\mathbb{R}}^c(A,B) \ = \ & \inf\{\|[\Delta_A, \Delta_B]\|_2 \mid \Delta_A \in \mathbb{R}^{n\times n}, \Delta_B \in \mathbb{R}^{n\times m}, \\
& (A+\Delta_A, B+\Delta_B) \text{ is uncontrollable}\}
\end{aligned}$$

In [3], the real controllability radius was shown to be given by the following formula:

$$r_{\mathbb{R}}^c(A,B) = \min_{s\in\mathbb{C}} \tilde{\tau}_n\left(\begin{bmatrix} A - sI, & B \end{bmatrix}\right)$$

In [10], the *structured* real controllability radius was defined as:

$$r_{\mathbb{R}}^{c,struct}(A,B,\mathcal{E},\mathcal{F},\mathcal{G}) \quad (15)$$
$$= \inf\{\|[\Delta_A,\Delta_B]\|_2 \mid \Delta_A \in \mathbb{R}^{n\times n}, \Delta_B \in \mathbb{R}^{n\times m}, (A+\mathcal{E}\Delta_A\mathcal{F}, B+\mathcal{E}\Delta_B\mathcal{G}) \text{ is uncontrollable}\}$$

where $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ were limited in [10] to being square nonsingular matrices. The invertibility of $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ allowed the structured real controllability radius to be reduced to the unstructured case, which can then easily be computed as follows [10]:

$$r_{\mathbb{R}}^{c,struct}(A,B,\mathcal{E},\mathcal{F},\mathcal{G}) = \min_{s\in\mathbb{C}} \tilde{\tau}_n\left(\mathcal{E}^{-1}\begin{bmatrix} A-sI, & B\end{bmatrix}\begin{bmatrix}\mathcal{F} & 0\\ 0 & \mathcal{G}\end{bmatrix}^{-1}\right)$$

Using the results of the generalized real perturbation values presented in Section III, we can remove the nonsingularity limitation of $\mathcal{F}$ and $\mathcal{G}$.

Theorem 4.1: Given the LTI system (1), and matrices $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$, where only $\mathcal{E}$ is required to be nonsingular, then the structured real controllability radius (15) is given as follows:

$$r_{\mathbb{R}}^{c,struct}(A,B,\mathcal{E},\mathcal{F},\mathcal{G}) \quad (16)$$
$$= \min_{s\in\mathbb{C}} \tilde{\tau}_n\left(\mathcal{E}^{-1}\begin{bmatrix} A-sI, & B\end{bmatrix}, \begin{bmatrix}\mathcal{F} & 0\\ 0 & \mathcal{G}\end{bmatrix}\right)$$

Proof: The proof is straightforward. Given $s \in \mathbb{C}$, the perturbed system $(A+\mathcal{E}\Delta_A\mathcal{F}, B+\mathcal{E}\Delta_B\mathcal{G})$ is uncontrollable at s if and only if

$$\text{rank}([A+\mathcal{E}\Delta_A\mathcal{F}-sI, B+\mathcal{E}\Delta_B\mathcal{G}]) < n$$
$$\Leftrightarrow \quad \text{rank}\left(\mathcal{E}^{-1}[A-sI,B]+[\Delta_A,\Delta_B]\begin{bmatrix}\mathcal{F} & 0\\ 0 & \mathcal{G}\end{bmatrix}\right) < n$$

Using the generalized real perturbation values (of the second kind, see Definition 3.1), the smallest real perturbations, $[\Delta_A,\Delta_B]$, such that the perturbed system is uncontrollable at s is given by $\tilde{\tau}_n\left(\mathcal{E}^{-1}[A-sI,B], \begin{bmatrix}\mathcal{F} & 0\\ 0 & \mathcal{G}\end{bmatrix}\right)$. The structured real controllability radius, $r_{\mathbb{R}}^{c,struct}$, is then the minimization of this function over the complex plane. ■

From the state space model of the pendulum system given by (14), it can be seen that the "upper" portion of the (A,B) matrices are hard 0's and 1's. This arises from the relationship between the state variables. Therefore in [10], only perturbations that disrupt the lower half of the (A,B) matrices were considered. Furthermore, the perturbations were normalized by the (A,B) matrices. As a result, perturbations of the following form were considered in [10]:

$$A \rightarrow \left(I+\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\Delta_A\right)A$$
$$B \rightarrow \left(I+\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\Delta_B\right)B$$

The structured real controllability radius of the pendulum system defined in [10] was hence given by (15), where $\mathcal{E} = \left[\begin{smallmatrix}0 & 0\\ 0 & I\end{smallmatrix}\right]$, $\mathcal{F}=A$, and $\mathcal{G}=B$. However, since $\mathcal{F}$ and $\mathcal{G}$ are not necessarily nonsingular, an ad-hoc normalization technique involving a random matrix was used in [10] to obtained an estimate of the actual radius. In this example, we will use equation (16) to compute the structured real controllability radius of the pendulum. Like the approach in [10], however, we will approximate $\mathcal{E}$ by $\tilde{\mathcal{E}} = \left[\begin{smallmatrix}\epsilon I & 0\\ 0 & I\end{smallmatrix}\right]$, where $\epsilon > 0$ is chosen as small as possible (e.g. $\epsilon = 10^{-6}$) such that $\tilde{\mathcal{E}}$ is invertible and not ill-conditioned.

TABLE I
THE STRUCTURED REAL CONTROLLABILITY RADIUS OF A MULTI-LINK INVERTED PENDULUM (REVISITED)

Number of Links v	$r_{\mathbb{R}}^{c,struct}$ obtained in [10]	$r_{\mathbb{R}}^{c,struct}$ obtained via (16)
1	$1.000_{10^{+0}}$	$1.000_{10^{+0}}$
2	$1.111_{10^{-1}}$	$1.085_{10^{-1}}$
3	$4.688_{10^{-2}}$	$4.655_{10^{-2}}$
4	$2.456_{10^{-2}}$	$2.450_{10^{-2}}$
5	$1.467_{10^{-2}}$	$1.466_{10^{-2}}$
6	$9.565_{10^{-3}}$	$9.559_{10^{-3}}$
7	$6.635_{10^{-3}}$	$6.633_{10^{-3}}$

Table I shows the values previously obtained in [10] and the new values computed using (16). The new values obtained using (16) are tighter and more accurate than those obtained in [10]. Note, however, that even though the new values obtained by (16) are more accurate than those obtained in [10], the new values are still an estimate of the actual structured real controllability radius due to the approximation introduced in $\tilde{\mathcal{E}} = \left[\begin{smallmatrix}\epsilon I & 0\\ 0 & I\end{smallmatrix}\right]$. To compute the true structured real controllability radius, we will need to remove the nonsingularity requirement of $\mathcal{E}$. This will be done in a future work.

B. Structured Transmission Zero at s Radius

In this example, we will use a new normalization technique and recompute the transmission zero at $s=0$ radius [6] of several industrial systems' linearized LTI models ([6]).

Using the definition found in [12], the transmission zeros (TZ) of a LTI system (1) are the set of $s \in \mathbb{C}$ such that

$$\text{rank}\left(\begin{bmatrix}A-sI & B\\ C & D\end{bmatrix}\right) < n+\min(r,m)$$

The *real transmission zero at s radius*, denoted by $r_{\mathbb{R}}^{TZ}$, is defined as [6]:

$$r_{\mathbb{R}}^{TZ}(C,A,B,D,s) = \inf\left\{\left\|\begin{bmatrix}\Delta_A & \Delta_B\\ \Delta_C & \Delta_D\end{bmatrix}\right\|_2 \,\middle|\, \Delta_A \in \mathbb{R}^{n\times n}, \Delta_B \in \mathbb{R}^{n\times m}, \Delta_C \in \mathbb{R}^{r\times n}, \Delta_D \in \mathbb{R}^{r\times m}, (C+\Delta_C, A+\Delta_A, B+\Delta_B, D+\Delta_D) \text{ has a TZ at } s\right\}$$

which can be computed by the following formula [6]:

$$r_{\mathbb{R}}^{TZ}(C,A,B,D,s) = \tilde{\tau}_{n+\min(r,m)}\left(\begin{bmatrix}A-sI & B\\ C & D\end{bmatrix}\right)$$

Recall from [13] that there exists a solution to the robust servomechanism problem for constant tracking and disturbance rejection if and only if the system has no transmission zeros at $s=0$. Hence, the real transmission zero at $s=0$

radius can be used to evaluate the robustness of the system's servomechanism controller. In particular, if a system's transmission zero at $s = 0$ radius is small, then the system is very close to having a transmission zero at the origin, which implies that the system's (current) servomechanism controller may be fragile and not robust.

The real transmission zero at $s = 0$ radii, $r_{\mathbb{R}}^{TZ}$, of a number of industrial systems' linearized LTI models were computed in [6]. In this section, we will consider a different normalization, which provides a relative measure of the perturbations with respect to the plant parameters, whereas the values obtained in [6] give an absolute measure. In particular, we will consider perturbations of the following form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow \begin{bmatrix} (I+\Delta_A)\,A & (I+\Delta_B)\,B \\ C & D \end{bmatrix} \qquad (17)$$

and we shall define the *structured real transmission zero at s radius* to be:

$$r_{\mathbb{R}}^{TZ,struct}(C,A,B,D,s) = \inf\{\|[\ \Delta_A \quad \Delta_B\ ]\|_2 \mid \Delta_A \in \mathbb{R}^{n\times n}, \Delta_B \in \mathbb{R}^{n\times n}, (C,A,B,D) \text{ as perturbed according to (17) has a TZ at } s\}$$

The motivation behind the perturbation structure (17) is due to the fact that the C and D matrices of the plants studied in [6] (given in Table II) contain hard 0's and 1's, which is a result of the systems' design structures. Hence, it would not be appropriate to perturb the C and D matrices.

It can be shown that the structured real transmission zero at s radius can be computed by:

$$r_{\mathbb{R}}^{TZ,struct}(C,A,B,D,s) = \qquad (18)$$
$$\tilde{\tau}_{n+\min(r,m)}\left(\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix}^{-1}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}, \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$$

for $\epsilon \rightarrow 0$.

Proof: The proof is similar to the proof of Theorem 4.1 and is omitted. ■

Table II displays the results obtained using the proposed normalization (18). It can be seen that using the proposed technique, there is a large difference in the structured TZ radius values between the various models, and it can be seen that some plants appear to be very fragile; i.e. plant F, the boiler system. On studying this system, it is found that the original unperturbed system has a transmission zero that is very close to the origin, namely, at $s = -9.55_{10^{-3}}$, which implies that a slight perturbation can cause this transmission zero to approach the origin, resulting in no solution existing to the servomechanism problem for this system. Plant D has a zero % radius because the plant's nominal system is degenerate and therefore has no solution to the servomechanism problem.

TABLE II

STRUCTURED REAL TZ AT $s = 0$ RADIUS FOR VARIOUS PLANTS

	Plant		$r_{\mathbb{R}}^{TZ,struct}$ (obtained via (18))
A.	distillation	$n = 11$, S, M	$2.215_{10^{-2}}$ (= 2.22%)
B.	gas turbine	$n = 4$, S, M	$2.672_{10^{-1}}$ (= 26.7%)
C.	helicopter	$n = 4$, US, M	$9.244_{10^{-2}}$ (= 9.24%)
D.	thermal	$n = 9$, S, NM	0 (= 0%)
E.	pilot	$n = 6$, US, M	$1.710_{10^{-3}}$ (= 0.171%)
F.	boiler	$n = 9$, US, NM	$2.043_{10^{-6}}$ (= $2.04_{10^{-4}}$%)
G.	mass	$n = 6$, US, M	$9.926_{10^{-2}}$ (= 9.93%)
H.	2-cart	$n = 8$, US, NM	$4.726_{10^{-1}}$ (= 47.3%)

LEGEND: n is the order of the plant, and S, US, M and NM respectively denote a stable, an unstable, a minimum-phase, and a nonminimum-phase plant.

V. Conclusions

In this paper, we generalized the real perturbation value problem to account for more general perturbations. Formulas were derived in the paper to compute the so-called generalized real perturbation values of a given complex matrix. Using these results, we introduced a new normalized version of the transmission zero at s radius, which is more appropriate for the industrial systems studied in [6]. We also revisited the pendulum problem in [10] and obtained better estimates of the structured real controllability radius of the multi-link inverted pendulum system. In the future, similar studies involving the other robustness measures (e.g. the structured real DFM radius and the structured real minimum-phase radius) can also be carried out using the results of this paper.

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