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Lyapunov Functions under LaSalle Conditions with an Application to Lotka-Volterra Systems

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Abstract— **We provide new techniques for building explicit global strict Lyapunov functions for broad classes of time varying nonlinear systems satisfying LaSalle conditions. Our new constructions are simpler than the designs available in the literature. We illustrate our work using the Lotka-Volterra model, which plays a fundamental role in bioengineering.**

I. Introduction

Lyapunov functions provide vital tools for the analysis of, and controller design for, nonlinear systems [5], [6], [15]. The two main types of Lyapunov functions are *strict* Lyapunov functions (also called *strong* Lyapunov functions, having negative definite time derivatives along the trajectories of the system) and *nonstrict* Lyapunov functions (also called *weak* Lyapunov functions, whose time derivatives along the trajectories are negative *semi*definite); see Section II below for precise definitions.

Strict Lyapunov functions are typically far more useful than nonstrict ones. In general, nonstrict Lyapunov functions can only be used to prove asymptotic stability, via e.g. the LaSalle invariance principle, while *strict* Lyapunov functions can often be used to show robustness properties, such as input-to-state stability (ISS). Robustness is essential in engineering, largely due to uncertainty in dynamical models and noise entering into controllers. For this reason, it is important to construct strict Lyapunov functions, even for systems that are already known to be asymptotically stable.

Moreover, many controller methods (such as backstepping [6], forwarding [13], [15] and universal stabilizing controllers [16]) require strict Lyapunov functions. For example, if V is a global strict Lyapunov function for a system $\dot{x} = f(x)$ for which $-\nabla V(x)f(x)$ is radially unbounded and g is locally Lipschitz, then $\dot{x} = f(x) + g(x)[K(x) + d]$ is input-to-state stable if $K(x) = -\nabla V(x)g(x)$. Consequently, when an explicit strict Lyapunov function is known, many important stabilization problems can be solved almost immediately.

In general, it is much easier to construct *nonstrict* Lyapunov functions, owing to the more restrictive decay condition for strict Lyapunov functions. For instance, when a passive nonlinear system is stabilized by linear output feedback, the energy (i.e., storage) function can typically be used as the weak Lyapunov function. When a system is stabilized via the Jurdjevic-Quinn theorem, nonstrict Lyapunov functions are typically available, e.g., using the Hamiltonian for Euler Lagrange systems [2], [4], [10], [14]. If a system is known to be asymptotically stable, then converse Lyapunov function theory typically guarantees the existence of a strict Lyapunov function. However, the Lyapunov functions provided by converse theory are often abstract and nonexplicit, and therefore may not always lend themselves to applications. This has motivated a significant literature on constructing strict Lyapunov functions, e.g., [1], [2].

In this note, we present two new strict Lyapunov function constructions, based on transforming nonstrict Lyapunov functions into strict ones, under Lie derivative conditions. The assumptions for our first construction are more general than those of [12] and different from those of [8, Corollary 2]. This is because we allow *time varying* systems, including cases where all of the higher order Lie derivatives are allowed to vanish at some points outside the equilibrium, on some time intervals. Our construction is simpler than the one in [12], even in the special case of time invariant systems.

Our second result uses the Matrosov approach. In general, Matrosov's method can be difficult to apply, because one needs to find the necessary auxiliary functions. Here we give simple sufficient conditions leading to a systematic design of auxiliary functions. This makes it possible to construct strict Lyapunov functions via the Matrosov theorem from [11]. We illustrate our approach by building a strict Lyapunov function for the Lotka-Volterra dynamics, which plays a fundamental role in bioengineering.

II. Definitions and Assumptions

Throughout this note, $\mathcal{X}$ is any open subset of $\mathbb{R}^n$ containing the origin. Consider a nonlinear time varying dynamics

$$\dot{x} = f(t, x), \quad x \in \mathcal{X} \tag{1}$$

where $f : [0, \infty) \times \mathcal{X} \to \mathbb{R}^n$ is C^∞, $\mathcal{X}$ is positively invariant for (1), and $f(t, 0) = 0$ for all $t \geq 0$. For convenience, we always assume that (1) is *periodic* of period T in t, meaning there is a constant $T > 0$ so that $f(t+T, x) = f(t, x)$ for all $(t, x) \in [0, \infty) \times \mathcal{X}$. We further assume that (1) is *forward complete*, meaning for each initial condition $x(t_0) = x_0$ with $t_0 \geq 0$ and $x_0 \in \mathcal{X}$, the solution $x(t, t_0, x_0)$ for the corresponding initial value problem for (1) is uniquely defined on $[t_0, \infty)$. Set $\mathbb{N} = \{1, 2, 3, \ldots\}$. Given a C^∞ function $V : [0, \infty) \times \mathcal{X} \to \mathbb{R}$, define $\{a_i : i \in \mathbb{N}\}$ by

$$\begin{aligned} a_1(t,x) &= -\tfrac{\partial V}{\partial x}(t,x)f(t,x) - \tfrac{\partial V}{\partial t}(t,x) \ \text{ and} \\ a_r(t,x) &= -\tfrac{\partial a_{r-1}}{\partial x}(t,x)f(t,x) - \tfrac{\partial a_{r-1}}{\partial t}(t,x) \end{aligned} \tag{2}$$

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for all $r \geq 2$. If V and f are time invariant, then $a_r = (-1)^r L_f^r V$ for all $r \geq 1$, where L_f^r is the usual iterated Lie derivative defined by $L_f^0 V = V$, $L_f V(x) = L_f^1 V(x) = (\partial V/\partial x)(x) f(x)$ and $L_f^k V = L_f(L_f^{k-1} V)$ for $k \geq 2$. If we use $\dot{G} = (\partial G/\partial t)(t,x) + (\partial G/\partial x)(t,x) f(t,x)$ for any C^1 function G, then $\dot{a}_r = -a_{r+1}$ for all r. A continuous function $k : [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}_\infty$ (written $k \in \mathcal{K}_\infty$) provided it is zero at zero, strictly increasing and unbounded. A continuous function $G : [0,\infty) \times \mathcal{X} \to \mathbb{R}$ is *positive definite* (resp., *positive semi-definite*) on $\mathcal{X}$ provided $G(t,0) = 0$ for all t and $\inf\{G(t,x) : t \geq 0\} > 0$ (resp., ≥ 0) for all $x \in \mathcal{X} \setminus \{0\}$. A function G is *negative (semi-)definite* provided $-G$ is positive (semi-)definite.

A function $V : [0,\infty) \times \mathcal{X} \to \mathbb{R}$ is called a *storage function* provided there exist positive definite functions $\alpha_1, \alpha_2 : \mathcal{X} \to [0,\infty)$ such that the following hold: (a) for each i, $\alpha_i(q) \to +\infty$ whenever $|q| \to +\infty$ with q remaining in $\mathcal{X}$; and (b) $\alpha_1(x) \leq V(t,x) \leq \alpha_2(x)$ for all $x \in \mathcal{X}$. A storage function V is called a *non-strict* (resp., *strict*) *Lyapunov-like function* for (5) provided it is C^1 and $\dot{V}(t,x)$ is negative semi-definite (resp., negative definite). If, in addition, for each i and each $\bar{q} \in \partial\mathcal{X}$, $\alpha_i(q) \to +\infty$ when $q \to \bar{q}$ then a non-strict (resp., strict) Lyapunov-like function is called a *non-strict* (resp., *strict*) *Lyapunov function*. The existence of strict Lyapunov functions is key to proving uniform global asymptotic stability (UGAS) [11].

III. First Construction: Iterated Lie Derivatives

To motivate our assumptions, suppose that a given C^∞ time invariant system $\dot{x} = f(x)$ evolving on $\mathbb{R}^n$ admits a time invariant C^∞ nonstrict Lyapunov function $V(x)$ such that for each $q \in \mathbb{R}^n \setminus \{0\}$, there is an $i \in \mathbb{N}$ such that $L_f^i V(q) \neq 0$. Then if $L_f V(x(t,x_0)) \equiv 0$ along some trajectory $x(\cdot, x_0)$ of the system, we can differentiate repeatedly in time to get $L_f^k V(x(t,x_0)) \equiv 0$ for all $t \geq 0$ and $k \in \mathbb{N}$. Hence, UGAS follows from the LaSalle invariance principle. However, it is not obvious how to construct a global *strict* Lyapunov function in this situation. This motivates our (more general) hypotheses in the following theorem:

Theorem 1: Consider the time varying system (1) with state space $\mathcal{X} = \mathbb{R}^n$ and period $T > 0$ in t. Assume that there exists a C^∞ storage function $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ having period T in t such that the following hold:

(i) $V(\cdot)$ is a nonstrict Lyapunov function for (1); and

(ii) there exist constants $\tau \in (0,T]$ and $\ell \in \mathbb{N}$ and a positive definite continuous function ρ such that for all $x \neq 0$ and all $t \in [0,\tau]$,

$$a_1(t,x) + \textstyle\sum_{m=2}^{\ell} a_m^2(t,x) \geq \rho(V(t,x)) \, . \qquad (3)$$

(See (2).) Then we can explicitly determine functions $\mathcal{F}_j$ and $\mathcal{G}$, with $\mathcal{G}$ periodic of period T in t, such that

$$\begin{aligned} &V^\sharp(t,x) = \textstyle\sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(t,x)\big) A_j(t,x) + \mathcal{G}\big(t, V(t,x)\big) \\ &\text{where } \ A_j(t,x) = \textstyle\sum_{m=1}^{j} a_{m+1}(t,x) a_m(t,x) \quad \forall j \end{aligned} \qquad (4)$$

is a strict Lyapunov function for (1), giving UGAS of (1).

Remark 1: Theorem 1 remains true if V is merely $C^{\ell+1}$ (instead of C^∞). The assumptions of Theorem 1 are related to, but more general than, the assumptions of the strict Lyapunov function construction from [12] and different from the assumptions of [8, Corollary 2]. In fact, the assumptions of [12] are the special case of (i)-(ii) in which f and V are time invariant; in that case, (3) is the requirement that there is a continuous positive definite function ρ so that $-L_f V(x) + \sum_{m=2}^{\ell} (L_f^m(x))^2 \geq \rho(V(x))$ for all $x \in \mathbb{R}^n$. Our result is new, even in this special case, because the strict Lyapunov function construction in our proof of Theorem 1 is simpler than the one in [12]; the construction in [12] uses a complicated smooth dynamic scaling that we do not need here. It is important to have strict Lyapunov functions that are as simple as possible when they are used for feedback design and robustness analysis. We prove Theorem 1 in Section V.

IV. Second Construction: Matrosov Conditions

To simplify the notation in our next theorem, we consider only time invariant systems

$$\dot{x} = f(x), \quad x \in \mathcal{X} \qquad (5)$$

for which $\mathcal{X} \subseteq \mathbb{R}^n$ is positively invariant, where $f(0) = 0$; see Remark 2 below for the generalization to (1). We use the Matrosov approach from [11] to construct global strict Lyapunov functions for (5). In addition to a nonstrict Lyapunov function, the Matrosov results from [11] require appropriate auxiliary functions, which can be difficult to find in practice. The paper [11] does not provide any general methods for constructing auxiliary functions. Hence, the theorem we give next sheds new light on the Matrosov theorems, because it gives a new mechanism for choosing auxiliary functions.

However, the most important features of our second construction are that (A) it applies to systems whose state space is only a subset of $\mathbb{R}^n$ and (B) it may yield Lyapunov functions that are simpler than the ones from Theorem 1, and that also have desirable local properties, such as local boundedness from below by positive definite quadratic functions; see Section VII. For the rest of this section, we assume that all of our functions are sufficiently smooth.

Assumption 1: There exist a storage function $V_1 : \mathcal{X} \to [0,\infty)$; functions $h_1, \ldots, h_m$ such that $h_j(0) = 0$ for all j; everywhere positive functions $r_1, \ldots, r_m$ and ρ; and an integer $N > 0$ for which

$$\nabla V_1(x) f(x) \ \leq \ -r_1(x) h_1^2(x) - \ldots - r_m(x) h_m^2(x) \qquad (6)$$

$$\text{and} \quad \sum_{l=0}^{N-1} \sum_{j=1}^{m} \left[L_f^l h_j(x)\right]^2 \ \geq \ \rho(V_1(x)) V_1(x) \qquad (7)$$

hold for all $x \in \mathcal{X}$. Moreover, f is defined on $\mathbb{R}^n$, and there is a function $\overline{\Gamma} \in \mathcal{K}_\infty$ such that

$$|f(x)| \leq \overline{\Gamma}(|x|) \ \ \forall x \in \mathbb{R}^n. \qquad (8)$$

Also, V_1 has a positive definite quadratic lower bound in some neighborhood of the origin.

In Section VI, we prove the following:

Theorem 2: If (5) satisfies Assumption 1, then one can determine explicit functions $k_i, \Omega_i \in \mathcal{K}_\infty \cap C^1$ such that

$$S(x) = \sum_{i=1}^{N} \Omega_i\big(k_i(V_1(x)) + V_i(x)\big) \tag{9}$$

with the choices

$$V_i(x) = -\sum_{l=1}^{m} L_f^{i-2} h_l(x) L_f^{i-1} h_l(x)\ ,\ \ i = 2, \ldots, N \tag{10}$$

satisfies $S(x) \geq V_1(x)$ for all $x \in \mathcal{X}$ and is such that $\nabla S(x) f(x)$ is negative definite. If, in addition, $\mathcal{X} = \mathbb{R}^n$, then the system (5) is globally asymptotically stable.

Throughout our proofs, all inequalities should be understood to hold globally unless otherwise indicated, and we leave out the arguments of our functions when they are clear.

V. Proof of Theorem 1

By enlarging $\ell \geq 1$ in Theorem 1 as necessary without relabeling, we can assume that $\ell \geq 3$. By minorizing ρ from (3) as necessary without relabeling, we assume that

$$\rho(r) = \frac{\omega(r)}{K(r)} \tag{11}$$

for some $\omega \in \mathcal{K}_\infty \cap C^1$ and some strictly increasing everywhere positive function $K \in C^1$, without loss of generality; see [9, Lemma A.8] for the construction of ω and K. Since the a_m's and V are periodic in t, we can find an everywhere positive increasing function $\Gamma \in C^1$ such that

$$\Gamma\big(V(t,x)\big) \geq (\ell+2)|a_m(t,x)|+1 \ \ \forall m \in \{1,...,\ell+1\} \tag{12}$$

holds for all $x \in \mathbb{R}^n$ and $t \geq 0$, e.g., by majorizing $s \mapsto 1 + \max\{(\ell+2)|a_m(t,x)| : t \geq 0, m \in \{1, 2, \ldots, \ell+1\}, V(t,x) \leq s\}$ by a C^1 function.

We use these functions, which have period T in t:

$$M_j(t,x) = \textstyle\sum_{m=1}^{j} a_{m+1}(t,x) a_m(t,x) + \int_0^{V(t,x)} \Gamma(r)\mathrm{d}r$$
$$\text{and}\ \ N_j(t,x) = \textstyle\sum_{m=2}^{j+1} a_m^2(t,x) + a_1(t,x)$$

for $j = 1, 2, \ldots \ell-1$. Since $a_1 = -\dot{V} \geq 0$, (12) gives

$$\begin{aligned} \dot{M}_1 &= \dot{a}_2(t,x)a_1(t,x) - a_2^2(t,x) \\ &\quad -\Gamma(V(t,x))a_1(t,x) \\ &\leq -a_2^2(t,x) - a_1(t,x) = -N_1(t,x)\ , \end{aligned} \tag{13}$$

since $\dot{a}_i = -a_{i+1}$ for all i. Also, for each $j \in \{2, ..., \ell-1\}$,

$$\begin{aligned} \dot{M}_j &\leq -\textstyle\sum_{m=1}^{j} a_{m+1}^2(t,x) \\ &\quad + \textstyle\sum_{m=2}^{j} |a_{m+2}(t,x)||a_m(t,x)| \\ &\quad + |a_3(t,x)|a_1(x) - \Gamma\big(V(t,x)\big)a_1(t,x) \\ &\leq -\textstyle\sum_{m=1}^{j} a_{m+1}^2(t,x) + |a_3(t,x)|a_1(t,x) \\ &\quad + \textstyle\sum_{m=2}^{j} |a_{m+2}(t,x)||a_m(t,x)| \\ &\quad - \big[(\ell+2)|a_3(t,x)| + 1\big]a_1(t,x). \end{aligned}$$

From (12), we deduce that for all $j \in \{2, ..., \ell-1\}$,

$$\begin{aligned} \dot{M}_j &\leq -\textstyle\sum_{m=1}^{j} a_{m+1}^2(t,x) \\ &\quad + \tfrac{\Gamma(V(t,x))}{\ell+2} \textstyle\sum_{m=2}^{j} |a_m(t,x)| \\ &\quad - \big[(\ell+1)|a_3(t,x)| + 1\big]a_1(t,x). \end{aligned} \tag{14}$$

It follows from the Cauchy Inequality that for all $j \in \{2, ..., \ell-1\}$,

$$\begin{aligned} \dot{M}_j &\leq -\textstyle\sum_{m=1}^{j} a_{m+1}^2(t,x) \\ &\quad + \Gamma\big(V(t,x)\big)\sqrt{\textstyle\sum_{m=2}^{j} a_m^2(t,x)} \\ &\quad - \big[(\ell+1)|a_3(t,x)| + 1\big]a_1(t,x) \\ &= -\textstyle\sum_{m=2}^{j+1} a_m^2(t,x) - a_1(t,x) \\ &\quad + \Gamma\big(V(t,x)\big)\sqrt{\textstyle\sum_{m=2}^{j} a_m^2(t,x)} \\ &\quad - (\ell+1)|a_3(t,x)|a_1(t,x) \\ &\leq -N_j(t,x) + \Gamma\big(V(t,x)\big)\sqrt{N_{j-1}(t,x)}, \end{aligned} \tag{15}$$

since $a_1 = -\dot{V} \geq 0$ everywhere. Set

$$\Omega(v) \ = \ \frac{2\tau\omega(v)}{3T\ell\Gamma^2(v)K(v)}, \tag{16}$$

$$\begin{aligned} k_{\ell-1}(v) &= \tfrac{6T}{\tau} K(v)\omega^{2^{\ell-1}}(v), \ \text{and} \\ k_p(v) &= k_{\ell-1}(v)\Omega^{1-2^{l-p-1}}(v) \end{aligned} \tag{17}$$

for $p = 1, 2, \ldots, \ell-2$. The functions $k_1, k_2, \ldots, k_{\ell-1} \in C^1$ are positive definite.

Pick a C^1 everywhere positive increasing function k_0 such that

$$\begin{aligned} & k_0\big(V(t,x)\big) + k_0'\big(V(t,x)\big)V(t,x) \\ & \geq \textstyle\sum_{p=1}^{\ell-1} \big|k_p'\big(V(t,x)\big)\big|\big|M_p(t,x)\big| + 1. \end{aligned} \tag{18}$$

This can be done because each M_p is periodic in t, and because V is a storage function that is also periodic in t. Let

$$S_1(t,x) = \sum_{p=1}^{\ell-1} k_p\big(V(t,x)\big)M_p(t,x) + k_0\big(V(t,x)\big)V(t,x).$$

Then

$$\begin{aligned} \dot{S}_1 &= \textstyle\sum_{p=1}^{\ell-1} k_p\big(V(t,x)\big)\dot{M}_p \\ &\quad + \Big[\textstyle\sum_{p=1}^{\ell-1} k_p'\big(V(t,x)\big)M_p(t,x)\Big]\dot{V} \\ &\quad + \big[k_0\big(V(t,x)\big) + k_0'\big(V(t,x)\big)V(t,x)\big]\dot{V} \\ &\leq \textstyle\sum_{p=1}^{\ell-1} k_p(V(t,x))\dot{M}_p \\ &\quad + \Big[\textstyle\sum_{p=1}^{\ell-1} \big|k_p'\big(V(t,x)\big)\big|\big|M_p(t,x)\big|\Big][-\dot{V}] \\ &\quad + \big[k_0\big(V(t,x)\big) + k_0'\big(V(t,x)\big)V(t,x)\big]\dot{V} \\ &\leq \textstyle\sum_{p=1}^{\ell-1} k_p\big(V(t,x)\big)\dot{M}_p, \end{aligned} \tag{19}$$

using (18) and the fact that $\dot{V}$ is nonpositive. Using the definition of a_1, (13) and (15), we deduce that

$$\begin{aligned} \dot{S}_1 &\leq -k_1(V)N_1 \\ &\quad + \textstyle\sum_{p=2}^{\ell-1} k_p(V)\big[-N_p + \Gamma(V)\sqrt{N_{p-1}}\,\big] \\ &\leq -\textstyle\sum_{p=1}^{\ell-1} k_p(V)N_p + \textstyle\sum_{p=2}^{\ell-1} k_p(V)\Gamma(V)\sqrt{N_{p-1}}. \end{aligned}$$

Let $q : \mathbb{R} \to [0,1]$ be any continuous function with period T that satisfies (a) $q(t) = 0$ when $t \in [\tau, T]$ and (b) $q(t) = 1$ when $t \in [\frac{\tau}{3}, \frac{2\tau}{3}]$. By (3), (11), and the nonnegativity of N_{l-1}, we deduce that

$$N_{\ell-1}(t,x) \ \geq \ q(t)\frac{\omega(V(t,x))}{K(V(t,x))} \tag{20}$$

for all $(t,x)\in[0,\infty)\times\mathbb{R}^n$. It follows that

$$\begin{aligned}\dot{S}_1 \;\leq\; & -k_{\ell-1}(V)q(t)\frac{\omega(V)}{K(V)}-\textstyle\sum_{p=1}^{\ell-2}k_p(V)N_p\\ & +\textstyle\sum_{p=1}^{\ell-2}k_{p+1}(V)\Gamma(V)\sqrt{N_p}.\end{aligned}\tag{21}$$

Let G be any C^1 function such that

$$G'(v)\geq T\left|k_{\ell-1}(v)\frac{\omega'(v)K(v)-\omega(v)K'(v)}{K^2(v)}+k'_{\ell-1}(v)\frac{\omega(v)}{K(v)}\right|$$

for all $v\geq 0$. Let

$$\begin{aligned}&S_2(t,x)=G\big(V(t,x)\big)+\\ &\tfrac{1}{T}\left(\int_{t-T}^t\int_s^t q(r)\mathrm{d}r\mathrm{d}s\right)k_{\ell-1}\big(V(t,x)\big)\frac{\omega\big(V(t,x)\big)}{K\big(V(t,x)\big)}.\end{aligned}\tag{22}$$

Since $\int_{t-T}^t\int_s^t q(r)\mathrm{d}r\mathrm{d}s\leq T^2$ and $\frac{d}{dt}\int_{t-T}^t\int_s^t q(r)\mathrm{d}r\mathrm{d}s=Tq(t)-\int_{t-T}^t q(r)\mathrm{d}r$ everywhere, we get

$$\begin{aligned}\dot{S}_2\;\leq\;& q(t)k_{\ell-1}\big(V(t,x)\big)\frac{\omega\big(V(t,x)\big)}{K\big(V(t,x)\big)}\\ &-\tfrac{1}{T}\left(\int_{t-T}^t q(r)\mathrm{d}r\right)k_{\ell-1}\big(V(t,x)\big)\frac{\omega\big(V(t,x)\big)}{K\big(V(t,x)\big)}.\end{aligned}\tag{23}$$

Consider the following function, which has period T in t:

$$S_3(t,x)=S_1(t,x)+S_2(t,x).\tag{24}$$

Since $\int_{t-T}^t q(r)\mathrm{d}r\geq\frac{\tau}{3}$ for all $t\in\mathbb{R}$, (21) and (23) give

$$\begin{aligned}\dot{S}_3\;\leq\;& -\tfrac{1}{T}\left(\int_{t-T}^t q(r)\mathrm{d}r\right)k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)}\\ &-\textstyle\sum_{p=1}^{\ell-2}k_p(V)N_p\\ &+\textstyle\sum_{p=1}^{\ell-2}k_{p+1}(V)\Gamma(V)\sqrt{N_p}\\ \leq\;& -\tfrac{\tau}{3T}k_{\ell-1}(V)\frac{\omega(V)}{K(V)}-\textstyle\sum_{p=1}^{\ell-2}k_p(V)N_p\\ &+\textstyle\sum_{p=1}^{\ell-2}k_{p+1}(V)\Gamma(V)\sqrt{N_p}.\end{aligned}\tag{25}$$

From the triangular inequality $c_1c_2\leq c_1^2+\frac{1}{4}c_2^2$ for nonnegative values c_1 and c_2, we deduce that

$$\begin{aligned}&k_{p+1}(V)\Gamma(V)\sqrt{N_p}=\left\{\sqrt{k_p(V)N_p}\right\}\left\{\tfrac{\Gamma(V)k_{p+1}(V)}{\sqrt{k_p(V)}}\right\}\\ &\leq\; k_p(V)N_p+\tfrac{\Gamma^2(V)k_{p+1}^2(V)}{4k_p(V)}\end{aligned}\tag{26}$$

for $p=1,2,\ldots,\ell-2$ when $V\neq 0$. Summing the inequalities in (26) over p and combining the result with (25), we get

$$\dot{S}_3\;\leq\;-\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)}+\textstyle\sum_{p=1}^{\ell-2}\tfrac{\Gamma^2(V)k_{p+1}^2(V)}{4k_p(V)}\tag{27}$$

when $x\neq 0$. By our choices (17) of the k_p's, (27) gives

$$\begin{aligned}\dot{S}_3\;\leq\;& -\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)}\\ &+\textstyle\sum_{p=1}^{\ell-2}\tfrac{\Gamma^2(V)k_{\ell-1}^2(V)\Omega^{2(1-2^{\ell-p-2})}(V)}{4k_{\ell-1}(V)\Omega^{1-2^{\ell-p-1}}(V)}\\ \leq\;& -\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)}\\ &+(\ell-2)\tfrac{\Gamma^2(V)k_{\ell-1}(V)\Omega(V)}{4}\\ \leq\;& -\tfrac{\tau}{6T}k_{\ell-1}(V(t,x))\tfrac{\omega(V(t,x))}{K(V(t,x))},\quad x\neq 0,\end{aligned}\tag{28}$$

where the last inequality is by our choice of Ω in (16). Recalling our choice (17) of $k_{\ell-1}$ now gives

$$\dot{S}_3(t,x)\;\leq\;-\omega(V(t,x))^{2^{\ell-1}+1},\tag{29}$$

which has a negative definite right hand side. However, S_3 is not necessarily positive definite and radially unbounded, and therefore may not be a strict Lyapunov function.

To obtain a strict Lyapunov function, consider

$$V^\sharp(t,x)=V(t,x)S_3(t,x)+\kappa\big(V(t,x)\big)V(t,x)\tag{30}$$

where $\kappa\in C^1$ is an everywhere positive function with an everywhere positive first derivative such that $\kappa(V(t,x))\geq|S_3(t,x)|+1$ for all $x\in\mathbb{R}^n$ and $t\geq 0$. Then $V^\sharp$ is positive definite and radially unbounded because $V^\sharp(t,x)\geq V(t,x)$. Since we also have $\dot{V}\leq 0$ everywhere, we get

$$\begin{aligned}\dot{V}^\sharp\;=\;& V(t,x)\dot{S}_3(t,x)+\dot{V}(t,x)S_3(t,x)\\ &+\left[\kappa'\big(V(t,x)\big)V(t,x)+\kappa\big(V(t,x)\big)\right]\dot{V}(t,x)\\ \leq\;& -\omega^{2^{\ell-1}+1}(V(t,x))V(t,x).\end{aligned}$$

The result readily follows from the formula (24) for S_3, by collecting the functions involving V in the formula for $V^\sharp$.

VI. PROOF OF THEOREM 2

In the following proof, we omit the dependencies of the functions on x when they are clear from the context. To simplify our notation, we introduce the functions

$$\begin{aligned}&\mathcal{N}_1(x)=R(x)\textstyle\sum_{l=1}^m h_l^2(x)\quad\text{and}\\ &\mathcal{N}_i(x)=\textstyle\sum_{l=1}^m\left[L_f^{i-1}h_l(x)\right]^2\quad\forall i\geq 2,\\ &\text{where }R(x)=\frac{\prod_{i=1}^m r_i(x)}{\prod_{i=1}^m[r_i(x)+1]}.\end{aligned}\tag{31}$$

Since R is everywhere positive and satisfies $R(x)\leq r_i(x)$ for all $x\in\mathbb{R}^n$ and all $i\in\{1,\ldots,m\}$, (6) and (10) give

$$\begin{aligned}\nabla V_1(x)f(x)\;&\leq\;-\mathcal{N}_1\quad\forall x\in\mathcal{X},\text{ and}\\ \nabla V_i(x)f(x)\;&=\;-\textstyle\sum_{l=1}^m\left[L_f^{i-1}h_l\right]^2\\ &\qquad-\textstyle\sum_{l=1}^m L_f^{i-2}h_lL_f^ih_l\\ &\leq\;-\mathcal{N}_i+\textstyle\sum_{l=1}^m|L_f^{i-2}h_l||L_f^ih_l|\end{aligned}\tag{32}$$

for $i=2,\ldots,N$ and all $x\in\mathcal{X}$. In particular, we have:

$$\begin{aligned}\nabla V_2(x)f(x)\;&\leq\;-\mathcal{N}_2(x)+\textstyle\sum_{l=1}^m\frac{|L_f^2h_l(x)|}{\sqrt{R(x)}}\sqrt{\mathcal{N}_1(x)};\\ \nabla V_i(x)f(x)\;&\leq\;-\mathcal{N}_i(x)+\left[\textstyle\sum_{l=1}^m|L_f^ih_l(x)|\right]\sqrt{\mathcal{N}_{i-1}(x)}\end{aligned}$$

for $i=3,4,\ldots,N$. Since $f(0)=0$, all of the functions $L_f^ih_l(x)$ are zero at the origin and sufficiently smooth for all $i\in\mathbb{N}$. Also, Assumption 1 provides a positive definite quadratic lower bound for V_1 near the origin. Moreover, the fact that V_1 is a storage function implies that there exists a function $\underline{\alpha}\in\mathcal{K}_\infty$ such that $V_1(x)\geq\underline{\alpha}(|x|)$ for all $x\in\mathcal{X}$.

Therefore, we can use (8) to determine a continuous everywhere positive function ϕ_1 such that for all $x\in\mathcal{X}$,

$$\begin{aligned}\textstyle\sum_{l=1}^m\frac{|L_f^2h_l(x)|}{\sqrt{R(x)}}\;&\leq\;\phi_1\big(V_1(x)\big)\sqrt{V_1(x)}\quad\text{and}\\ \textstyle\sum_{l=1}^m|L_f^ih_l(x)|\;&\leq\;\phi_1\big(V_1(x)\big)\sqrt{V_1(x)}\end{aligned}\tag{33}$$

for $i = 3, \ldots, N$; see [9, Chapter 5]. It follows that for all $i \geq 2$,

$$\begin{array}{rcl}\nabla V_i(x) f(x) & \leq & -\mathcal{N}_i(x) \\ & & +\phi_1\big(V_1(x)\big)\sqrt{\mathcal{N}_{i-1}(x)}\sqrt{V_1(x)}.\end{array} \quad (34)$$

Arguing as we did to get (33) gives a continuous everywhere nonnegative function p_1 such that $|V_i(x)| \leq p_1(V_1(x))V_1(x)$ for $i = 1, \ldots, N$ on $x \in \mathcal{X}$ (by first finding an increasing everywhere positive function $\tilde{\alpha}$ so that $|V_i(x)| \leq \tilde{\alpha}(|x|)|x|^2$ for all $i \geq 2$ and all x near 0, using the fact that $h_r(0) = 0$ for all r to get the estimate for the $i = 2$ case.) Finally, we can find a decreasing everywhere positive function $\underline{\rho}$ so that $R(x) \geq \underline{\rho}(\underline{\alpha}(|x|)) \geq \underline{\rho}(V_1(x))$ on $\mathcal{X}$, and so also a continuous everywhere positive function $\tilde{\rho}$ so that

$$\sum_{i=1}^N \mathcal{N}_i(x) \geq \tilde{\rho}\big(V_1(x)\big)V_1(x)$$

on $\mathcal{X}$, by (7). Hence, the assumptions of [11, Theorem 1] hold with $a_i \equiv \frac{1}{2}$, so [11, Theorem 1] constructs the necessary strict Lyapunov-like function.

Remark 2: We can prove an analog of Theorem 2 for (1), under a time varying version of Assumption 1. The time varying analog of Assumption 1 is obtained by (A) replacing the arguments of f and the V_i's by (t, x) and (B) replacing $\nabla V_i(x) f(x)$ with $(\partial V_i/\partial t)(t, x) + (\partial V_i/\partial x)(t, x) f(t, x)$. The proof then proceeds as before, using a simple extension of [11, Theorem 1] to time varying systems [11].

VII. Lotka-Volterra Example

We illustrate Theorem 2 using the celebrated Lotka-Volterra Predator-Prey system

$$\left\{\begin{array}{rcl}\dot{\chi} & = & \gamma\chi\left(1 - \frac{\chi}{L}\right) - a\chi\zeta, \\ \dot{\zeta} & = & \beta\chi\zeta - \Delta\zeta\end{array}\right. \quad (35)$$

with positive constants a, β γ, Δ, and L. System (35) is a simple model of one predator feeding on one prey. The population of the predator is ζ, χ is the population of the prey, and the constants are related to the birth and death rates; see [3], [7] for an extensive analysis of this model and generalizations to several predators. We assume that the population levels are positive. While there are many Lyapunov constructions for Lotka-Volterra models available (based on computing the LaSalle invariant set), to the best of our knowledge, the result to follow is original and significant because we provide a *global strict Lyapunov function*. We only sketch the construction; see [9] for details.

The change of coordinates and constants

$$\begin{array}{l} x(t) = \frac{1}{L}\chi\left(\frac{t}{\gamma}\right), \quad y(t) = \frac{a}{\beta L}\zeta\left(\frac{t}{\gamma}\right), \\ \alpha = \frac{\beta L}{\gamma} \text{ and } d = \frac{\Delta}{\gamma}\end{array} \quad (36)$$

produce the dynamics

$$\left\{\begin{array}{rcl}\dot{x} & = & x(1 - x) - \alpha x y, \\ \dot{y} & = & \alpha x y - d y.\end{array}\right. \quad (37)$$

For simplicity, we assume that the model parameters are such that $\alpha > d$. Then the constants

$$x_* = \frac{d}{\alpha} \text{ and } y_* = \frac{1}{\alpha} - \frac{d}{\alpha^2} \quad (38)$$

satisfy $x_* \in (0, 1)$ and $y_* > 0$. Moreover, the error variable $(\tilde{x}, \tilde{y}) = (x - x_*, y - y_*)$ has the dynamics

$$\left\{\begin{array}{rcl}\dot{\tilde{x}} & = & -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \dot{\tilde{y}} & = & \alpha\tilde{x}(\tilde{y} + y_*),\end{array}\right. \quad (39)$$

evolving on $\mathcal{X} = (-x_*, +\infty) \times (-y_*, +\infty)$. We do our Lyapunov construction for the system (39), which gives

$$f(\tilde{x}, \tilde{y}) = \left[\begin{array}{c} -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \alpha\tilde{x}(\tilde{y} + y_*)\end{array}\right]. \quad (40)$$

We verify the assumptions of Theorem 2 using $m = 1$, $N = 2$, $r_1 \equiv 1$, $h_1(\tilde{x}, \tilde{y}) = \tilde{x}$, and

$$V_1(\tilde{x}, \tilde{y}) = \tilde{x} - x_* \ln\left(1 + \frac{\tilde{x}}{x_*}\right) + \tilde{y} - y_* \ln\left(1 + \frac{\tilde{y}}{y_*}\right). \quad (41)$$

Simple calculations show that V_1 is a storage function whose time derivative along the trajectories of (39) satisfies

$$\begin{array}{rcl}\dot{V}_1 & = & \left[1 - \frac{x_*}{x_* + \tilde{x}}\right]\dot{\tilde{x}} + \left[1 - \frac{y_*}{y_* + \tilde{y}}\right]\dot{\tilde{y}} \\ & = & -\frac{\tilde{x}}{x_* + \tilde{x}}[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) + \frac{\alpha\tilde{y}}{y_* + \tilde{y}}\tilde{x}(\tilde{y} + y_*) \\ & = & -\tilde{x}[\tilde{x} + \alpha\tilde{y}] + \alpha\tilde{y}\tilde{x} \; = \; -\tilde{x}^2;\end{array} \quad (42)$$

and $L_f h_1(\tilde{x}, \tilde{y}) = -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*)$. Taking the $\mathcal{N}_i$'s as in (31), we get $\mathcal{N}_1(\tilde{x}, \tilde{y}) = \frac{1}{2}h_1^2(\tilde{x}, \tilde{y})$ and $\mathcal{N}_2(\tilde{x}, \tilde{y}) = (L_f h_1(\tilde{x}, \tilde{y}))^2$. One can therefore find a constant $\underline{d} > 0$ such that

$$\sum_{i=1}^2 \mathcal{N}_i(\tilde{x}, \tilde{y}) \geq \underline{d}\frac{V_1(\tilde{x}, \tilde{y})}{1 + V_1^2(\tilde{x}, \tilde{y})} \quad (43)$$

on $\mathcal{X}$ [9]. Also, V_1 has a positive definite quadratic lower bound near the origin [9, Lemma A.9]. Hence, Assumption 1 holds with $\rho(r) = \underline{d}/(1 + r^2)$, so Theorem 2 constructs the global strict Lyapunov function for (39).

Let us construct the strict Lyapunov function guaranteed by the theorem, using the notation from (10) and (31). We have $V_2(\tilde{x}, \tilde{y}) = -h_1(\tilde{x}, \tilde{y})L_f h_1(\tilde{x}, \tilde{y})$, $L_f V_1(\tilde{x}, \tilde{y}) \leq -\mathcal{N}_1(\tilde{x}, \tilde{y})$, and

$$\begin{array}{rcl}L_f V_2(\tilde{x}, \tilde{y}) & = & -(L_f h_1(\tilde{x}, \tilde{y}))^2 - h_1(\tilde{x}, \tilde{y})L_f^2 h_1(\tilde{x}, \tilde{y}) \\ & = & -\mathcal{N}_2(\tilde{x}, \tilde{y}) - h_1(\tilde{x}, \tilde{y})L_f^2 h_1(\tilde{x}, \tilde{y}).\end{array} \quad (44)$$

Substituting

$$\begin{array}{rcl}L_f^2 h_1(\tilde{x}, \tilde{y}) & = & -(\dot{\tilde{x}} + \alpha\dot{\tilde{y}})(\tilde{x} + x_*) - (\tilde{x} + \alpha\tilde{y})\dot{\tilde{x}} \\ & = & -(x_* + 2\tilde{x} + \alpha\tilde{y})\,\dot{\tilde{x}} - (x_* + \tilde{x})\,\alpha\dot{\tilde{y}} \\ & = & -(x_* + 2\tilde{x} + \alpha\tilde{y})\,L_f h_1(\tilde{x}, \tilde{y}) \\ & & -\alpha^2(x_* + \tilde{x})\,h_1(\tilde{x}, \tilde{y})(\tilde{y} + y_*)\end{array} \quad (45)$$

into (44), we get

$$\begin{array}{rcl}L_f V_2(\tilde{x}, \tilde{y}) & \leq & -\mathcal{N}_2(\tilde{x}, \tilde{y}) \\ & & +(x_* + 2|\tilde{x}| + \alpha|\tilde{y}|)\,|h_1(\tilde{x}, \tilde{y})||L_f h_1(\tilde{x}, \tilde{y})| \\ & & +\alpha^2(x_* + |\tilde{x}|)(|\tilde{y}| + y_*)h_1^2(\tilde{x}, \tilde{y}) \\ & \leq & -\mathcal{N}_2(\tilde{x}, \tilde{y}) \\ & & +(x_* + 2|\tilde{x}| + \alpha|\tilde{y}|)\,|h_1(\tilde{x}, \tilde{y})||L_f h_1(\tilde{x}, \tilde{y})| \\ & & +\alpha^2 x_* y_*\left(1 + \frac{|\tilde{x}|}{x_*}\right)\left(1 + \frac{|\tilde{y}|}{y_*}\right)h_1^2(\tilde{x}, \tilde{y}).\end{array}$$

A simple calculation gives

$$\begin{array}{l}\left(\frac{1}{x_*} + \frac{1}{y_*}\right)V_1(\tilde{x}, \tilde{y}) \;\geq \\ \frac{\tilde{x}}{x_*} - \ln\left(1 + \frac{\tilde{x}}{x_*}\right) + \frac{\tilde{y}}{y_*} - \ln\left(1 + \frac{\tilde{y}}{y_*}\right)\end{array} \quad (46)$$

so the inequality $1+A^2 \geq \frac{1}{2}(1+|A|)$ and [9, Lemma A.9] yield

$$\begin{array}{rcl} e^{\left(\frac{1}{x_*}+\frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})} & \geq & \left(\frac{e^{\frac{\tilde{x}}{x_*}}}{1+\frac{\tilde{x}}{x_*}}\right)\left(\frac{e^{\frac{\tilde{y}}{y_*}}}{1+\frac{\tilde{y}}{y_*}}\right) \\ & \geq & \frac{1}{36}\left(1+\frac{\tilde{x}^2}{x_*^2}\right)\left(1+\frac{\tilde{y}^2}{y_*^2}\right) \\ & \geq & \frac{1}{144}\left(1+\frac{|\tilde{x}|}{x_*}\right)\left(1+\frac{|\tilde{y}|}{y_*}\right). \end{array} \quad (47)$$

In particular, we get

$$\begin{array}{l} |\tilde{x}| \leq 144 x_* e^{\left(\frac{1}{x_*}+\frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})} \text{ and} \\ |\tilde{y}| \leq 144 y_* e^{\left(\frac{1}{x_*}+\frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})} \quad \forall(\tilde{x},\tilde{y}) \in \mathcal{X}. \end{array}$$

The function $\mathcal{M}(r) = (289x_* + 144\alpha y_*)\, e^{\left(\frac{1}{x_*}+\frac{1}{y_*}\right)r}$ therefore satisfies

$$\begin{array}{rcl} L_f V_2(\tilde{x},\tilde{y}) & \leq & -\mathcal{N}_2(\tilde{x},\tilde{y}) \\ & & +2\mathcal{M}\big(V_1(\tilde{x},\tilde{y})\big)\sqrt{\mathcal{N}_1(\tilde{x},\tilde{y})}\sqrt{\mathcal{N}_2(\tilde{x},\tilde{y})} \\ & & +288\alpha^2 x_* y_* e^{\left(\frac{1}{x_*}+\frac{1}{y_*}\right)V_1(\tilde{x},\tilde{y})}\mathcal{N}_1(\tilde{x},\tilde{y}). \end{array}$$

By the inequality $c_1 c_2 \leq \frac{1}{4}c_1^2 + c_2^2$ with $c_1 = \sqrt{\mathcal{N}_2}$,

$$\begin{array}{l} \mathcal{M}(V_1)\sqrt{\mathcal{N}_1}\sqrt{\mathcal{N}_2} \\ \leq \frac{1}{4}\mathcal{N}_2 + (289x_* + 144\alpha y_*)^2\, e^{2\left(\frac{1}{x_*}+\frac{1}{y_*}\right)V_1}\mathcal{N}_1 \end{array} \quad (48)$$

where we omit the dependencies on $(\tilde{x},\tilde{y})$, so also

$$L_f V_2(\tilde{x},\tilde{y}) \;\leq\; -\frac{1}{2}\mathcal{N}_2(\tilde{x},\tilde{y}) + \phi_1\big(V_1(\tilde{x},\tilde{y})\big)\mathcal{N}_1(\tilde{x},\tilde{y}), \quad (49)$$

where

$$\phi_1(r) = 2\left[(289x_* + 144\alpha y_*)^2 + 144\alpha^2 x_* y_*\right] e^{2\left(\frac{1}{x_*}+\frac{1}{y_*}\right)r}.$$

Since $V_2(\tilde{x},\tilde{y}) = \tilde{x}[\tilde{x}+\alpha\tilde{y}](\tilde{x}+x_*)$, we have

$$|V_2(\tilde{x},\tilde{y})| \leq 2(x_*+1)(1+\alpha)\left[\tilde{y}^4 + |\tilde{x}|^3 + \tilde{x}^2 + \tilde{y}^2\right]. \quad (50)$$

Applying [9, Lemma A.9] readily gives

$$\begin{array}{rcl} |\tilde{x}/x_*| & \leq & 2\left\{[V_1/x_*] + [V_1/x_*]^2\right\}^{1/2} \\ & \leq & 2\big[\max\{1/x_*, 1/x_*^2\}\{V_1 + V_1^2\}\big]^{1/2}, \end{array}$$

and similarly for y, where we omit the dependence of V_1 on $(\tilde{x},\tilde{y})$. In combination with (50), and with the choice $\bar{d} = 1 + x_* + y_*$, one easily checks that

$$\begin{array}{rcl} |V_2(\tilde{x},\tilde{y})| & \leq & 8(x_*+1)(1+\alpha)\sum_{i=2}^{4}\left\{2\bar{d}\sqrt{V_1+V_1^2}\right\}^i \\ & \leq & p_1(V_1(\tilde{x},\tilde{y}))V_1(\tilde{x},\tilde{y}), \end{array}$$

where $p_1(r) = 1536(x_*+1)(\alpha+1)\bar{d}^4(1+r)^3$.

The desired strict Lyapunov-like function is then

$$\begin{array}{rcl} S(\tilde{x},\tilde{y}) & = & V_2(\tilde{x},\tilde{y}) + \int_0^{V_1(\tilde{x},\tilde{y})}\phi_1(r)\,\mathrm{d}r \\ & & +\big[p_1\big(V_1(\tilde{x},\tilde{y})\big)+1\big]V_1(\tilde{x},\tilde{y}) \end{array} \quad (51)$$

This is because $S(\tilde{x},\tilde{y}) \geq V_1(\tilde{x},\tilde{y})$ and

$$L_f S(\tilde{x},\tilde{y}) \leq -\frac{1}{2}[\mathcal{N}_1(\tilde{x},\tilde{y}) + \mathcal{N}_2(\tilde{x},\tilde{y})]$$

hold everywhere. In fact, S is a strict Lyapunov function, using the fact that $V_1(\tilde{x},\tilde{y})$ goes to infinity when $\tilde{x}$ goes to $-x_*$ or $+\infty$, or when $\tilde{y}$ goes to $-y_*$ or $+\infty$.

VIII. CONCLUSIONS

The construction of global strict Lyapunov functions is a central problem in nonlinear control, owing to the value of strict Lyapunov functions in robustness analysis, feedback design, and other important situations. Even when a system is known to be GAS, it is often still important to have closed form expressions for strict Lyapunov functions, rather than the abstract strict Lyapunov function constructions provided by converse theory. We gave new methods for building global strict Lyapunov functions under LaSalle conditions. As a byproduct, we exhibited a general class of auxiliary functions for which the Matrosov theorem from [11] can be applied.

We illustrated our work using the celebrated Lotka-Volterra model, which plays a fundamental role in bio-engineering. Our global strict Lyapunov function for the Lotka-Volterra example allows us to quantify the effects of uncertainty in the birth and death rates, using ISS [9]. Due to space constraints, we omit this robustness analysis.

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