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Objective-Driven ILC for Point-to-Point Movement Tasks

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***Abstract*— A framework is developed which allows a general class of ILC algorithm to be applied to tasks which require the plant output to reach a given point in a set time. It is shown that superior convergence and robustness properties are obtained compared with those associated with using the original ILC law to track an arbitrary reference trajectory satisfying the end-point conditions. Experimental results are given to confirm the theoretical findings, and verify the favourable qualities of this novel approach to point-to-point movement control.**

I. Introduction

Apart from a small number of exceptions dealing with specialised cases (for example [1], [2], [3]), little work has been conducted to incorporate and practically assess Iterative Learning Control (ILC) algorithms in which the repeated operation may consist of a more general objective, which need not comprise the tracking of a static pre-defined reference. A framework is therefore proposed in which the high level of performance that ILC has been shown to provide, can be combined with a greater degree of flexibility in terms of how the tracking control task is defined, and the way in which it is achieved. This has obvious application to industrial robotic point-to-point movement operations, and is partly motivated by the recent successful use of ILC in stroke rehabilitation. Here electrical stimulation was mediated by ILC to assist repeated limb movements, but the exact path to be followed by the patient's arm was not critical, rather the start and end-point positions of the arm [4]. This application requires precise control over the degree of arm extension produced by electrical stimulation during the reaching movement, thus precluding use of the initialisation mechanism (consisting of a robotic arm) to reach the end-point. The application area necessiates use of a smooth reference which is as close to the patient's natural arm movement as possible, a requirement that may not be met by controllers which simply adjust the input at samples close to the end-point (the arm may be far away from the desired position). As well as satisfying these issues, the proposed method benefits from the ability to learn from experience gained over previous trials of the task which is provided by the ILC framework (and sets it aside from alternative end-point tracking techniques (see [5], [6], [7] and references therein). Furthermore, in seeking a global solution to the problem, it is shown that the robustness to uncertainty of the class of ILC considered can be increased.

II. Objective-driven ILC Development

The key departure from the standard ILC framework that will be considered is to permit the reference to change between iterations. It will be shown that this leads to a simple method which has the ability to speed up learning when the intention is not to track a fixed reference, but instead wish to perform a specified point-to-point movement task. The initial stage of this process is to define the intended point-to-point task. A SISO linear time-invariant system is considered, and given, on trial k, by

$$\begin{aligned} x_k(t+1) &= Ax_k(t)+Bu_k(t) \\ y_k(t) &= Cx_k(t) \end{aligned} \quad (1)$$

where A, B and C are matrices of suitable dimension, and a sample time of unity is assumed for notational simplicity. The tracking task is only defined at times 0 and T when the output must equal 0 and r_N respectively for all k, where T denotes the length of the task. An initial reference could be selected in the form of any trajectory which achieves the point-to-point task (e.g. a straight line trajectory connecting 0, at time 0 to r_N at time T). A sensible approach is to use the solution u^* to the optimisation

$$\begin{aligned} &\text{minimise } \|u\|_2^2 \\ &\text{subject to } [CA^{N-1}B \; CA^{N-2}B \ldots CAB \; CB]u = r_N \end{aligned} \quad (2)$$

where $u = [u(1), u(2), \ldots, u(N)]^T$ and the integer $N = \frac{T}{T_s}$. This satisfies the end-points whilst minimising the norm of the input signal, and can be solved using constrained quadratic minimisation (e.g. through application of the Newton method). The solution and corresponding plant output are shown in Figure 1 for the non-minimum phase plant (introduced in Section V) using the end-point constraint $r_N = 10$. The optimal time solution u^* is then set as the initial

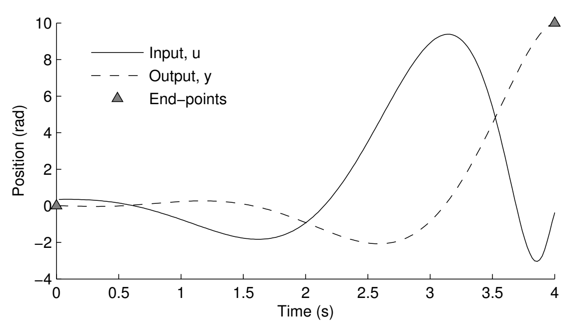

Fig. 1. Initial reference selection for non-minimum phase plant.

input vector $u_1 = u^*$ and initial reference vector $r_1 = Gu^*$, with the lifted system matrix

$$G = \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \ldots & CB \end{bmatrix} \quad (3)$$

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Having defined the task and an initial reference, a controller is required to robustly achieve the tracking task. An ILC algorithm of the general form

$$u_{k+1} = u_k + \beta K e_k \tag{4}$$

will be considered, where K is a suitable linear operator which may be non-causal (e.g. the adjoint algorithm [8] which has been shown to provide a high degree of performance and robustness [9]), and β is a positive scalar used to influence algorithm convergence. However, at the end of the k^{th} trial, y_k is taken and instead of calculating $e_k = r - y_k$, with a fixed reference, r, for use in the update (4), the reference is allowed to change, and is replaced with r_{k+1}, leading to

$$e_k^* = r_{k+1} - y_k \tag{5}$$

So the ILC law (4) becomes

$$u_{k+1} = u_k + \beta K e_k^* \tag{6}$$

The time domain relationships which then arise are

$$\begin{aligned} y_{k+1} &= y_k + \beta G K e_k^* &(7)\\ r_{k+1} - y_{k+1} &= r_{k+1} - y_k - \beta G K e_k^* &(8)\\ r_{k+1} - y_{k+1} &= r_{k+1} - y_k - \beta G K (r_{k+1} - y_k) &(9)\\ e_{k+1} &= (r_{k+1} - y_k)(I - \beta G K) &(10) \end{aligned}$$

where I is the identity operator. So the criterion for monotonic convergence becomes

$$||e_{k+1}||^2 = \|(r_{k+1} - y_k)(I - \beta G K)\|^2 < ||e_k||^2 \tag{11}$$

and since

$$\|(r_{k+1} - y_k)(I - \beta G K)\|^2 \leq \|r_{k+1} - y_k\|^2 \|I - \beta G K\|^2 \tag{12}$$

a sufficient condition for monotonic convergence is

$$||I - \beta G K||^2 < 1 \tag{13}$$

together with the requirement

$$\frac{\|r_{k+1} - y_k\|^2}{\|e_k\|^2} = \frac{\|r_{k+1} - y_k\|^2}{\|r_k - y_k\|^2} \leq 1 \tag{14}$$

which equates to

$$||r_{k+1} - y_k||^2 \leq ||r_k - y_k||^2 \tag{15}$$

The first condition, (13), is the monotonic convergence criterion associated with the ILC algorithm (4) using a static reference (note that if K is chosen to be a scalar, G must have full rank, and the system must consequently be relative degree 1). If this is satisfied it is then necessary to choose r_{k+1} such that (15) is satisfied. Now this can also be expressed as

$$||r_{k+1} - y_k||^2 = ||\Delta r_k + e_k||^2 \leq ||e_k||^2 \tag{16}$$

where $\Delta r_k = r_{k+1} - r_k$. The initial reference already satisfies the end-point constraints so it is possible to choose Δr_k from a suitable set of functions with end-points equal to zero for $k \geq 2$. In this paper the set considered is the set of harmonic sinewaves, since this leads to simplification in how Δr_k is chosen, although it also limits it to be an even function. This choice is equivalent to taking the Discrete Fourier Transform (DFT) of Δr_k, with components, $\Delta R_{k,i}$, and requiring that they are all real (i.e. $\text{Im}\{\Delta R_{k,i}\} = 0$ for $i = \{0, 1, \ldots N-1\}$). It is the property that the components of ΔR_k are all real that simplifies how they can be selected.

III. CHOICE OF TRAJECTORY

At the end of the k^{th} iteration, ΔR_k must be chosen to satisfy (16). Having limited the set from which it may belong in the way described, it is easy to choose Δr_k to minimise $||\Delta r_k + e_k||^2$ since there is no global constraint and the optimisation can be conducted frequency-wise. The transparency this affords is used to confirm favourable properties of this choice of update in Section IV, and, although alternative selections may results in similar properties, the lack of a frequency-wise optimisation means they cannot be analysed in such a straight-forward manner. Taking the inner product

$$\begin{aligned} ||e_k^*||^2 &= ||\Delta r_k + e_k||^2 = ||\Delta r_k||^2 + ||e_k||^2 + 2\text{Re}\langle \Delta r_k, e_k \rangle \\ &= \frac{1}{N}\sum_{i=0}^{N-1} \Delta R_{k,i}^2 + \frac{1}{N}\sum_{i=0}^{N-1} |E_{k,i}|^2 + \frac{2}{N}\text{Re}\left\{\sum_{i=0}^{N-1} \Delta R_{k,i} E_{k,i}\right\} \\ &= \frac{1}{N}\sum_{i=0}^{N-1} \Delta R_{k,i}^2 + \frac{1}{N}\sum_{i=0}^{N-1} |E_{k,i}|^2 + \frac{2}{N}\sum_{i=0}^{N-1} \Delta R_{k,i}\text{Re}\{E_{k,i}\} \\ &= \frac{1}{N}\sum_{i=0}^{N-1} |E_{k,i}|^2 + \frac{1}{N}\sum_{i=0}^{N-1} \Delta R_{k,i}\left(\Delta R_{k,i} + 2\text{Re}\{E_{k,i}\}\right) \end{aligned} \tag{17}$$

Each frequency component of the reference change, $\Delta R_{k,i}$, can be chosen to minimise this in order to best satisfy (16). The differential with respect to ΔR_k is

$$\frac{\partial}{\partial \Delta R_k}||\Delta r_k + e_k||^2 = \frac{2}{N}\sum_{i=0}^{N-1} \Delta R_{k,i} + \frac{2}{N}\sum_{i=0}^{N-1} \text{Re}\{E_{k,i}\} \tag{18}$$

Setting all elements to zero yields the optimal solution

$$\Delta R_{k,i} = -\text{Re}\{E_{k,i}\} \quad \text{for } i = 0, 1 \ldots N-1 \tag{19}$$

Note that it can be shown that this is also the optimal solution to directly minimising the error norm over the next trial

$$\begin{aligned} ||e_{k+1}||^2 &= ||\,(\Delta r_k + e_k)(I - \beta G K)\,||^2 &(20)\\ &= \frac{1}{N}\sum_{i=0}^{N-1} |\Delta R_{k,i} + E_{k,i}|^2 |1 - \beta G_i K_i|^2 &(21) \end{aligned}$$

where G_i, K_i denote the i^{th} DFT components of G, K respectively. Using (19) the corresponding optimal value of (16) on trial k is given by

$$\begin{aligned} ||\Delta r_k + e_k||^2 &= \frac{1}{N}\sum_{i=0}^{N-1} |E_{k,i}|^2 - \frac{1}{N}\sum_{i=0}^{N-1} \left(\text{Re}\{E_{k,i}\}\right)^2 &(22)\\ &= \frac{1}{N}\sum_{i=0}^{N-1} \left(\text{Im}\{E_{k,i}\}\right)^2 \leq ||e_k||^2 &(23)\\ &< ||e_k||^2 \text{ iff } \exists\, i \text{ s.t } \text{Re}\{E_{k,i}\} \neq 0 &(24) \end{aligned}$$

This therefore ensures the sufficient condition given by (16) is satisfied. Note that the reference is being updated using

$$r_{k+1} = r_k + \Delta r_k \tag{25}$$

which is chosen to speed up learning of the final trajectory. It is always possible to ensure $||\Delta r_k + e_k||^2 \leq ||e_k||^2$ since

$$
\begin{aligned}
||\Delta r_k + e_k||^2 - ||e_k||^2 &= -\frac{1}{N}\sum_{i=0}^{N-1}\left(\mathrm{Re}\{E_{k,i}\}\right)^2 \leq 0 \\
&< 0 \text{ iff } \exists\, i \text{ s.t } \mathrm{Re}\{E_{k,i}\} \neq 0 \quad (26)
\end{aligned}
$$

From (10) it is possible to state

$$
\begin{aligned}
||e_{k+1}||^2 &= \left\| \frac{(\Delta r_k + e_k)}{e_k}(I - \beta GK) \star e_k \right\|^2 && (27) \\
&\leq \left\| \frac{(\Delta r_k + e_k)}{e_k}(I - \beta GK) \right\|^2 ||e_k||^2 && (28)
\end{aligned}
$$

in which the division, and multiplication, $\star$, are executed component-wise. The norm error reduction ratio is

$$
\begin{aligned}
\frac{||e_{k+1}||^2}{||e_k||^2} &\leq \left\| \frac{(\Delta r_k + e_k)}{e_k}(I - \beta GK) \right\|^2 \\
&= \frac{1}{N}\sum_{i=0}^{N-1} \frac{\left(\mathrm{Im}\{E_{k,i}\}\right)^2 |1 - \beta G_i K_i|^2}{|E_k|^2} \\
&= \frac{1}{N}\sum_{i=0}^{N-1} \frac{1}{1 + \frac{\left(\mathrm{Re}\{E_{k,i}\}\right)^2}{\left(\mathrm{Im}\{E_{k,i}\}\right)^2}} |1 - \beta G_i K_i|^2 \\
&= \frac{1}{N}\sum_{i=0}^{N-1} \sin^2\left(\angle E_{k,i}\right)|1 - \beta G_i K_i|^2 && (29)
\end{aligned}
$$

whereas it is easy to show that in the case of a static reference

$$
\frac{||e_{k+1}||^2}{||e_k||^2} \leq \frac{1}{N}\sum_{i=0}^{N-1} |1 - \beta G_i K_i|^2 \qquad (30)
$$

so the use of objective-driven ILC has introduced the multiplier $\sin^2\left(\angle E_{k,i}\right)$ on the i^{th} frequency component. This multiplier can be used to relax the monotonic convergence condition (13). From (29) a sufficient condition to produce trial-to-trial error reduction is

$$
\sin^2\left(\angle E_{k,i}\right)|1 - \beta G_i K_i|^2 < 1 \qquad (31)
$$

for each frequency, i, so that

$$
|1 - \beta G_i K_i|^2 < \frac{1}{\sin^2\left(\angle E_{k,i}\right)} \qquad (32)
$$

so

$$
|1 - \beta G_i K_i| < \frac{1}{|\sin\left(\angle E_{k,i}\right)|} \qquad (33)
$$

compared with the equivalent for the static reference case

$$
|1 - \beta G_i K_i| < 1 \qquad (34)
$$

Since $0 \leq |\sin\left(\angle E_{k,i}\right)| \leq 1$, this clearly provides the possibility of additional robustness at each frequency, but this robustness is with respect to tracking the constantly modified reference. To examine the system robustness, a multiplicative plant uncertainty, expressed as $G_i = G_{0,i}M_i$ where $G_{0,i}$ is the nominal plant, can be substituted in (33) and (34) and the region of the uncertainty space examined in which M_i must lie to satisfy the monotonic convergence criterion. This is illustrated in Figure 2 for both cases, and the larger region of convergence for the new method is clear. The effect of this bound on the convergence and robustness associated with the complete dynamic system will now be investigated.

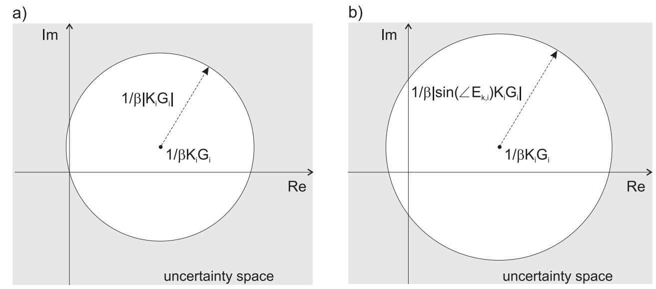

Fig. 2. Uncertainty bound for i^{th} frequency component using a) static reference, and b) modified reference.

IV. SYSTEM CONVERGENCE AND ROBUSTNESS PROPERTIES

From the reference change $\Delta R_{k,i}$ given by (19), it is possible to write

$$
\begin{aligned}
R_{k+1,i} &= R_{k,i} - \mathrm{Re}\{\mathrm{E_{k,i}}\} \\
&= R_{k,i} - \mathrm{Re}\{\mathrm{R_{k,i} - G_i U_{k,i}}\} \\
&= \mathrm{Re}\{\mathrm{G_i U_{k,i}}\} + \mathrm{jIm}\{\mathrm{R_{k,i}}\} \\
&= \mathrm{Re}\{\mathrm{G_i U_{k,i}}\} + \mathrm{jIm}\{\mathrm{R_{0,i}}\} \qquad (35)
\end{aligned}
$$

Taking real and imaginary components of the frequency-transformed ILC algorithm (6) (with $E^*_{k,i} = R_{k+1,i} - Y_{k,i} = R_{k+1,i} - G_i U_{k,i}$)

$$
\begin{cases}
\hat{U}_{k+1,i} = \hat{U}_{k,i} + \bar{K}_i \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} - \mathrm{\bar{K}_i Im}\{\mathrm{R_{0,i}}\} \\
\bar{U}_{k+1,i} = \bar{U}_{k,i} - \hat{K}_i \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} + \mathrm{\hat{K}_i Im}\{\mathrm{R_{0,i}}\}
\end{cases} \qquad (36)
$$

where β has been absorbed into each K_i for conciseness, and $\hat{\cdot}$ and $\bar{\cdot}$ denote $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ respectively. The real and imaginary components of the output can be expressed as

$$
\begin{aligned}
\bar{Y}_{k+1,i} &= \mathrm{Im}\{\mathrm{G_i U_{k+1,i}}\} = \mathrm{\hat{G}_i \bar{U}_{k+1,i}} + \mathrm{\bar{G}_i \hat{U}_{k+1,i}} \\
&= \hat{G}_i \bar{U}_{k,i} - \hat{G}_i \hat{K}_i \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} + \mathrm{\hat{G}_i \hat{K}_i Im}\{\mathrm{R_{0,i}}\} \\
&\quad + \bar{G}_i \hat{U}_{k,i} + \bar{G}_i \bar{K}_i \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} - \mathrm{\bar{G}_i \bar{K}_i Im}\{\mathrm{R_{0,i}}\} \\
&= \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} + \mathrm{Im}\{\mathrm{G_i U_{k,i}}\}\left(\mathrm{\bar{G}_i \bar{K}_i - \hat{G}_i \hat{K}_i}\right) + \\
&\quad \mathrm{Re}\{\mathrm{G_i K_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \\
&= \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} - \mathrm{Im}\{\mathrm{G_i U_{k,i}}\}\mathrm{Re}\{\mathrm{G_i K_i}\} \\
&\quad + \mathrm{Re}\{\mathrm{G_i K_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \\
&= \mathrm{Im}\{\mathrm{G_i U_{k,i}}\}\left(1 - \mathrm{Re}\{\mathrm{G_i K_i}\}\right) + \mathrm{Re}\{\mathrm{G_i K_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \\
& \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{Y}_{k+1,i} &= \mathrm{Re}\{\mathrm{G_i U_{k+1,i}}\} = \mathrm{\hat{G}_i \hat{U}_{k+1,i}} - \mathrm{\bar{G}_i \bar{U}_{k+1,i}} \\
&= \hat{G}_i \hat{U}_{k,i} + \hat{G}_i \bar{K}_i \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} - \mathrm{\bar{G}_i \bar{U}_{k,i}} \\
&\quad + \bar{G}_i \hat{K}_i \mathrm{Im}\{\mathrm{G_i U_{k,i}}\} \\
&= \mathrm{Re}\{\mathrm{G_i U_{k,i}}\} + \mathrm{Im}\{\mathrm{G_i U_{k,i}}\}\left(\mathrm{\hat{G}_i \bar{K}_i + \bar{G}_i \hat{K}_i}\right) - \\
&\quad \mathrm{Im}\{\mathrm{G_i K_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \\
&= \mathrm{Re}\{\mathrm{G_i U_{k,i}}\} + \mathrm{Im}\{\mathrm{G_i U_{k,i}}\}\mathrm{Im}\{\mathrm{G_i K_i}\} \\
&\quad - \mathrm{Im}\{\mathrm{G_i K_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \qquad (38)
\end{aligned}
$$

To find the output value as k increases, it is possible to express the output components (37) and (38) as iterative sequences

$$\bar{Y}_{k+1,i} = \mathrm{Im}\{\mathrm{G_iU_{k+1,i}}\} = (1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{k+1}}\bar{\mathrm{Y}}_{0,\mathrm{i}} + \mathrm{Im}\{\mathrm{R_{0,i}}\}\mathrm{Re}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\mathrm{k}}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}} \quad (39)$$

which, if $|1-\mathrm{Re}\{\mathrm{G_iK_i}\}| < 1$, converges to

$$\bar{Y}_{\infty,i} = \frac{\mathrm{Im}\{\mathrm{R_{0,i}}\}\mathrm{Re}\{\mathrm{G_iK_i}\}}{1-(1-\mathrm{Re}\{\mathrm{G_iK_i}\})} = \mathrm{Im}\{\mathrm{R_{0,i}}\} \quad (40)$$

and similarly

$$\begin{aligned}\hat{Y}_{k+1,i} &= \mathrm{Re}\{\mathrm{G_iU_{k+1,i}}\} \\ &= \hat{Y}_{0,i} + \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\mathrm{k}}\left(\bar{\mathrm{Y}}_{\mathrm{p,i}} - \mathrm{Im}\{\mathrm{R_{0,i}}\}\right) \\ &= \hat{Y}_{0,i} + \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{j=0}}^{\mathrm{k}}\Big\{(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{j}}\bar{\mathrm{Y}}_{0,\mathrm{i}} + \\ &\quad \mathrm{Im}\{\mathrm{R_{0,i}}\}\mathrm{Re}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\mathrm{j-1}}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}} - \mathrm{Im}\{\mathrm{R_{0,i}}\}\Big\}\end{aligned} \quad (41)$$

Now this can be simplified using

$$\begin{aligned}1-a\sum_{p=0}^{k}(1-a)^p &= 1-a-a(1-a)-a(1-a)^2- \\ &\quad a(1-a)^3-\cdots-a(1-a)^k \\ &= (1-a)(1-a-(1-a)-(1-a)^2- \\ &\quad (1-a)^3\cdots-(1-a)^k) \\ &= (1-a)^2(1-a-(1-a)-(1-a)^2- \\ &\quad (1-a)^3\cdots-(1-a)^{k-1}) \\ &= (1-a)^{k+1}\end{aligned} \quad (42)$$

so that (41) can be written as

$$\begin{aligned}\hat{Y}_{k+1,i} &= \hat{Y}_{0,i} + \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{j=0}}^{\mathrm{k}}\Big\{(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{j}}\bar{\mathrm{Y}}_{0,\mathrm{i}} - \\ &\quad \mathrm{Im}\{\mathrm{R_{0,i}}\}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{j}}\Big\} \\ &= \hat{Y}_{0,i} + (\bar{Y}_{0,i} - \mathrm{Im}\{\mathrm{R_{0,i}}\})\,\mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{j=0}}^{\mathrm{k}}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{j}}\end{aligned} \quad (43)$$

and this converges to

$$\begin{aligned}\hat{Y}_{\infty,i} &= \hat{Y}_{0,i} + (\bar{Y}_{0,i} - \mathrm{Im}\{\mathrm{R_{0,i}}\})\frac{\mathrm{Im}\{\mathrm{G_iK_i}\}}{\mathrm{Re}\{\mathrm{G_iK_i}\}} \quad (44) \\ &= \hat{Y}_{0,i} + (\bar{Y}_{0,i} - \mathrm{Im}\{\mathrm{R_{0,i}}\})\tan(\angle G_iK_i) \quad (45)\end{aligned}$$

Having found, through (40) and (45), the final output as $k \to \infty$, the stability of the system will now be investigated. To do this, let the reference, R^*, be introduced and equated with the final converged output values, so that

$$\begin{cases}\bar{R}_i^* = \bar{Y}_{\infty,i} = \mathrm{Im}\{\mathrm{R_{0,i}}\} \\ \hat{R}_i^* = \hat{Y}_{\infty,i} = \hat{Y}_{0,i} + (\bar{Y}_{0,i} - \mathrm{Im}\{\mathrm{R_{0,i}}\})\tan(\angle G_iK_i)\end{cases} \quad (46)$$

then using (37)

$$\begin{aligned}\bar{E}_{k+1,i}^* &= \bar{R}_i^* - \bar{Y}_{k+1,i} \\ &= \mathrm{Im}\{\mathrm{R_{0,i}}\} - (1-\mathrm{Re}\{\mathrm{G_iK_i}\})\bar{\mathrm{Y}}_{\mathrm{k,i}} - \\ &\quad \mathrm{Re}\{\mathrm{G_iK_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \\ &= (1-\mathrm{Re}\{\mathrm{G_iK_i}\})\left\{\mathrm{Im}\{\mathrm{R_{0,i}}\} - \bar{\mathrm{Y}}_{\mathrm{k,i}}\right\} \\ &= (1-\mathrm{Re}\{\mathrm{G_iK_i}\})\bar{\mathrm{E}}_{\mathrm{k,i}}^*\end{aligned} \quad (47)$$

and similarly (38) gives

$$\begin{aligned}\hat{E}_{k+1,i}^* &= \hat{R}_i^* - \hat{Y}_{k+1,i} = \hat{R}_i^* - \hat{Y}_{k,i} - \mathrm{Im}\{\mathrm{G_iK_i}\}\bar{\mathrm{Y}}_{\mathrm{k,i}} + \\ &\quad \mathrm{Im}\{\mathrm{G_iK_i}\}\mathrm{Im}\{\mathrm{R_{0,i}}\} \\ &= \hat{E}_{k,i}^* + \mathrm{Im}\{\mathrm{G_iK_i}\}\left(\mathrm{Im}\{\mathrm{R_0}\} - \bar{\mathrm{Y}}_{\mathrm{k,i}}\right) \\ &= \hat{E}_{k,i}^* + \mathrm{Im}\{\mathrm{G_iK_i}\}\left(\bar{\mathrm{R}}_{\mathrm{i}}^* - \bar{\mathrm{Y}}_{\mathrm{k,i}}\right) \\ &= \hat{E}_{k,i}^* + \mathrm{Im}\{\mathrm{G_iK_i}\}\bar{\mathrm{E}}_{\mathrm{k,i}}\end{aligned} \quad (48)$$

Using (47) and (48), the dynamic system for frequency i is

$$\begin{bmatrix}\bar{E}_{k+1,i}^* \\ \hat{E}_{k+1,i}^*\end{bmatrix} = \begin{bmatrix}1-\mathrm{Re}\{\mathrm{G_iK_i}\} & 0 \\ \mathrm{Im}\{\mathrm{G_iK_i}\} & 1\end{bmatrix}\begin{bmatrix}\bar{E}_{k,i}^* \\ \hat{E}_{k,i}^*\end{bmatrix} \quad (49)$$

and it is then straightforward to show that

$$\hat{E}_{\infty,i}^* = \tan(\angle G_iK_i)\bar{E}_{0,i}^* + \hat{E}_{0,i}^* \quad (50)$$

which equals 0 using the reference defined by (46), so the system converges to this reference with zero error, but this signal is itself determined by initial conditions. The system (49) has eigenvalues at $+1$, $(1-\mathrm{Re}\{\mathrm{G_iK_i}\})$ making it marginally stable. Instability is therefore avoided if $|1-\mathrm{Re}\{\mathrm{G_iK_i}\}| < 1$. When using a static reference it can be shown that the dynamic error system is instead given by

$$\begin{bmatrix}\bar{E}_{k+1,i}^* \\ \hat{E}_{k+1,i}^*\end{bmatrix} = \begin{bmatrix}1-\mathrm{Re}\{\mathrm{G_iK_i}\} & -\mathrm{Im}\{\mathrm{G_iK_i}\} \\ \mathrm{Im}\{\mathrm{G_iK_i}\} & 1-\mathrm{Re}\{\mathrm{G_iK_i}\}\end{bmatrix}\begin{bmatrix}\bar{E}_{k,i}^* \\ \hat{E}_{k,i}^*\end{bmatrix} \quad (51)$$

with eigenvalues $1-\mathrm{Re}\{\mathrm{G_iK_i}\} \pm \mathrm{j}\mathrm{Im}\{\mathrm{G_iK_i}\}$, that is $1-G_iK_i, 1-\overline{G_iK_i}$. Instability is avoided if $|1-G_iK_i| < 1$ which is more difficult to satisfy than the requirement for the objective-driven ILC approach. This benefit is paid for by the eigenvalue at $+1$ which allows $\hat{Y}_{k+1,i}$ to increase depending on the magnitude of $\mathrm{Re}\{\mathrm{G_iK_i}\}$. In other words, rather than causing instability, values of $\mathrm{Im}\{\mathrm{G_iK_i}\}$ increase the magnitude of the final value to which $\hat{Y}_{k,i}$ converges. The stability regions for the system eigenvalues are shown in Figure 3. A final output bound has been inserted in Figure 3 b) to reflect the practical limitation of an increasing $\hat{Y}_{k+1,i}$. In terms of bounds on the absolute error,

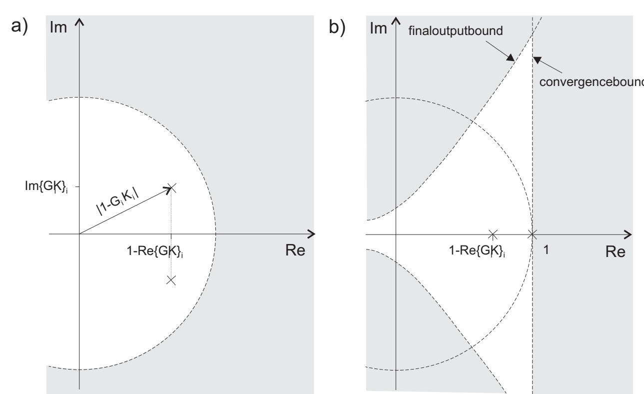

Fig. 3. Stability regions for a) static reference ILC, and b) new objective-driven method.

$$\begin{cases}\bar{E}_{k,i}^* = (1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{k}}\bar{\mathrm{E}}_{0,\mathrm{i}}^* \\ \hat{E}_{k,i}^* = \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\mathrm{k}}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}}\bar{\mathrm{E}}_{0,\mathrm{i}}^* + \hat{\mathrm{E}}_{0,\mathrm{i}}^*\end{cases} \quad (52)$$

Since

$$\hat{E}^*_{0,i} = -\mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\infty}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}}\bar{\mathrm{E}}^*_{0,\mathrm{i}} \quad (53)$$

it is possible to write

$$\begin{aligned}\hat{E}^*_{k,i} &= \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\mathrm{k}}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}}\bar{\mathrm{E}}^*_{0,\mathrm{i}} - \\ &\quad \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=0}}^{\infty}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}}\bar{\mathrm{E}}^*_{0,\mathrm{i}} \quad (54)\\ &= \mathrm{Im}\{\mathrm{G_iK_i}\}\sum_{\mathrm{p=k+1}}^{\infty}(1-\mathrm{Re}\{\mathrm{G_iK_i}\})^{\mathrm{p}}\bar{\mathrm{E}}^*_{0,\mathrm{i}} \quad (55)\end{aligned}$$

provided $|1-\mathrm{Re}\{\mathrm{G_iK_i}\}| < 1$. From (52) and (54), it is clear that $|\bar{E}^*_{k+1,i}| < |\bar{E}^*_{k,i}|$ and $|\hat{E}^*_{k+1,i}| < |\hat{E}^*_{k,i}|$ so that the error, $|E_{k,i}|$, converges monotonically. Figure 4 shows the region in which the frequency-wise multiplicative plant uncertainty, M_i, can exist and the stability condition $|1-\mathrm{Re}\{\mathrm{G_iK_i}\}| < 1$ assured, again using $G_i = G_{0,i}M_i$. The figure also shows

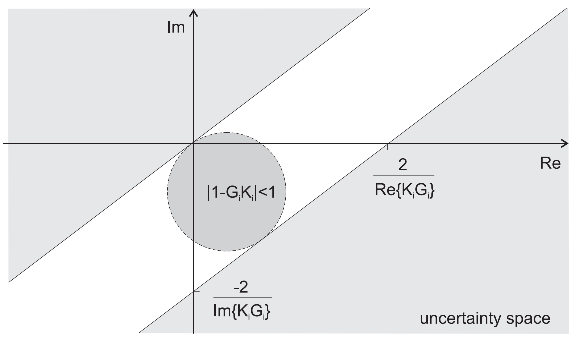

Fig. 4. Uncertainty space for complete system.

the corresponding region for the static reference case (with criterion $|1-G_iK_i| < 1$). As this is contained within the former region, the superior robustness capability of the proposed method is clearly evident.

V. Experimental Results

The test facility used to provide the experimental results is shown in Figure 5, and has previously been used to evaluate a number of ILC, and, by an allowable reconfiguration, repetitive control schemes (see [10], [9] for details). It

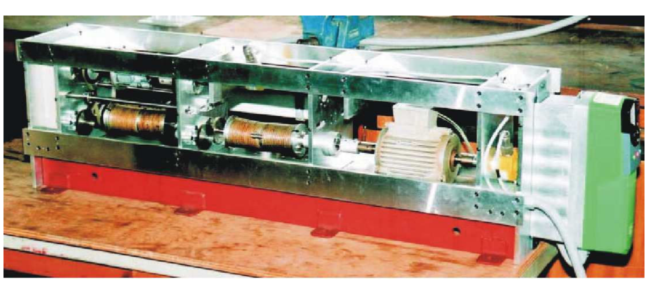

Fig. 5. Non-minimum phase experimental test facility.

consists of a rotary mechanical system of inertias, dampers, torsional springs, a timing belt, pulleys and gears. The plant uses a PID loop in order to act as a pre-stabiliser and provide greater stability, and the resulting closed-loop system constitutes the system to be controlled. The system can be represented using the continuous time plant transfer function

$$G(s) = \frac{165.95(4-s)}{s^4+21.5s^3+170.28s^2+368.52s+663.82} \quad (56)$$

which has been identified in previous work. The adjoint ILC algorithm is selected as a well known member of the class considered, and is given in discrete form by

$$u_{k+1}(z) = u_k(z) + \beta G^*(z)e_k(z) \quad (57)$$

where $G^*(z)$ is the adjoint of the plant model used (see [8] for theoretical background). An attractive feature of the method is that, with a sufficiently small positive scalar multiplier, β, it is guaranteed to satisfy the condition for monotonic convergence over all frequencies, and hence ensure a satisfactory transient response [11]. The static reference monotonic convergence criterion corresponding to (34) in this case is

$$\left|1-\beta G(e^{j\omega T_s})G^*(e^{j\omega T_s})\right| < 1 \quad (58)$$

which leads to

$$0 < \beta|G(e^{j\omega T_s})|^2 < 2 \quad (59)$$

for ω up to the Nyquist frequency. Figures 6 and 7 show experimental tracking results for the objective-driven approach using the adjoint algorithm with $\beta = 0.7$ and $\beta = 0.9$ respectively. The values $T = 5.12$s and $r_N = 12$rad are used, together with $T_s = 0.01s$. The ILC algorithm is implemented using (6), and the reference is updated using (25) and (19), the latter operation being conducted in the frequency domain. Figures 6a) and 7a) show the reference quickly converges to a fixed signal which is markedly different from the reference used during the first trial. Figures 6b) and 7b) show the plant output over the course of the same trials, and Figures 6c) and 7c) show the error norm. For both values of β it is clear that high accuracy tracking is achieved within 5 trials. For comparison, the error norm when using a static reference (equal to r_1) is also shown in each figure, and it is clear the objective-driven approach can produce more accurate tracking in a reduced number of trials. The motivation for the proposed technique was to ensure the plant output equaled the end-point value r_N at time T. To examine whether this has been achieved, Figures 8 and 9 provide the final error value $|r_N - r_k(N)|$ using $\beta = 0.7$ and $\beta = 0.9$ respectively. In both cases the results using a static reference are also shown, and it is clear that the proposed method has provided the capability to reach the end-point with greater accuracy in fewer trials.

VI. Future Work

The objective-driven framework developed in this paper will be applied to stroke rehabilitation to confirm its effectiveness in this area. It will then be expanded to address multiple point-to-point movements, where the timing between points is prescribed. The framework will subsequently be generalised to incorporate the case in which the time-points used are not fixed, and must be provided through optimisation of a suitable cost.

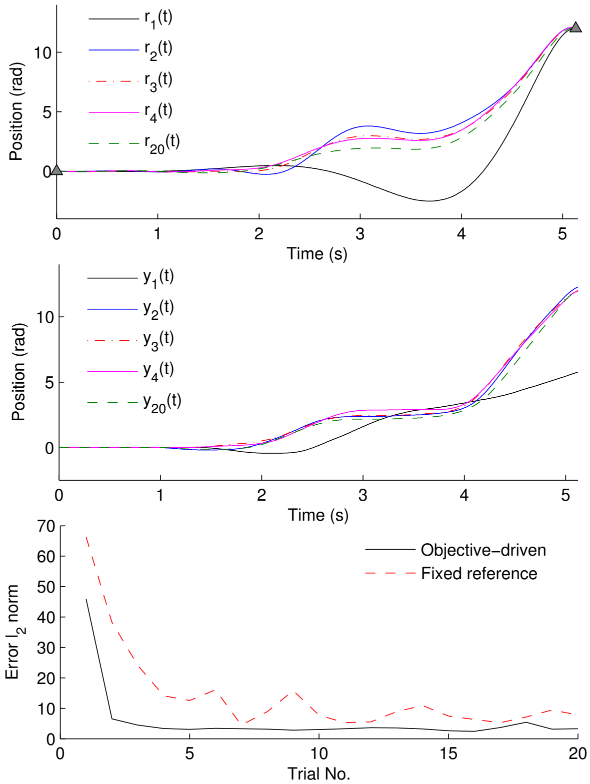

Fig. 6. Adjoint algorithm with changing reference using $\beta = 0.7$.

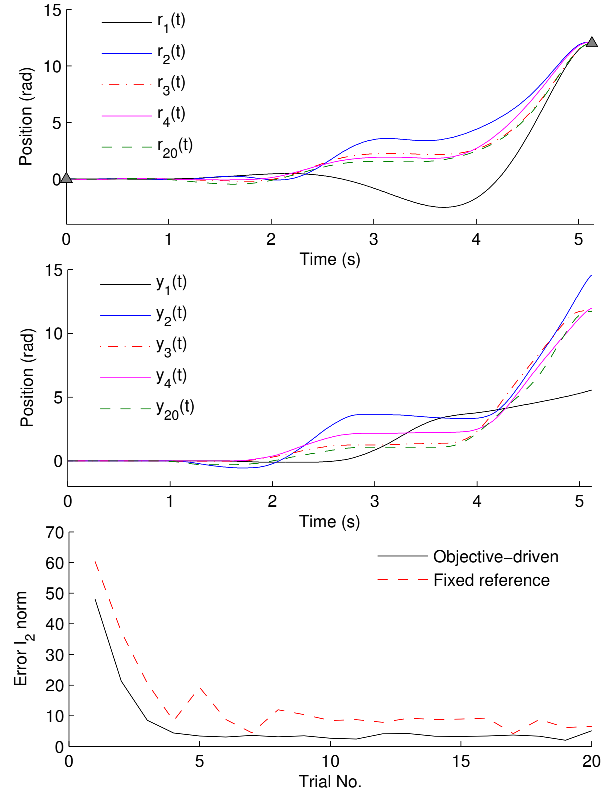

Fig. 7. Adjoint algorithm with changing reference using $\beta = 0.9$.

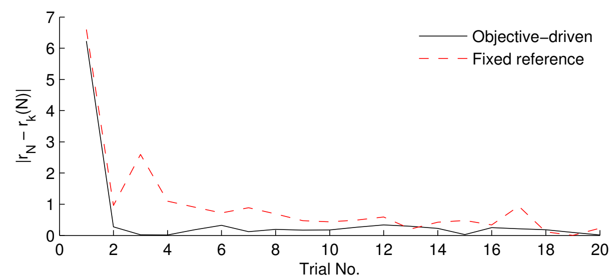

Fig. 8. Adjoint algorithm with changing reference using $\beta = 0.7$.

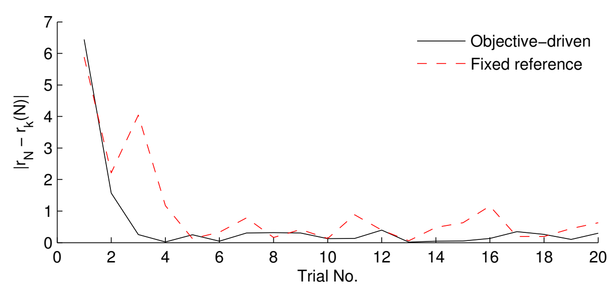

Fig. 9. Adjoint algorithm with changing reference using $\beta = 0.9$.

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