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Multi-Decision Decentralized Prognosis of Failures in Discrete Event Systems

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Abstract— **An architecture for the decentralized prognosis of discrete event systems (DES) was recently proposed in [1], where several local prognosers cooperate for predicting failures in a system modeled as a DES. In this paper, we first formulate the proposition of [1] as a disjunctive architecture, and we develop a conjunctive architecture which is dual and complementary to the disjunctive one. Then, we develop a so-called *multi-decision* prognosis which generalizes the conjunctive and disjunctive prognosis architectures. The basic principle of multi-decision prognosis consists in using several decentralized prognosis architectures working in parallel. We finally show that our work can be easily extended for predicting a failure at least k steps before its occurrence, for a given $k \geq 1$.**

I. INTRODUCTION

Prognosis of a discrete event system (DES) aims at predicting failure events of a DES before their occurrences. Note the contrast with *diagnosis*, which aims at detecting failures *after* their occurrence [2], [3]. Failure prognosis is an active area of research (e.g., [4] and its bibliography). In this study, we consider that a failure prognosis is issued when a failure will *certainly* occur, contrary to the statistical approach where a failure prognosis is issued when a failure will *very probably* occur [5].

The authors of [6] have proposed a prognosis framework in the case of a partially-observed DES. A so-called prognoser issues a prognosis on whether a failure will occur, based on the partial observation it has of the plant. The notion of predictability (we will say *prognosability*) was defined formally for characterizing the class of languages for which : 1) every failure is prognosed before its occurrence, and 2) after a failure has been prognosed, it will certainly occur. In [7], an off-line polynomial-time and an on-line algorithms were proposed for checking prognosability.

More recently, the authors of [1] proposed a framework for the *decentralized* prognosis, where several local prognosers cooperate in their tasks of failure prediction. It is assumed that the prognosers do not communicate directly among each other. Based on the partial observation it has of the plant, a local prognoser issues a prognosis "1" when it is *certain* that a failure will occur. Otherwise, a prognosis "0" is issued. Because of their limited observation capability, the prognosers cooperate by fusing their local prognoses, in order to synthesize a global prognosis. In [1], the notion of *coprognosability* is defined for generalizing prognosability to the decentralized case.

The prognosis method of [1] is the starting point of our study. Being inspired by methods developed in decentralized supervisory control [8], [9], [10], [11], [12] and decentralized diagnosis [2], [13], [14], [15], our essential contribution consists of the following points :

1) We formulate the proposition of [1] as a disjunctive architecture. We define the notion of $\vee$-COPROG, which is equivalent to the coprognosability of [1].
2) We develop a conjunctive prognosis architecture which is complementary to the disjunctive one. We define the notion of $\wedge$-COPROG, which characterizes the class of languages prognosable with the conjunctive architecture. We also propose the idea of a general architecture which combines the disjunctive and conjunctive architectures.
3) We develop a *multi-decision* prognosis which generalizes the conjunctive and disjunctive architectures. The basic principle of multi-decision consists in using several decentralized architectures working in parallel. We define the notions of $\vee$-$\wedge^m$-COPROG and $\wedge$-$\vee^m$-COPROG which generalize $\vee$-COPROG and $\wedge$-COPROG, respectively. Note that the multi-decision approach has also been developed in supervisory control and diagnosis of DES [11], [12], [15].
4) In the above points 1 to 3, a failure could be prognosed *just* before its occurrence. We will show that our work can be easily extended for predicting a failure at least k steps before its occurrence, for a given $k \geq 1$.

This paper is structured as follows. In Section II, we present some preliminaries and notations. Section III formulates the prognosis method of [1] as a disjunctive architecture. In Section IV, we develop a conjunctive prognosis architecture and propose a general architecture which combines and generalizes the disjunctive and conjunctive architectures. Sections V and VI use a multi-decision approach for developing Conj-Disj and Disj-Conj prognosis architectures, which generalize the disjunctive and conjunctive architectures, respectively. In Section VII, we discuss the idea of a general multi-decision architecture which combines and generalizes the Conj-Disj and Disj-Conj architectures. We also explain how all our work can be easily extended for predicting a failure at least k steps before its occurrence, for a given $k \geq 1$. Section VIII recapitulates the contributions of the paper and proposes some relevant future work.

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II. Preliminaries and Notations

We denote by Σ the finite set of events (also called *alphabet*) that can be executed by the *plant* (i.e., the DES to be prognosed). Σ^* is the set of all finite sequences of events of Σ, including the empty sequence ε. For any event sequence $\lambda \in \Sigma^*$, $|\lambda|$ denotes the length of λ. $\Sigma^{\geq k}$ (resp. $\Sigma^{\leq k}$) is the subset of Σ^* consisting of the sequences whose length is greater (resp. smaller) than or equal to k. For any event sequences $\lambda, \mu \in \Sigma^*$, $\mu \leq \lambda$ means that μ is a prefix of λ, i.e., $\exists \nu \in \Sigma^*$ s.t. $\lambda = \mu\nu$. The set of all prefixes of a language K is denoted $\overline{K}$, and K is said prefix-closed if $K = \overline{K}$. Let $\mathbf{N}^+$ denote the set of strictly positive integers.

The plant is assumed modeled by a prefix-closed language $\mathcal{L}$, which consists of a failure part and a non-failure (or healthy) part, modeled by $\mathcal{F}$ and $\mathcal{H}$, respectively. We have $\mathcal{L} = \mathcal{H} \cup \mathcal{F}$ and $\mathcal{H} \cap \mathcal{F} = \emptyset$. Typically, $\mathcal{F}$ and $\mathcal{H}$ are related to an unobservable failure event $f \in \Sigma$ as follows : $\mathcal{F}$ (resp. $\mathcal{H}$) consists of the sequences of $\mathcal{L}$ containing (resp. without) f. In this case, $\mathcal{H}$ is prefix-closed. Without loss of generality, we study uniquely the case of a single failure. In the presence of several failures, the same study is done for each failure.

This article has used [1] as starting point, where the prognosis is based on the following two notions.

Definition 2.1: (*Boundary, Non-Indicator*) Given two prefix-closed languages $\mathcal{L}$ and $\mathcal{H}$ such that $\mathcal{H} \subseteq \mathcal{L}$, we define the following sequences and languages :

– A *boundary* sequence of $\mathcal{H}$ w.r.t $\mathcal{L}$ is a sequence of $\mathcal{H}$ for which a failure in a next step is possible. The set of boundary sequences of $\mathcal{H}$ w.r.t $\mathcal{L}$ is formally defined by : $\partial = \{\lambda \in \mathcal{H} \mid \{\lambda\}\Sigma \cap \mathcal{F} \neq \emptyset\}$.

– A *non-indicator* sequence of $\mathcal{H}$ w.r.t $\mathcal{L}$ is a sequence of $\mathcal{H}$ for which a failure in future is not guaranteed. The set of non-indicator sequences of $\mathcal{H}$ w.r.t $\mathcal{L}$ is defined by : $\Upsilon = \{\lambda \in \mathcal{H} \mid \forall k \in \mathbf{N}^+ : \{\lambda\}\Sigma^{\geq k} \cap \mathcal{H} \neq \emptyset\}$.

For simplicity of notation, $\mathcal{L}$ and $\mathcal{H}$ are implicit in ∂ and Υ.

As indicated in [1], we consider the case where the objective of a prognosis system is to observe the plant and respect the following two properties, where $\mathrm{X}(\lambda)$ denotes the global prognosis taken after the execution of a sequence $\lambda \in \mathcal{H}$, it takes the value "1" when a fault is predicted :

Property 2.1: Each failure is prognosed before its occurrence. We propose the equivalent expression : a failure is prognosed if its occurrence is possible in the next step, i.e. :

$$\forall \lambda \in \partial : \mathrm{X}(\lambda) = 1 \tag{1}$$

Property 2.2: A failure prognosis guarantees that a failure will occur in future. We propose the equivalent expression : a failure is not prognosed if its occurrence in future is not guaranteed, i.e. :

$$\forall \lambda \in \Upsilon : \mathrm{X}(\lambda) = 0 \tag{2}$$

Respecting Equations (1,2) w.r.t (∂, Υ) will be the objective of all the proposed decentralized architectures.

In decentralized prognosis, n local prognosers $(Prog_i)_{1 \leq i \leq n}$ observe the plant and cooperate with each other in order to synthesize a correct prognosis. Each $Prog_i$ has a partial view of the plant, that is, its set of observable events is $\Sigma_{o,i} \subset \Sigma$. Let $\Sigma_o = \bigcup_{1 \leq i \leq n} \Sigma_{o,i}$ and $\Sigma_{uo} = \Sigma \setminus \Sigma_o$. Therefore, an event of Σ_o is observable by at least one prognoser, and no prognoser can observe an event of Σ_{uo}. We denote by P_i the natural projection that hides the events of $\Sigma \setminus \Sigma_{o,i}$ from any sequence $\lambda \in \Sigma^*$.

III. Disjunctive Prognosis Architecture of [1]

We consider two prefix-closed languages $\mathcal{L}$ and $\mathcal{H}$ modeling the plant and its healthy part, respectively. The failure part is deduced by $\mathcal{F} = \mathcal{L} \setminus \mathcal{H}$.

A. Disjunctive Prognosers

In decentralized prognosis, n local prognosers $(Prog_i)_{1 \leq i \leq n}$ observe the plant and cooperate with each other in order to predict a failure before its occurrence. After the execution of a sequence $\lambda \in \mathcal{H}$, each $Prog_i$ makes a local prognosis $X_i(P_i(\lambda))$ depending on what it has observed, i.e., $P_i(\lambda)$. Then, the global prognosis is synthesized from the local prognoses. A global prognosis "1" means that a failure is prognosed. For the purpose of our study, let us present the prognosis method of [1] as a *disjunctive* architecture.

Each local prognoser $Prog_i$ $(i = 1, \cdots, n)$ makes a local prognosis $X_i(P_i(\lambda)) \in \{0, 1\}$. Then, the global prognosis $\mathrm{X}(\lambda)$ is obtained by fusing $X_i(P_i(\lambda))$ $(i = 1, \cdots, n)$ *disjunctively*, that is :

$$\mathrm{X}(\lambda) = \bigvee_{i=1,\cdots,n} X_i(P_i(\lambda)) \tag{3}$$

Definition 3.1: (*Disjunctive prognoser, $\vee$-prognoser*) A set of local prognosers satisfying Equation 3 is called *disjunctive prognoser* or more shortly $\vee$-prognoser.

A particular rule for computing the local decisions $X_i(P_i(\lambda))$ in the disjunctive architecture, is to issue a local prognosis "1" if and only if the local prognoser is certain that a failure in future is guaranteed.

$$\forall i \in \{1, \cdots, n\} : \\ X_i(P_i(\lambda)) = \left\{ \begin{array}{ll} 0, & \text{if } P_i(\lambda) \in P_i(\Upsilon) \\ 1, & \text{otherwise} \end{array} \right\} \tag{4}$$

Let us consider the example of Fig. 1, where $\Sigma_{o,1} = \{a_1, \sigma\}$, $\Sigma_{o,2} = \{a_2, \sigma\}$, $\Sigma_{uo} = \{f\}$. $\mathcal{H} = \overline{\sigma^*(a_1 + a_2)} = \sigma^*(\varepsilon + a_1 + a_2)$ contains the sequences without f, and $\mathcal{F} = \sigma^*(a_1 + a_2)ff^*$ contains the sequences with f. $\partial = \sigma^*(a_1 + a_2)$ contains the sequences of $\mathcal{H}$ for which f is possible in the next step. $\Upsilon = \sigma^*$ contains the sequences of $\mathcal{H}$ for which f is not guaranteed in future (because σ can be executed indefinitely).

Let us show that the $\vee$-prognoser defined by Eqs. (3,4) satisfies Eqs. (1,2). Table I outlines the local and global prognoses for all sequences *before* f (since the aim is to *predict* f). We see that the global prognosis "$\mathrm{X}(\lambda) = 1$" is issued only for the sequences $\sigma^* a_1$ and $\sigma^* a_2$, which is conform to Eqs. (1,2).

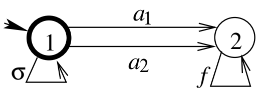

Fig. 1. Example for illustrating the disjunctive prognosis of [1]

λ	$P_1(\lambda)$	$X_1(P_1(\lambda))$	$P_2(\lambda)$	$X_2(P_2(\lambda))$	$\mathrm{X}(\lambda)$
σ^*	σ^*	0	σ^*	0	0
$\sigma^* a_1$	$\sigma^* a_1$	1	σ^*	0	1
$\sigma^* a_2$	σ^*	0	$\sigma^* a_2$	1	1

TABLE I

DISJUNCTIVE PROGNOSIS RESULTS FOR THE EXAMPLE OF FIG. 1

B. Existence Results

We consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages such that $\mathcal{H} \subseteq \mathcal{L}$, and their corresponding (∂, Υ). In order to determine whether there exists a $\vee$-prognoser defined by Eq. 3 and satisfying Eqs. (1,2), the authors of [1] have defined the notion of coprognosability. Intuitively, (∂, Υ) is coprognosable if before a failure occurs, at least one local prognoser $Prog_i$ is certain that the failure is inevitable. Since we will need to define *coprognosability* also for the conjunctive architecture, we rename the coprognosability of [1] as *disjunctive coprognosability*, or more shortly $\vee$-COPROG, which can be expressed formally as follows :

Definition 3.2: ($\vee$-*COPROG*) Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, the corresponding (∂, Υ) is $\vee$-COPROG if :

$$\bigcap_{i=1,\cdots n} [P_i^{-1} P_i(\Upsilon)] \cap \partial = \emptyset \qquad (5)$$

The following theorem relates $\vee$-COPROG to the existence of $\vee$-prognoser.

Theorem 3.1: Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, the following two points are equivalent :

1) (∂, Υ) is $\vee$-COPROG.
2) There exists a $\vee$-prognoser defined by Eq. 3 and satisfying Eqs. (1,2).

The following proposition implies that it is not restrictive to consider as solution uniquely the $\vee$-prognoser respecting Eqs. (3,4), instead of any $\vee$-prognoser respecting Eqs. 3.

Proposition 3.1: Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, the following Point 1 implies Point 2 :

1) There exists a $\vee$-prognoser defined by Eq. 3 and satisfying Eqs. (1,2)
2) The $\vee$-prognoser defined by Eqs. (3,4) satisfies Eqs. (1,2).

Let us return to the example of Fig. 1, where $\Upsilon = \sigma^*$ and $\partial = \sigma^*(a_1 + a_2)$. Let us show that (∂, Υ) is $\vee$-COPROG. We compute : $P_i(\Upsilon) = \sigma^*$, $[P_i^{-1} P_i(\Upsilon)] \cap \partial = \sigma^* a_j$, for $i = 1, 2$ and $i \neq j$, and thus $\bigcap_{i=1,2} [P_i^{-1} P_i(\Upsilon)] \cap \partial = \emptyset$. From Def. 3.2, (∂, Υ) is $\vee$-COPROG. And from Theorem 3.1, there exists a $\vee$-prognoser defined by Eq. 3 and satisfying Eqs. (1,2). Proposition 3.1 confirms that a solution is the $\vee$-prognoser defined by Eqs. (3,4) and outlined in Table I.

IV. CONJUNCTIVE AND GENERAL PROGNOSIS ARCHITECTURES

A. Conjunctive Prognosers

The disjunctive architecture was defined as the one where the global prognosis is obtained by fusing disjunctively the local prognoses. In the same way, in the conjunctive architecture, the local prognoses are fused *conjunctively* :

$$\mathrm{X}(\lambda) = \bigwedge_{i=1,\cdots,n} X_i(P_i(\lambda)) \qquad (6)$$

Definition 4.1: (*Conjunctive prognoser,* $\wedge$-*prognoser*) A set of local prognosers satisfying Eq. 6 is called *conjunctive prognoser* or more shortly $\wedge$-prognoser.

A particular interesting rule for computing the local decisions $X_i(P_i(\lambda))$ in the conjunctive architecture, consists in issuing a local prognosis "0" when the local prognoser is certain that the next event is not a failure. Otherwise, the local prognosis is "1". Formally :

$$\forall i \in \{1, \cdots, n\} :$$
$$X_i(P_i(\lambda)) = \left\{ \begin{array}{ll} 1, & \text{if } P_i(\lambda) \in P_i(\partial) \\ 0, & \text{otherwise} \end{array} \right\} \qquad (7)$$

Consider the example of Fig. 2, where $\Sigma_{o,1} = \{a_1, \sigma\}$, $\Sigma_{o,2} = \{a_2, \sigma\}$, $\Sigma_{uo} = \{f\}$. $\mathcal{H} = \overline{\sigma^*(a_1 \sigma^* a_2 + a_2 \sigma^* a_1)}$ contains the sequences without f, and $\mathcal{F} = \sigma^*(a_1 \sigma^* a_2 + a_2 \sigma^* a_1) f f^*$ contains the sequences with f. $\partial = \sigma^*(a_1 \sigma^* a_2 + a_2 \sigma^* a_1)$ contains the sequences of $\mathcal{H}$ for which f is possible in the next step. $\Upsilon = \overline{\sigma^*(a_1 + a_2)}\sigma^* = \sigma^* + \sigma^* a_1 \sigma^* + \sigma^* a_2 \sigma^*$ contains the sequences of $\mathcal{H}$ for which f is not guaranteed in future.

Let us show that the $\wedge$-prognoser defined by Eqs. (6,7) satisfies Eqs. (1,2). Table II outlines the local and global prognoses for all sequences *before* f. We see that the global prognosis "$\mathrm{X}(\lambda) = 1$" is issued only for the sequences $\sigma^* a_1 \sigma^* a_2$ and $\sigma^* a_2 \sigma^* a_1$, which is conform to Eqs. (1,2).

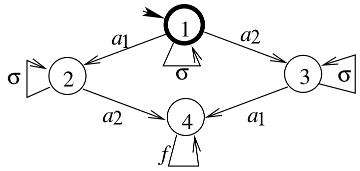

Fig. 2. Example for illustrating the conjunctive prognosis

λ	$P_1(\lambda)$	$X_1(P_1(\lambda))$	$P_2(\lambda)$	$X_2(P_2(\lambda))$	$\mathrm{X}(\lambda)$
σ^*	σ^*	0	σ^*	0	0
$\sigma^* a_1 \sigma^*$	$\sigma^* a_1 \sigma^*$	1	σ^*	0	0
$\sigma^* a_1 \sigma^* a_2$	$\sigma^* a_1 \sigma^*$	1	$\sigma^* a_2$	1	1
$\sigma^* a_2 \sigma^*$	σ^*	0	$\sigma^* a_2 \sigma^*$	1	0
$\sigma^* a_2 \sigma^* a_1$	$\sigma^* a_1$	1	$\sigma^* a_2 \sigma^*$	1	1

TABLE II

CONJUNCTIVE PROGNOSIS RESULTS FOR THE EXAMPLE OF FIG. 2

B. Existence Results

We consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages such that $\mathcal{H} \subseteq \mathcal{L}$, and their corresponding (∂, Υ). In order to determine whether there exists a $\wedge$-prognoser defined by Eq. 6 and satisfying Eqs. (1,2), we define the notion of *conjunctive coprognosability*, or more shortly $\wedge$-COPROG. Intuitively, (∂, Υ) is $\wedge$-COPROG if while a failure is not guaranteed in future, at least one local prognoser $Prog_i$ is certain that a failure is impossible in the next step. Formally :

Definition 4.2: ($\wedge$-*COPROG*) Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, the corresponding (∂, Υ) is $\wedge$-COPROG if :

$$\bigcap_{i=1,\cdots n} [P_i^{-1} P_i(\partial)] \cap \Upsilon = \emptyset \qquad (8)$$

The following theorem is an important result that relates $\wedge$-COPROG to the existence of $\wedge$-prognoser.

Theorem 4.1: Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, the following two points are equivalent :

1) (∂, Υ) is $\wedge$-COPROG.
2) There exists a $\wedge$-prognoser defined by Eq. 6 and satisfying Eqs. (1,2).

The following proposition implies that it is not restrictive to consider as solution uniquely the $\wedge$-prognoser respecting Eqs. (6,7), instead of any $\wedge$-prognoser respecting Eq. 6.

Proposition 4.1: Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, the following Point 1 implies Point 2 :

1) There exists a $\wedge$-prognoser defined by Eq. 6 and satisfying Eqs. (1,2)
2) The $\wedge$-prognoser defined by Eqs. (6,7) satisfies Eqs. (1,2).

Let us return to the example of Fig. 2, where $\Upsilon = \sigma^* + \sigma^* a_1 \sigma^* + \sigma^* a_2 \sigma^*$, $\partial = \sigma^*(a_1 \sigma^* a_2 + a_2 \sigma^* a_1)$. Let us show that (∂, Υ) is $\wedge$-COPROG. We compute : $P_i(\partial) = \sigma^* a_i \sigma^*$, $[P_i^{-1} P_i(\partial)] \cap \Upsilon = \sigma^* a_i \sigma^* a_j$, for $i = 1, 2$ and $i \neq j$, and thus $\bigcap_{i=1,2} [P_i^{-1} P_i(\partial)] \cap \Upsilon = \emptyset$. Therefore, from Def. 4.2, (∂, Υ) is $\wedge$-COPROG. We deduce from Theorem 4.1 that there exists a $\wedge$-prognoser defined by Eq. 6 and satisfying Eqs. (1,2). And Prop. 4.1 confirms that a solution is the $\wedge$-prognoser defined by Eqs. (6,7) and outlined in Table II.

C. Idea of General Architecture

We have the following Proposition :

Proposition 4.2: $\vee$-COPROG and $\wedge$-COPROG are incomparable, i.e., none of them implies the other.

Let us illustrate Prop. 4.2 using the two examples of Figures (1,2). We have shown in Sect. III that the example of Fig. 1 is $\vee$-COPROG ; it can be checked that it is not $\wedge$-COPROG by computing $\bigcap_{i=1,2} [P_i^{-1} P_i(\partial)] \cap \Upsilon = \Upsilon \neq \emptyset$. And we have shown that the example of Fig. 2 is $\wedge$-COPROG ; it can be checked that it is not $\vee$-COPROG by computing $\bigcap_{i=1,2} [P_i^{-1} P_i(\Upsilon)] \cap \partial = \partial \neq \emptyset$.

To recapitulate, we have found an example which is $\vee$-COPROG and not $\wedge$-COPROG (Fig. 1), and an example which is $\wedge$-COPROG and not $\vee$-COPROG (Fig. 2). We can think of a general prognosis architecture for systems with two categories of failures : failures for which the system is $\vee$-COPROG, and failures for which the system is $\wedge$-COPROG. The disjunctive (resp. conjunctive) architecture is applied for predicting the first (resp. second) category of faults. Such a general architecture is not detailed here for lack of space.

V. Conj-Disj Architecture : Generalization of the Disjunctive Architecture

Let us present a Conj-Disj architecture which generalizes the disjunctive architecture using an approach called multi-decision. The generalization of the conjunctive architecture is studied in Section VI.

A. $\wedge$-$\vee$-*prognoser*

The multi-decision when applied to the disjunctive architecture consists in using several (say m) decentralized disjunctive architectures working in parallel and whose global decisions are combined *conjunctively* into a "final" global decision which must satisfy Equations (1,2). The resulting architecture is thus qualified as Conj-Disj. The m $\vee$-prognosers are indexed by $j = 1 \cdots m$. The global decision $X^j(\lambda)$ of the j^{th} $\vee$-prognoser is computed by combining *disjunctively* its local decisions $(X_i^j(P_i(\lambda)))_{i=1\cdots n}$ as in Eq. 3, that is :

$$\forall j \in \{1, \cdots, m\} : X^j(\lambda) = \bigvee_{i=1,\cdots,n} X_i^j(P_i(\lambda)) \qquad (9)$$

Then the global prognoses of the m $\vee$-prognosers are fused conjunctively in order to obtain the global actual prognosis $\mathrm{X}(\lambda)$. Formally :

$$\mathrm{X}(\lambda) = \bigwedge_{j=1,\cdots,m} X^j(\lambda) \qquad (10)$$

Definition 5.1: ($\wedge$-$\vee^m$*-prognoser*, $\wedge$-$\vee$*-prognoser*) Given $m \in \mathbf{N}^+$, a set of m $\vee$-prognosers which are fused conjunctively (i.e., satisfying Equations (9,10) is called $\wedge$-$\vee^m$-prognoser. When m is not specified, we say $\wedge$-$\vee$-prognoser.

B. Decomposing Υ for computing the local decisions

We have explained that, given $m \in \mathbf{N}^+$, a $\wedge$-$\vee^m$-prognoser is obtained by using m $\vee$-prognosers (Eq. 9) and combining them conjunctively (Eq. 10). Let us propose a rule for computing the local decisions $(X_i^j(P_i(\lambda)))_{i=1\cdots n}$ of the j^{th} architecture, for each $j = 1 \cdots m$. We assume we are given a decomposition $(\Upsilon^j)_{j=1\cdots m}$ of Υ, that is, $\Upsilon = \Upsilon^1 \cup \cdots \cup \Upsilon^m$. We say "decomposition" instead of "partition", because we may have $\Upsilon^j \cap \Upsilon^k \neq \emptyset$ for $j \neq k$. For each $j = 1 \cdots m$, the local decisions are computed like in Eq. 4, but w.r.t Υ^j instead of Υ. That is :

$$\forall j \in \{1, \cdots, m\}, \forall i \in \{1, \cdots, n\} :$$
$$X_i^j(P_i(\lambda)) = \left\{ \begin{array}{ll} 0, & \text{if } P_i(\lambda) \in P_i(\Upsilon^j) \\ 1, & \text{otherwise} \end{array} \right\} \qquad (11)$$

The idea behind this approach is that for each $j = 1 \cdots m$, the global decision of the j^{th} disjunctive architecture respects Eqs (1,2), but w.r.t (∂, Υ^j) instead of (∂, Υ). And by fusing conjunctively these m global decisions, we obtain a "final" global decision that respects Eqs (1,2) w.r.t (∂, Υ).

Consider the example of Fig. 3, where $\Sigma_{uo} = \{f\}$, $\Sigma_{o,1} = \{a_1, \sigma\}$, $\Sigma_{o,2} = \{a_2, \sigma\}$. $\mathcal{F} = \sigma^*(a_1 + a_2)\sigma f f^*$ contains the sequences with f, and $\mathcal{H} = \overline{\sigma^*(a_1 + a_2 + a_1 a_2 + a_2 a_1)\sigma}$ contains the sequences without f. $\Upsilon = \overline{\sigma^*(a_1 a_2 + a_2 a_1)\sigma} = \sigma^*(\varepsilon + a_1 \overline{a_2 \sigma} + a_2 \overline{a_1 \sigma})$ corresponds to states 1, 2, 3, 5, 6 ; it contains the sequences of $\mathcal{H}$ for which f is not guaranteed in future. $\partial = \sigma^*(a_1 + a_2)\sigma$ corresponds to state 4 ; it contains the sequences of $\mathcal{H}$ for which f is possible in the next step. Υ is decomposed into $\Upsilon^1 = \sigma^*(a_1 a_2 + a_2 a_1)\sigma$ and $\Upsilon^2 = \overline{\sigma^*(a_1 a_2 + a_2 a_1)}$. Υ^1 corresponds to state 6, and Υ^2 corresponds to states 1, 2, 3, 5.

Let us show that with this decomposition of Υ, the $\wedge$-$\vee^2$-prognoser defined by Eqs. (9,10,11) satisfies Eqs. (1,2). Table III outlines $X_i^j(P_i(\lambda))$, $X^j(\lambda)$ and $\mathrm{X}(\lambda)$ computed using Eqs. (9,10,11) for all sequences λ without f. We see that "$\mathrm{X}(\lambda) = 1$" is issued only for the sequences $\sigma^* a_1 \sigma$ and $\sigma^* a_2 \sigma$, which is conform to Eqs. (1,2).

λ	$P_1(\lambda)$	$X_1^1(P_1(\lambda))$	$X_1^2(P_1(\lambda))$	$P_2(\lambda)$	$X_2^1(P_2(\lambda))$	$X_2^2(P_2(\lambda))$	$X^1(\lambda)$	$X^2(\lambda)$	$\mathrm{X}(\lambda)$
σ^*	σ^*	1	0	σ^*	1	0	1	0	0
σ^*a_1	σ^*a_1	1	0	σ^*	1	0	1	0	0
$\sigma^*a_1\sigma$	$\sigma^*a_1\sigma$	0	1	$\sigma\sigma^*$	1	0	1	1	1
$\sigma^*a_1a_2$	σ^*a_1	1	0	σ^*a_2	1	0	1	0	0
$\sigma^*a_1a_2\sigma$	$\sigma^*a_1\sigma$	0	1	$\sigma^*a_2\sigma$	0	1	0	1	0
σ^*a_2	σ^*	1	0	σ^*a_2	1	0	1	0	0
$\sigma^*a_2\sigma$	$\sigma\sigma^*$	1	0	$\sigma^*a_2\sigma$	0	1	1	1	1
$\sigma^*a_2a_1$	σ^*a_1	1	0	σ^*a_2	1	0	1	0	0
$\sigma^*a_2a_1\sigma$	$\sigma^*a_1\sigma$	0	1	$\sigma^*a_2\sigma$	0	1	0	1	0

TABLE III

CONJ-DISJ PROGNOSIS RESULTS FOR THE EXAMPLE OF FIG. 3

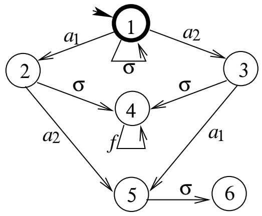

Fig. 3. Example for illustrating the Conj-Disj prognosis

C. Existence Results

We are given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages such that $\mathcal{H} \subseteq \mathcal{L}$, and $m \in \mathbf{N}^+$. An important question that arises in the Conj-Disj architecture is :

Question 1: Does there exist a $\wedge$-$\vee^m$-prognoser defined by Eqs. (9,10) and satisfying Eqs. (1,2) ?

We will show further that answering this question amounts to answering the following question :

Question 2: Does there exist a decomposition $(\Upsilon^1)_{j=1\cdots m}$ of Υ such that each (∂, Υ^j) is $\vee$-COPROG, that is :

$$\forall j = 1, \cdots, m : \bigcap_{i=1,\cdots n} [P_i^{-1} P_i(\Upsilon^j)] \cap \partial = \emptyset \qquad (12)$$

Since Υ is in general infinite, we are confronted to the problem of decomposing an infinite set, which is known to be a challenging and open problem. We propose to transform the problem of decomposing the infinite regular language Υ [1] into a problem of decomposing the *finite* set of states of the corresponding finite state automation (FSA). The solution consists in considering the finite equivalence relation Nerode $\mathcal{N}_\Upsilon$ over Υ. That is, if we consider the minimal FSA $A_{\mathcal{N}_\Upsilon}$ accepting Υ, then each equivalence class of $\mathcal{N}_\Upsilon$ is the set of traces leading to the same state of $A_{\mathcal{N}_\Upsilon}$.

Definition 5.2: (*Stronger equivalence relation*) Consider two finite equivalence relations $\mathcal{R}_E^1$ and $\mathcal{R}_E^2$ on a set E. $\mathcal{R}_E^1$ is said stronger than $\mathcal{R}_E^2$, noted $\mathcal{R}_E^1 \leq \mathcal{R}_E^2$, if every equivalence class of $\mathcal{R}_E^2$ consists of one or several equivalence classes of $\mathcal{R}_E^1$.

Instead of $\mathcal{N}_\Upsilon$, we may also use any *finite* equivalence relation (i.e., having a *finite* number of equivalence classes) $\mathcal{R}_\Upsilon$ which is stronger than $\mathcal{N}_\Upsilon$ That is, $\mathcal{R}_\Upsilon$ can be defined like $\mathcal{N}_\Upsilon$, but by using a FSA $A_{\mathcal{R}_\Upsilon}$ accepting Υ which is not necessarily minimal. Given a finite relation $\mathcal{R}_\Upsilon \leq \mathcal{N}_\Upsilon$, we consider uniquely the decompositions $(\Upsilon^1, \cdots, \Upsilon^m)$ of Υ that respect the following assumption :

A1 : For every $j = 1, \cdots, m$, Υ^j consists of one or several equivalence classes of $\mathcal{R}_\Upsilon$.

Therefore, finding a decomposition of Υ respecting Assumption **A1** w.r.t $\mathcal{R}_\Upsilon$ amounts to finding a decomposition of the *finite* state set of the FSA $A_{\mathcal{R}_\Upsilon}$.

For the FSA of Fig. 3, we obtain $A_{\mathcal{N}_\Upsilon}$ by removing state 4. Decomposing Υ amounts therefore to decomposing the finite state set $\{1, 2, 3, 5, 6\}$. The number of decompositions is therefore finite. The following decomposition has been used :

– $\Upsilon^1 = \sigma^* \underline{(a_1a_2 + a_2a_1)}\sigma$ corresponds to state 6,

– $\Upsilon^2 = \sigma^* \underline{(a_1a_2 + a_2a_1)}$ corresponds to states $1, 2, 3, 5$.

We have defined the relation "Stronger than" because, when $\mathcal{R}_\Upsilon^1 \leq \mathcal{R}_\Upsilon^2 \leq \mathcal{N}_\Upsilon$, $\mathcal{R}_\Upsilon^1$ permits more decompositions of Υ than $\mathcal{R}_\Upsilon^2$ respecting Assumption **A1**.

Given a finite equivalence relation $\mathcal{R}_\Upsilon$ s.t. $\mathcal{R}_\Upsilon \leq \mathcal{N}_\Upsilon$ ($\mathcal{R}_\Upsilon$ may be $\mathcal{N}_\Upsilon$), the previous Question 2 is strengthened into checking the following notion of $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$:

Definition 5.3: ($\wedge$-$\vee^m$*-COPROG w.r.t* $\mathcal{R}_\Upsilon$) Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\Upsilon$ such that $\mathcal{R}_\Upsilon \leq \mathcal{N}_\Upsilon$. (∂, Υ) is said $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$ if there exists a decomposition $(\Upsilon^1, \cdots, \Upsilon^m)$ of Υ respecting Assumption **A1** and such that each (∂, Υ^j) is $\vee$-COPROG (i.e., Eq. 12).

Proposition 5.1: Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\Upsilon$ such that $\mathcal{R}_\Upsilon \leq \mathcal{N}_\Upsilon$. If (∂, Υ) is $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$ for some $m \in \mathbf{N}^+$, then for every equivalence class A of $\mathcal{R}_\Upsilon$ $(A \subseteq \Upsilon)$: $\bigcap_{i=1,\cdots n} [P_i^{-1} P_i(A)] \cap \partial = \emptyset$.

The link between previous Questions 1 and 2 is stated by the following theorem :

Theorem 5.1: Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\Upsilon$ such that $\mathcal{R}_\Upsilon \leq \mathcal{N}_\Upsilon$. The following Point 1 ensures Point 2 :

1) (∂, Υ) is $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$.
2) There exists a $\wedge$-$\vee^m$-prognoser defined by Eqs. (9,10) and satisfying Eqs. (1,2).

The following proposition implies that it is not restrictive to consider as solution uniquely the $\wedge$-$\vee^m$-prognoser respecting Eqs. (9,10,11), instead of any $\wedge$-$\vee^m$-prognoser respecting Eqs. (9,10).

Proposition 5.2: Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\Upsilon$ such that $\mathcal{R}_\Upsilon \leq \mathcal{N}_\Upsilon$. The following Point 1 ensures Point 2 :

1) (∂, Υ) is $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$.

[1] $\mathcal{L}$ and $\mathcal{H}$, and thus Υ, are assumed to be *regular* languages.

2) We consider any decomposition of Υ satisfying Assumption **A1** and Eq. 12 (exists by definition of $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$). The corresponding $\wedge$-$\vee^m$-prognoser defined by Eqs. (9,10,11) satisfies Eqs. (1,2).

Let us return to the example of Fig. 3, where $\partial = \sigma^*(a_1+a_2)\sigma$, and Υ was decomposed into $\Upsilon^1 = \sigma^*(a_1a_2+a_2a_1)\sigma$ and $\Upsilon^2 = \sigma^*\overline{(a_1a_2+a_2a_1)}$. Let us show that (∂,Υ) is $\wedge$-$\vee^2$-COPROG w.r.t $\mathcal{N}_\Upsilon$. $\mathcal{N}_\Upsilon$ contains five equivalence classes corresponding to states 1, 2, 3, 5, 6, respectively. Υ^1 contains the class corresponding to state 6, and Υ^2 contains the classes corresponding to states 1, 2, 3, 5. We compute :

$P_i(\Upsilon^1) = \sigma^* a_i \sigma,\ P_i(\Upsilon^2) = \sigma^* + \sigma^* a_i$, for $i = 1,2$,

$[P_i^{-1}P_i(\Upsilon^1)] \cap \partial = \sigma^* a_i \sigma$, for $i = 1,2$,

$[P_i^{-1}P_i(\Upsilon^2)] \cap \partial = \sigma^* a_j \sigma$, for $i,j = 1,2$ and $i \neq j$.

$\bigcap_{i=1,2}[P_i^{-1}P_i(\Upsilon^1)] \cap \partial = \bigcap_{i=1,2}[P_i^{-1}P_i(\Upsilon^2)] \cap \partial = \emptyset$.

Therefore, from Def. 5.3, (∂,Υ) is $\wedge$-$\vee^2$-COPROG w.r.t $\mathcal{N}_\Upsilon$. We deduce from Theor. 5.1 that there exists a $\wedge$-$\vee^2$-prognoser defined by Eqs. (9,10) and satisfying Eqs. (1,2). Prop. 5.2 confirms that a solution is the $\wedge$-$\vee^2$-prognoser defined by Eqs. (9,10,11) and outlined in Table III.

The following proposition implies that the Conj-Disj architecture is more general than the disjunctive architecture.

Proposition 5.3: Given a pair $(\mathcal{L},\mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, if (∂,Υ) is $\vee$-COPROG then there exists $m \in \mathbf{N}^+$ such that (∂,Υ) is $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{N}_\Upsilon$.

Proposition 5.3 is confirmed by the fact that $\vee$-COPROG is equivalent to $\wedge$-$\vee^1$-COPROG w.r.t $\mathcal{N}_\Upsilon$. The example of Fig. 3 proves that the converse of Prop. 5.3 is not true. Indeed, we have shown that (∂,Υ) is $\wedge$-$\vee^2$-COPROG w.r.t $\mathcal{N}_\Upsilon$, and now we show that it is not $\vee$-COPROG as follows :

$P_i(\Upsilon) = \sigma^* + \sigma^* a_i + \sigma^* a_i \sigma$ for $i = 1,2$,

$[P_i^{-1}P_i(\Upsilon)] \cap \partial = \partial$, for $i = 1,2$.

Thus, $\bigcap_{i=1,2}[P_i^{-1}P_i(\Upsilon)] \cap \partial = \partial \neq \emptyset$.

Note that this example is not $\wedge$-COPROG as well, because $\bigcap_{i=1,2}[P_i^{-1}P_i(\partial)] \cap \Upsilon = \sigma^*(a_1a_2+a_2a_1+\varepsilon)\sigma \neq \emptyset$.

The following proposition is due to the fact that if $\mathcal{R}^1_\Upsilon \leq \mathcal{R}^2_\Upsilon \leq \mathcal{N}_\Upsilon$, then $\mathcal{R}^1_\Upsilon$ permits more decompositions than $\mathcal{R}^2_\Upsilon$.

Proposition 5.4: Consider a pair of finite equivalence relations $(\mathcal{R}^1_\Upsilon,\mathcal{R}^2_\Upsilon)$ such that $\mathcal{R}^1_\Upsilon \leq \mathcal{R}^2_\Upsilon \leq \mathcal{N}_\Upsilon$, a pair $(\mathcal{L},\mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and an integer $m \in \mathbf{N}^+$. If (∂,Υ) is $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}^2_\Upsilon$, then it is also $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}^1_\Upsilon$.

VI. Disj-Conj Architecture : Generalization of the Conjunctive Architecture

In Sect. V, we proposed a Conj-Disj architecture which generalizes the disjunctive architecture. Let us now propose a Disj-Conj architecture which generalizes the conjunctive architecture. The two architectures of Sects. (V,VI) are dual, in the sense that one is obtained from the other essentially by switching : 1) between the OR and AND boolean operators, and 2) between decomposing ∂ and decomposing Υ.

A. $\vee$-$\wedge$-*prognoser,*

Now, m *conjunctive* architectures indexed by $j = 1,\cdots,m$ are working in parallel and their global decisions are combined *disjunctively* into a "final" global decision which must satisfy Equations (1,2). The resulting architecture is thus qualified as Disj-Conj. The global decision $X^j(\lambda)$ of the j^{th} $\wedge$-prognoser is computed using Eq. 13 and the final global decision using Eq. 14.

$$\forall j \in \{1,\cdots,m\} : X^j(\lambda) = \bigwedge_{i=1,\cdots,n} X_i^j(P_i(\lambda)) \qquad (13)$$

$$\mathrm{X}(\lambda) = \bigvee_{j=1,\cdots,m} X^j(\lambda) \qquad (14)$$

Definition 6.1: ($\vee$-$\wedge^m$*-prognoser,* $\vee$-$\wedge$*-prognoser*) Given $m \in \mathbf{N}^+$, a set of m $\vee$-prognosers which are fused disjunctively is called $\wedge$-$\vee^m$-prognoser. When m is not specified, we say $\wedge$-$\vee$-prognoser.

B. Decomposing ∂ for computing the local decisions

We assume we are given a decomposition $(\partial^1,\cdots\partial^m)$ of ∂. That is, $\partial = \partial^1 \cup \cdots \cup \partial^m$, where the ∂^j are not necessarily disjoint with each other. A rule for computing the local decisions $(X_i^j(P_i(\lambda)))_{i=1\cdots n}$ of the j^{th} architecture, for each $j = 1,\cdots,m$, is to adapt Eq. 7 as follows :

$$\forall j \in \{1,\cdots,m\}, \forall i \in \{1,\cdots,n\} :$$
$$X_i^j(P_i(\lambda)) = \left\{ \begin{array}{ll} 1, & \text{if } P_i(\lambda) \in P_i(\partial^j) \\ 0, & \text{otherwise} \end{array} \right\} \qquad (15)$$

The idea of this approach is that for each $j = 1\cdots m$, the global decision of the j^{th} conjunctive architecture respects Eqs (1,2), but w.r.t (∂^j,Υ) instead of (∂,Υ). And by fusing disjunctively these m global decisions, we obtain a "final" global decision that respects Eqs (1,2) w.r.t (∂,Υ).

Let us consider the example of Fig. 4, where $\Sigma_{uo} = \{f\}$, $\Sigma_{o,1} = \{a_1,\sigma\}$, $\Sigma_{o,2} = \{a_2,\sigma\}$. $\mathcal{F} = a_1\sigma f a_1^* + a_2\sigma f a_2^*$, and $\mathcal{H} = (a_1+a_2)\sigma + \overline{(\sigma + a_1a_2 + a_2a_1)}\sigma^*$. $\Upsilon = \sigma^* + \overline{(a_1a_2+a_2a_1)}\sigma^*$ corresponds to states 1, 2, 3, 4. $\partial = (a_1+a_2)\sigma$ is decomposed into $\partial^1 = a_1\sigma$ and $\partial^2 = a_2\sigma$, corresponding to states 5 and 6, respectively.

Let us show that with this decomposition of ∂, the $\vee$-$\wedge^2$-prognoser defined by Eqs. (13,14,15) satisfies Eqs. (1,2). Table IV outlines $X_i^j(P_i(\lambda))$, $X^j(\lambda)$ and $\mathrm{X}(\lambda)$ computed using Eqs. (15,13,14) for all sequences λ without f. We see that "$\mathrm{X}(\lambda) = 1$" is issued only for the sequences $a_1\sigma$ and $a_2\sigma$, which is conform to Eqs. (1,2).

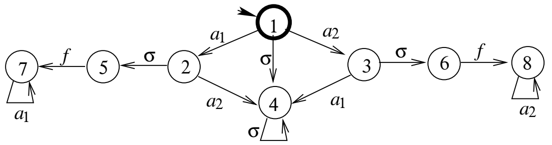

Fig. 4. Example for illustrating the Disj-Conj prognosis

C. Existence Results

Using the same approach as for the Conj-Disj architecture in Sect. V, we consider decompositions of ∂ based on the Nerode relation $\mathcal{N}_\partial$ over ∂, or any finite relation $\mathcal{R}_\partial$ such that $\mathcal{R}_\partial \leq \mathcal{N}_\partial$, and respecting the following assumption **A2** :

A2 : For every $j = 1,\cdots,m$, ∂^j consists of one or several equivalence classes of $\mathcal{R}_\partial$.

We thus define Equation (16) and the notion of $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$:

λ	$P_1(\lambda)$	$X_1^1(P_1(\lambda))$	$X_1^2(P_1(\lambda))$	$P_2(\lambda)$	$X_2^1(P_2(\lambda))$	$X_2^2(P_2(\lambda))$	$X^1(\lambda)$	$X^2(\lambda)$	X(λ)
σ	σ	0	1	σ	1	0	0	0	0
$\sigma\sigma^*$	$\sigma\sigma^*$	0	0	$\sigma\sigma^*$	0	0	0	0	0
a_1	a_1	0	0	ε	0	0	0	0	0
$a_1\sigma$	$a_1\sigma$	1	0	σ	1	0	1	0	1
a_1a_2	a_1	0	0	a_2	0	0	0	0	0
$a_1a_2\sigma$	$a_1\sigma$	1	0	$a_2\sigma$	0	1	0	0	0
$a_1a_2\sigma\sigma\sigma^*$	$a_1\sigma\sigma\sigma^*$	0	0	$a_2\sigma\sigma\sigma^*$	0	0	0	0	0
a_2	ε	0	0	a_2	0	0	0	0	0
$a_2\sigma$	σ	0	1	$a_2\sigma$	0	1	0	1	1
a_2a_1	a_1	0	0	a_2	0	0	0	0	0
$a_2a_1\sigma$	$a_1\sigma$	1	0	$a_2\sigma$	0	1	0	0	0
$a_2a_1\sigma\sigma\sigma^*$	$a_1\sigma\sigma\sigma^*$	0	0	$a_2\sigma\sigma\sigma^*$	0	0	0	0	0

TABLE IV

DISJ-CONJ PROGNOSIS RESULTS FOR THE EXAMPLE OF FIG. 4

$$\forall j = 1, \cdots, m : \bigcap_{i=1,\cdots n} [P_i^{-1} P_i(\partial^j)] \cap \Upsilon = \emptyset \qquad (16)$$

Definition 6.2: ($\vee$-$\wedge^m$*-COPROG w.r.t* $\mathcal{R}_\partial$) Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\partial$ such that $\mathcal{R}_\partial \leq \mathcal{N}_\partial$. (∂) is said $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$ if there exists a decomposition $(\partial^1, \cdots, \partial^m)$ of ∂ respecting Assumption **A2** and such that each (∂^j, Υ) is $\wedge$-COPROG (i.e., Eq. 16).

Proposition 6.1: Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\partial$ such that $\mathcal{R}_\partial \leq \mathcal{N}_\partial$. If (∂, Υ) is $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$ for some $m \in \mathbf{N}^+$, then for every equivalence class A of $\mathcal{R}_\partial$ $(A \subseteq \partial)$: $\bigcap_{i=1,\cdots n}[P_i^{-1}P_i(A)] \cap \Upsilon = \emptyset$.

The following theorem relates $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$ to the existence of $\vee$-$\wedge^m$-prognoser.

Theorem 6.1: Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\partial$ such that $\mathcal{R}_\partial \leq \mathcal{N}_\partial$. The following Point 1 ensures Point 2 :

1) (∂, Υ) is $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$.
2) There exists a $\vee$-$\wedge^m$-prognoser defined by Eqs. (13,14) and satisfying Eqs. (1,2).

The following proposition implies that it is not restrictive to consider as solution uniquely the $\vee$-$\wedge^m$-prognoser respecting Eqs. (13,14,15), instead of any $\vee$-$\wedge^m$-prognoser respecting Eqs. (13,14).

Proposition 6.2: Consider a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and a finite equivalence relation $\mathcal{R}_\partial$ such that $\mathcal{R}_\partial \leq \mathcal{N}_\partial$. The following Point 1 implies Point 2 :

1) (∂, Υ) is $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$.
2) We consider any decomposition of ∂ satisfying Assumption **A2** and Eq. (16) (exists by definition of $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$). The corresponding $\vee$-$\wedge^m$-prognoser defined by Eqs. (13,14,15) satisfies Eqs. (1,2).

Let us return to the example of Fig. 4, where $\Upsilon = \sigma^* + (a_1a_2 + a_2a_1)\sigma^*$, and ∂ was decomposed into $\partial^1 = a_1\sigma$ and $\partial^2 = a_2\sigma$. Let us show that (∂, Υ) is $\vee$-$\wedge^2$-COPROG w.r.t $\mathcal{N}_\partial$. $\mathcal{N}_\partial$ contains two equivalence classes, ∂^1 and ∂^2, corresponding to states 5 and 6, respectively. We compute :

$P_1(\partial^1) = a_1\sigma, P_2(\partial^1) = \sigma, \quad P_1(\partial^2) = \sigma, P_2(\partial^2) = a_2\sigma.$

$[P_1^{-1}P_1(\partial^1)] \cap \Upsilon = [P_2^{-1}P_2(\partial^2)] \cap \Upsilon = (a_1a_2 + a_2a_1)\sigma.$

$[P_1^{-1}P_1(\partial^2)] \cap \Upsilon = [P_2^{-1}P_2(\partial^1)] \cap \Upsilon = \sigma.$

Therefore, $\bigcap_{i=1,2}[P_i^{-1}P_i(\partial^j)] \cap \Upsilon = \emptyset$, for $j = 1, 2$. From Def. 6.2, (∂, Υ) is $\vee$-$\wedge^2$-COPROG w.r.t $\mathcal{N}_\partial$. And from Theor. 6.1, we deduce that there exists a $\vee$-$\wedge^2$-prognoser defined by Eqs. (13,14) and satisfying Eqs. (1,2). Prop. 6.2 confirms that a solution is the $\vee$-$\wedge^2$-prognoser defined by Eqs. (13,14,15) and outlined in Table IV.

The following proposition implies that the Disj-Conj architecture is more general than the conjunctive architecture.

Proposition 6.3: Given a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, if (∂, Υ) is $\wedge$-COPROG then there exists $m \in \mathbf{N}^+$ such that (∂, Υ) is $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{N}_\partial$.

Proposition 6.3 is confirmed by the fact that $\wedge$-COPROG is equivalent to $\vee$-$\wedge^1$-COPROG w.r.t $\mathcal{N}_\partial$. The example of Fig. 4 proves that the converse of Prop. 6.3 is not true. Indeed, we have shown that (∂, Υ) is $\vee$-$\wedge^2$-COPROG w.r.t $\mathcal{N}_\partial$, and now we show that it is not $\wedge$-COPROG as follows :

$P_i(\partial) = a_i\sigma + \sigma$, for $i = 1, 2$,

$[P_i^{-1}P_i(\partial)] \cap \Upsilon = \sigma + (a_1a_2 + a_2a_1)\sigma$, for $i = 1, 2$.

Thus, $\bigcap_{i=1,2}[P_i^{-1}P_i(\partial)] \cap \Upsilon = \sigma + (a_1a_2 + a_2a_1)\sigma \neq \emptyset$.

Note that this example is not $\vee$-COPROG as well, because we compute $\bigcap_{i=1,2}[P_i^{-1}P_i(\Upsilon)] \cap \partial = \partial \neq \emptyset$.

The following proposition is due to the fact that if $\mathcal{R}_\partial^1 \leq \mathcal{R}_\partial^2 \leq \mathcal{N}_\partial$, then $\mathcal{R}_\partial^1$ permits more decompositions than $\mathcal{R}_\partial^2$.

Proposition 6.4: Consider a pair of finite equivalence relations $(\mathcal{R}_\partial^1, \mathcal{R}_\partial^2)$ such that $\mathcal{R}_\partial^1 \leq \mathcal{R}_\partial^2 \leq \mathcal{N}_\partial$, a pair $(\mathcal{L}, \mathcal{H})$ of prefix-closed languages with $\mathcal{H} \subseteq \mathcal{L}$, and an integer $m \in \mathbf{N}^+$. If (∂, Υ) is $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial^2$, then it is also $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial^1$.

VII. GENERAL MULTI-DECISION ARCHITECTURE, PREDICTION IN ADVANCE

A. General Multi-Decision Architecture

In Section IV-C, we proposed a general architecture that combines and generalizes the disjunctive and conjunctive architectures. We propose here a general multi-decision architecture that combines and generalizes the Conj-Disj and Disj-Conj architectures.

Proposition 7.1: Given two finite equivalence relations $\mathcal{R}_\Upsilon$ and $\mathcal{R}_\partial$, $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{R}_\Upsilon$ and $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{R}_\partial$ are incomparable, i.e., none of them guarantees the other.

Let us illustrate Proposition 7.1 using the two examples of Figures (3,4). We have shown in Section V that the example of Fig. 3 is $\wedge$-$\vee^2$-COPROG w.r.t $\mathcal{N}_\Upsilon$. On the other hand, we have seen that : 1) it is not $\wedge$-COPROG and 2) ∂ consists of a single equivalence class A w.r.t $\mathcal{N}_\partial$. From these two points and Prop. 5.1, we deduce that, for every $m \in \mathbf{N}^+$, the example is not $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{N}_\partial$.

We have shown in Section VI that the example of Fig. 4 is $\vee$-$\wedge^2$-COPROG w.r.t $\mathcal{N}_\partial$. We have also seen that it is not $\vee$-COPROG and that Υ consists of four equivalence classes w.r.t $\mathcal{N}_\Upsilon$ (states 1, 2, 3, 4). If we consider the equivalence class A corresponding to state 4, we compute $A = \sigma\sigma^* + (a_1a_2+a_2a_1)\sigma\sigma^*$, $P_i(A) = \sigma\sigma^*+a_i\sigma\sigma^*$, $[P_i^{-1}P_i(A)]\cap\partial = \partial$, and thus $\bigcap_{i=1,2}[P_i^{-1}P_i(A)]\cap\partial = \partial \neq \emptyset$. From Prop. 6.1, we deduce that, for every $m \in \mathbf{N}^+$, the example is not $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{N}_\Upsilon$.

To recapitulate, we have found an example which is $\wedge$-$\vee^2$-COPROG w.r.t $\mathcal{N}_\Upsilon$ and not $\vee$-$\wedge^m$-COPROG w.r.t $\mathcal{N}_\partial$ (Fig. 3), and an example which is $\vee$-$\wedge^2$-COPROG w.r.t $\mathcal{N}_\partial$ and not $\wedge$-$\vee^m$-COPROG w.r.t $\mathcal{N}_\Upsilon$ (Fig. 4). Therefore, the Conj-Disj and Disj-Conj architectures can be respectively used for these two examples. We can think of a general multi-decision prognosis architecture for systems with two categories of failures : failures which can be predicted using the Conj-Disj architecture, and failures which can be predicted using the Disj-Conj architecture. It can be shown that such a general multi-decision architecture is more general than the Conj-Disj and Disj-Conj architectures. For lack of space, we do not give more details on this general architecture.

B. Failure Prediction in Advance

The four architectures of Sections (III,IV,V,VI) are based on the use of Υ and ∂. Remind that $\partial = \{\lambda \in \mathcal{H} \mid \{\lambda\}\Sigma \cap \mathcal{F} \neq \emptyset\}$. This definition of ∂ guarantees that the failure is predicted at the latest just before its occurrence.

Let us now consider that the objective is to predict a failure at least k steps before its occurrence, for a given $k \geq 1$. The interest of our framework is that it remains applicable with this new objective by just replacing ∂ by $\partial(k) = \{\lambda \in \mathcal{H} \mid \{\lambda\}\Sigma^{\leq k} \cap \mathcal{F} \neq \emptyset\}$. Note that $\partial(1) = \partial$.

VIII. Conclusion

We have studied the decentralized prognosis, where local prognosers cooperate in order to predict failures. Our essential contributions can be summarized by the following points :

1) We formulate the architecture of [1] as a disjunctive prognosis architecture.
2) We develop a conjunctive prognosis architecture which is dual and complementary to the disjunctive one. We also propose and discuss the idea of a general architecture which combines and generalizes the disjunctive and conjunctive architectures.
3) We develop a *multi-decision* framework whose basic principle consists in using several decentralized architectures working in parallel. We use the multi-decision to develop Conj-Disj and Disj-Conj architectures, which generalize the disjunctive and conjunctive architectures, respectively.
4) In the above points 1, 2 an 3, a failure can be predicted at the latest *just* before its occurrence. We show that our work can be very easily extended for predicting a failure at least k steps before its occurrence, for a given $k \geq 1$.

In a near future, we plan to study the complexity of our framework in terms of computation time and used memory.

References

[1] R. Kumar and S. Takai, "Decentralized Prognosis of Failures in Discrete Event Systems," in *Proc. 9th Intern. Workshop on Discrete Event Systems (WODES)*, Göteborg, Sweden, May 2008.

[2] Y. Wang, T. Yoo, and S. Lafortune, "Decentralized Diagnosis of Discrete Event Systems Using Unconditional and Conditional Decisions," in *Conference on Decision and Control (CDC)*, Seville, Spain, December 2005.

[3] S. Lafortune, Y. Wang, and T. Yoo, "Diagnostic décentralisé des systèmes à événements discrets," in *5th Colloque Francophone sur la Modélisation des Systèmes Réactifs (MSR)*, Grenoble, France, October 2005.

[4] G. Vachtsevanos, F. Lewis, M. Roemer, A. Hess, and B. Wu, *Intelligent Fault Diagnosis and Prognosis for Engineering Systems*. Hoboken, NJ : John Wiley and Sons, 2006.

[5] H. Fadel and L. Holloway, "Using SPC and Template Monitoring Method for Fault Detection and Prediction in Discrete Event Manufacturing Systems," in *Intelligent Control/Intelligent Systems and Semiotics*, Cambridge, MA, USA, 1999.

[6] S. Genc and S. Lafortune, "Predictability in Discrete-Event Systems Under Partial Observation," in *IFAC Symposium on Fault Detection, Supervision, and Safety of Technical Processes*, Beijing, China, 2006.

[7] T. Jéron, H. Marchand, S. Genc, and S. Lafortune, "Predictability of Sequence Patterns in Discrete Event Systems," IRISA, Rennes, France, Tech. Rep. 1834, 2007.

[8] T.-S. Yoo and S. Lafortune, "A General Architecture for Decentralized Supervisory Control of Discrete-Event Systems," *Discrete Event Dyna. Syst. : Theory Applicat.*, vol. 12, pp. 335–377, 2002.

[9] ——, "Decentralized Supervisory Control With Conditional Decisions : Supervisor Existence," *IEEE Transactions on Automatic Control*, vol. 49, no. 11, pp. 1886–1904, November 2004.

[10] R. Kumar and S. Takai, "Inference-Based Ambiguity Management in Decentralized Decision-Making : Decentralized Control of Discrete Event Systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1783–1794, 2007.

[11] H. Chakib and A. Khoumsi, "Multi-Decision Decentralized Control of Discrete Event Systems : Application to the C&P Architecture," in *Proc. 9th Intern. Workshop on Discrete Event Systems (WODES)*, Göteborg, Sweden, May 2008.

[12] ——, "Multi-decision C&PvD&A Architecture for the Decentralized Control of Discrete Event Systems," in *Proc. 4th Annual IEEE. Confernce on Automation Science and Engineering (CASE)*, Washington DC, USA, August 2008.

[13] R. Kumar and S. Takai, "Inference-Based Ambiguity Management in Decentralized Decision-Making : Decentralized Diagnosis of Discrete Event Systems," in *American Control Conference (ACC)*, Minneapolis, Minnesota, USA, June 2006.

[14] S. Takai and R. Kumar, "Decentralized Diagnosis for Nonfailures of Discrete Event Systems Using Inference-Based Ambiguity Management," in *Proc. 8th Intern. Workshop on Discrete Event Systems (WODES)*, Ann Arbor, Michigan, USA, July 2006.

[15] A. Khoumsi and H. Chakib, "A Multi-Decision Approach for Decentralized Diagnosis of the Presence and Absence of Faults in Discrete Event Systems," in *Proc. 16th Mediterranean Conference on Control and Automation (MED)*, Ajaccio, Corsica, France, June 2008.