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Necessary and sufficient conditions for the observability of linear motion quantities in strapdown navigation systems

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Abstract— **Navigation Systems are a key element of a large variety of mobile platforms, whether manned or unmanned, autonomous or human-operated. This paper dwells on the observability of linear motion quantities (position, linear velocity, linear acceleration, and bias), in 3-D, of mobile platforms, and presents necessary and sufficient conditions, with physical insight, for the observability of these variables. The analysis provided is based on kinematic models, which are exact and intrinsic to the motion of a rigid-body, and different cases are presented depending on the assumptions made on the sensor suite installed on-board.**

I. Introduction

The design of Integrated Navigation Systems arises naturally in the development of a large variety of vehicles and other mobile platforms, whether manned or unmanned, autonomous or human-operated, since the knowledge of the position, attitude, and other quantities is a basic requirement for its successful operation. Dead-reckoning navigation systems such as Inertial Navigation Systems (INS) provide all these quantities. However, the estimation of the position and attitude of the vehicle is necessarily obtained in this type of systems by integrating higher-order derivatives such as the linear acceleration and the angular velocity, which are measured using, e.g., an Inertial Measurement Unit (IMU). As such, and regardless of the accuracy and precision of the IMU, the errors in the position and attitude estimates grow unbounded due to the noise and bias of the IMU sensors [1]. These intrinsic limitations of dead-reckoning navigation systems are usually tackled by using aiding sensors such as position and attitude sensors, e.g., the popular Global Positioning System (GPS), inclinometers, and magnetometers. However, even with the inclusion of aiding sensors, not all states are always observable, in particular, if biases are considered and the acceleration of gravity is not known with enough accuracy. This paper investigates the observability of linear motion quantities of mobile platforms.

Previous work on the study of observability of Navigation Systems can be found in the literature. In [2] the observability of INS during initial alignment and calibration at rest is analyzed. The nominal nonlinear navigation dynamics are perturbed yielding linearized error dynamics and it is then shown that the system is not completely observable. In [3] the observability of a linearized INS error model is also examined for a stationary vehicle and it is reported, among other results concerning the leveling errors, that the unobservable states, which are distributed in two decoupled subspaces, can be systematically determined. In-flight alignment of INS is studied in [4] where it is shown that its observability is improved by maneuvering. In [5] sufficient conditions for the observability of stationary Strapdown Inertial Navigation System (SDINS) are analytically derived. In [6] an observability analysis of a GPS/INS system during two types of maneuvers, linear acceleration and steady turn, is presented. The analysis is based on a perturbation model of the INS and it is shown that the observability improves when the vehicle maneuvers. Observability properties of the errors in an integrated navigation system are studied in [7], where it is shown that acceleration changes improve the estimates of attitude and rate-gyro bias and changes of the angular velocity enhance the lever arm estimate. However, no theoretical results for non-trivial trajectories are given and only simulation results are provided, which confirm that the degree of observability of the system increases with the richness of the trajectories described by the vehicle.

It is well known that the observability improves when the vehicle maneuvers. This papers details such maneuvers and provides necessary and sufficient conditions for the observability of linear motion quantities (position, linear velocity, and linear acceleration) of mobile platforms assuming exact angular measurements. Four different sensor suites are considered and definite results are provided for all of them. The analysis is based on kinematic models, which are exact and intrinsic to the motion of the vehicle, and builds on well established observability results for general linear time-varying (LTV) and time invariant (LTI) systems. All the results presented in the paper are related to the concept of complete observability. Nevertheless, extensions for differential observability or instantaneous observability (see [8] for details) conditions are trivially obtained from the results derived in the paper.

The paper is organized as follows. Section II introduces the classes of dynamic systems whose observability will be studied. The main results of the paper are presented in Section III, where physical interpretations are also offered. Finally, Section IV summarizes the main conclusions of the paper. Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate

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dimensions.

II. Linear Motion Kinematics

Let $\{I\}$ be an inertial coordinate frame and $\{B\}$ the body-fixed coordinate frame, whose origin coincides with the center of mass of the vehicle. Let ${}^{I}\mathbf{p}(t)$ denote the position of the origin of $\{B\}$, described in $\{I\}$, and $\mathbf{v}(t)$ the velocity of the vehicle relative to $\{I\}$, expressed in body-fixed coordinates. The linear motion kinematics of the vehicle are given by

$$\frac{d}{dt}{}^{I}\mathbf{p}(t) = \mathbf{R}(t)\mathbf{v}(t), \tag{1}$$

where $\mathbf{R}(t)$ is the rotation matrix from body-fixed to inertial coordinates, i.e., from $\{B\}$ to $\{I\}$, that satisfies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}(t)\right],$$

where $\boldsymbol{\omega}(t)$ is the angular velocity of the vehicle, expressed in body-fixed coordinates, and

$$\mathbf{S}\left[\boldsymbol{\omega}(t)\right] = \begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix}, \; \boldsymbol{\omega}(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}.$$

The position of the vehicle in inertial coordinates is often available, e.g., when there is a GPS receiver installed on-board. However, in underwater robotics, for instance, GPS is unavailable and alternative positioning sensors are required [9]. Acoustic positioning systems are common, e.g., long baseline (LBL) or ultra-short baseline (USBL) sensors. In the latter case the USBL (in the so-called inverse configuration) typically measures the position of an external fixed mark relative to the position of the vehicle, expressed in body-fixed coordinates, and thus the position of the vehicle is only available indirectly. Indeed, if $\mathbf{p}(t)$ denotes the measurement of the USBL as it was just described, it satisfies

$$\mathbf{p}(t) = \mathbf{R}^T(t)\left[{}^{I}\mathbf{p}_m(t) - {}^{I}\mathbf{p}(t)\right],$$

where ${}^{I}\mathbf{p}_m(t)$ denotes the position of the mark relative to $\{I\}$, expressed in inertial coordinates. In this framework, the kinematics of the vehicle can be described, indirectly, by

$$\dot{\mathbf{p}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t) - \mathbf{v}(t). \tag{2}$$

A essential element of Navigation Systems is the IMU, which usually contains a triad of orthogonal accelerometers and rate-gyros. Assuming that the IMU is mounted in the center of mass of the vehicle and aligned with the body-fixed coordinate frame $\{B\}$, the rate-gyros provide the angular velocity of the vehicle, $\boldsymbol{\omega}(t)$, and the accelerometers measure an acceleration quantity $\mathbf{a}(t)$ given by

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) + \mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) - \mathbf{g}(t) + \mathbf{b}(t), \tag{3}$$

where $\mathbf{g}(t)$ denotes the acceleration of gravity and $\mathbf{b}(t)$ the bias of the accelerometer, both expressed in body-fixed coordinates. Ideal accelerometers would not measure the gravitational term but in practice this term must be considered due to the inherent physics of the accelerometers, see [10] for further details. The term $\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t)$ corresponds to the Coriolis acceleration of the vehicle and must also be considered.

In the remainder of this section four different systems will be introduced to describe the linear motion of the vehicle and its relation with the various sensors. The differences between the proposed dynamics depend on the sensor suite considered. As it was seen, both (1) and (2) describe the evolution of the position of the vehicle given the sensors installed on-board. In what concerns observability properties they are equivalent assuming exact angular measurements. Throughout the paper (2) is preferred due to its particular structure without loss of generality.

A. Navigation with calibrated accelerometer

In the first case considered in the paper it is assumed that the vehicle is equipped with a positioning sensor and a calibrated accelerometer, aside from the triad of rate-gyros or an Attitude and Heading Reference System (AHRS), that provides the angular velocity of the vehicle. The derivative of the linear position is given by (2), whereas the derivative of the velocity may be obtained from (3). The acceleration of gravity is locally constant in inertial coordinates. Thus, the derivative of this quantity when expressed in body-fixed coordinates is given by

$$\dot{\mathbf{g}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t).$$

The system dynamics can then be written as

$$\begin{cases} \dot{\mathbf{p}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t) - \mathbf{v}(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) + \mathbf{g}(t) + \mathbf{a}(t) \\ \dot{\mathbf{g}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t) \\ \mathbf{y}_1(t) = \mathbf{p}(t) \end{cases}, \tag{4}$$

where $\mathbf{a}(t)$ is here considered as a deterministic input and $\mathbf{y}_1(t)$ denotes the system output, available for the estimation of the system state.

B. Dynamic Accelerometer Bias Estimation

The previous system dynamics were derived assuming that the accelerometer was calibrated. This section introduces a class of systems suitable for the estimation of the bias of an accelerometer assuming exact angular and velocity measurements, in body-fixed coordinates. This is particularly interesting, for example, if one has available a calibration table which permits the generation of high-resolution trajectories with known velocities. This system reads as

$$\begin{cases} \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) + \mathbf{g}(t) - \mathbf{b}(t) + \mathbf{a}(t) \\ \dot{\mathbf{g}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t) \\ \dot{\mathbf{b}}(t) = \mathbf{0} \\ \mathbf{y}_2(t) = \mathbf{v}(t) \end{cases}, \tag{5}$$

where $\mathbf{a}(t)$ is again assumed to be a deterministic input, which is in fact available from the triad of accelerometers, and the output of the system is the velocity of the origin of the body-fixed coordinate frame.

C. Navigation with known gravity

In Section II-A it was assumed that the accelerometer was calibrated and the gravity unknown. In this section the accelerometer measurements are assumed corrupted by an unknown bias but the gravity is supposed to be known. This is not a very practical situation as, even if the magnitude of the gravity is known with great accuracy, any misalignment in the estimation of the gravity acceleration vector in body-fixed coordinates results in severe problems in the acceleration compensation. Nevertheless, it presents an interesting theoretical problem and provides insight to the more general setup, which will be presented in the next subsection. The system dynamics that reflect these assumptions are given by

$$\left\{\begin{array}{l}\dot{\mathbf{p}}(t)=-\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t)-\mathbf{v}(t)\\ \dot{\mathbf{v}}(t)=-\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t)-\mathbf{b}(t)+\mathbf{a}(t)+\mathbf{g}(t)\\ \dot{\mathbf{b}}(t)=\mathbf{0}\\ \mathbf{y}_3(t)=\mathbf{p}(t)\end{array}\right., \tag{6}$$

where $\mathbf{a}(t)$ and $\mathbf{g}(t)$ are assumed to be deterministic inputs and the system output is the position of the vehicle.

D. Navigation with dynamic accelerometer bias determination in the presence of unknown gravity

The most general setup regarding the estimation of linear motion quantities of mobile platforms is presented in this section. Both the gravity and the bias of the accelerometer are supposed unknown and the system dynamics read as

$$\left\{\begin{array}{l}\dot{\mathbf{p}}(t)=-\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t)-\mathbf{v}(t)\\ \dot{\mathbf{v}}(t)=-\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t)+\mathbf{g}(t)-\mathbf{b}(t)+\mathbf{a}(t)\\ \dot{\mathbf{g}}(t)=-\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t)\\ \dot{\mathbf{b}}(t)=\mathbf{0}\\ \mathbf{y}_4(t)=\mathbf{p}(t)\end{array}\right.. \tag{7}$$

III. Main Results

A. Navigation with calibrated accelerometer

This section examines the observability of the dynamic system (4), which has been derived in the past by the authors to propose a navigation filter with a calibrated accelerometer. The result provided in this section is not new, see [11], but it is presented here in preparation for the results that will follow.

Theorem 1: The dynamic system (4) is observable.

Proof: In compact form, the dynamic system (4) can be rewritten as

$$\left\{\begin{array}{l}\dot{\mathbf{x}}_1(t)=\mathbf{A}_1(t)\mathbf{x}_1(t)+\mathbf{B}_1\mathbf{u}_1(t)\\ \mathbf{y}_1(t)=\mathbf{C}_1\mathbf{x}_1(t)\end{array}\right.,$$

where $\mathbf{x}_1(t)=\left[\mathbf{p}^T(t)\,\mathbf{v}(^Tt)\,\mathbf{g}^T(t)\right]^T\in\mathbb{R}^9$ is the vector of system states, $\mathbf{u}_1(t)=\mathbf{a}(t)\in\mathbb{R}^3$ is the input of the system,

$$\mathbf{A}_1(t)=\begin{bmatrix}-\mathbf{S}\left[\boldsymbol{\omega}(t)\right] & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{S}\left[\boldsymbol{\omega}(t)\right] & \mathbf{I}\\ \mathbf{0} & \mathbf{0} & -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\end{bmatrix}, \mathbf{B}_1=\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\\ \mathbf{0}\end{bmatrix},$$

and $\mathbf{C}_1=[\mathbf{I}\;\mathbf{0}\;\mathbf{0}]$. Consider the Lyapunov transformation

$$\mathbf{x}_{\bar{1}}(t):=\mathbf{T}_1(t)\mathbf{x}_1(t),$$

with

$$\mathbf{T}_1(t):=\operatorname{diag}\left(\mathbf{R}(\mathrm{t}),\,\mathbf{R}(\mathrm{t}),\,\mathbf{R}(\mathrm{t})\right)$$

and define an equivalent output $\mathbf{y}_{\bar{1}}(t):=\mathbf{R}(t)\mathbf{y}_1(t)$. Then, the new system dynamics can be written as

$$\left\{\begin{array}{l}\dot{\mathbf{x}}_{\bar{1}}(t)=\mathbf{A}_{\bar{1}}\mathbf{x}_{\bar{1}}(t)+\mathbf{B}_{\bar{1}}(t)\mathbf{u}_1(t)\\ \mathbf{y}_{\bar{1}}(t)=\mathbf{C}_{\bar{1}}\mathbf{x}_{\bar{1}}(t)\end{array}\right.,$$

where

$$\mathbf{A}_{\bar{1}}=\begin{bmatrix}\mathbf{0} & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}, \mathbf{B}_{\bar{1}}(t)=\begin{bmatrix}\mathbf{0}\\ \mathbf{R}(t)\\ \mathbf{0}\end{bmatrix},$$

and $\mathbf{C}_{\bar{1}}=[\mathbf{I}\;\mathbf{0}\;\mathbf{0}]$. Since the pair $(\mathbf{A}_{\bar{1}},\,\mathbf{C}_{\bar{1}})$ is observable, it follows that (4) is also observable as both systems are related through a Lyapunov transformation, with equivalent outputs [12]. ∎

B. Dynamic Accelerometer Bias Estimation

This section presents observability conditions for dynamic accelerometer bias estimation. Before going into the details, some simpler but very useful and inspiring properties regarding the observability of the system are presented and discussed.

In compact form, the system dynamics (5) can be written as

$$\left\{\begin{array}{l}\dot{\mathbf{x}}_2(t)=\mathbf{A}_2(t)\mathbf{x}_2(t)+\mathbf{B}_2\mathbf{u}_2(t)\\ \mathbf{y}_2(t)=\mathbf{C}_2\mathbf{x}_2(t)\end{array}\right., \tag{8}$$

where $\mathbf{x}_2(t)=\left[\mathbf{v}^T(t)\,\mathbf{g}^T(t)\,\mathbf{b}^T(t)\right]^T\in\mathbb{R}^9$ is the vector of states of the system, $\mathbf{u}_2(t)=\mathbf{a}(t)\in\mathbb{R}^3$ is the input of the system,

$$\mathbf{A}_2(t)=\begin{bmatrix}-\mathbf{S}\left[\boldsymbol{\omega}(t)\right] & \mathbf{I} & -\mathbf{I}\\ \mathbf{0} & -\mathbf{S}\left[\boldsymbol{\omega}(t)\right] & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}, \mathbf{B}_2=\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix},$$

and $\mathbf{C}_2=[\mathbf{I}\;\mathbf{0}\;\mathbf{0}]$. Within this framework, suppose that the angular velocity $\boldsymbol{\omega}$ is constant. In this situation, the dynamic system (8) is LTI and therefore its observability can be directly assessed from the analysis of the rank of the observability matrix $\mathcal{O}_2$ associated to the pair $(\mathbf{A}_2,\mathbf{C}_2)$. It is a simple matter of computation to show that

- for constant $\boldsymbol{\omega}=\mathbf{0}$, $\operatorname{rank}\left[\mathcal{O}_2\right]=6$ and
- for constant $\boldsymbol{\omega}\neq\mathbf{0}$, $\operatorname{rank}\left[\mathcal{O}_2\right]=8$.

From this first result it is already possible to say that the system (8) is not observable for, at least, some trajectories of $\boldsymbol{\omega}$, and this is not a surprise. Indeed, for $\boldsymbol{\omega}=\mathbf{0}$, both the gravity and the bias are constant in body-fixed coordinates (and inertial coordinates too) and it is impossible to distinguish between them based on the velocity measurements. However, in this situation, it is straightforward to show that it would be possible to design an observer for both $\mathbf{v}$ and the quantity $\mathbf{g}-\mathbf{b}$. When $\boldsymbol{\omega}$ is constant but nonzero, the degree of observability of the system increases. In this situation it is also straightforward to show that it is still possible to estimate both $\mathbf{v}$ and $\mathbf{g}-\mathbf{b}$. This fact is important and will be exploited shortly as it suggests that $\mathbf{g}-\mathbf{b}$ is an observable mode regardless of the trajectory described by the angular velocity. Moreover, the non-observable subspace is related to the direction of the angular velocity, which suggests that, if the axis of rotation changes, the system becomes observable. The following proposition is required before presenting the main result of this section, which confirms this suspicion.

Proposition 1: The dynamic system (8) is observable on $[t_0,\ t_f]$ if and only if

$$\begin{aligned}\mathcal{S}_{\mathcal{W}_{\bar{2}_\mathbf{A}}}(t_f,t_0) \;:=\;& \mathbf{R}^{[2,2]}(t_f,t_0)\\ &-\frac{4}{T}\left[\mathbf{R}^{[2]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[2]}(t_f,t_0)\right]\\ &+\frac{6}{T^2}\left[\mathbf{R}^{[2]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[3]}(t_f,t_0)\right]\\ &+\frac{6}{T^2}\left[\mathbf{R}^{[3]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[2]}(t_f,t_0)\right]\\ &-\frac{12}{T^3}\left[\mathbf{R}^{[3]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[3]}(t_f,t_0)\right] \quad (9)\end{aligned}$$

is positive definite, where $T := t_f - t_0$,

$$\mathbf{R}^{[1]}(t,t_0) := \int_{t_0}^{t}\mathbf{R}(\sigma)\,d\sigma,$$

$$\mathbf{R}^{[2]}(t,t_0) := \int_{t_0}^{t}\int_{t_0}^{\sigma_1}\mathbf{R}(\sigma_2)\,d\sigma_2 d\sigma_1,$$

$$\mathbf{R}^{[3]}(t,t_0) := \int_{t_0}^{t}\int_{t_0}^{\sigma_1}\int_{t_0}^{\sigma_2}\mathbf{R}(\sigma_3)\,d\sigma_3 d\sigma_2 d\sigma_1,$$

and

$$\mathbf{R}^{[2,2]}(t,t_0) := \int_{t_0}^{t}\left[\mathbf{R}^{[1]}(\sigma,t_0)\right]^T\left[\mathbf{R}^{[1]}(\sigma,t_0)\right]d\sigma.$$

Proof: To examine the observability of the dynamic system (8) it is convenient to compute the observability Gramian $\mathcal{W}_2(t_0,\ t_f)$, given by

$$\mathcal{W}_2(t_0,\ t_f) := \int_{t_0}^{t_f}\boldsymbol{\phi}_2^T(t,\ t_0)\,\mathbf{C}_2^T\mathbf{C}_2\boldsymbol{\phi}_2(t,\ t_0)\,dt,$$

where $\boldsymbol{\phi}_2(t,\ t_0)$ denotes the transition matrix associated to $\mathbf{A}_2(t)$. The transition matrix and the observability Gramian are trivially obtained for linear time invariant systems. However, when that is not the case, the task usually becomes much more intricate, depending on the complexity of the system at hand. The present system is linear time-varying and the transition matrix is not trivially obtained. However, it is possible to tackle the problem resorting to an appropriate Lyapunov coordinate transformation, known to preserve the observability properties of linear systems.

In Section III-A the observability of the system was assessed trough the use of an orthogonal Lyapunov transformation which rendered the system dynamics LTI. Although the application of this trick to (8) does not render the dynamics LTI, it is still useful as it reduces the number of time varying elements of the new dynamics. Coupled with this, it has been shown that both $\mathbf{v}$ and $\mathbf{g}-\mathbf{b}$ are observable for a constant angular velocity. This suggests that no restrictions for full observability should arise on these two quantities. These two ideas motivate the coordinate change

$$\mathbf{x}_{\bar{2}}(t) := \mathbf{T}_2(t)\mathbf{x}_2(t), \quad (10)$$

with

$$\mathbf{T}_2(t) := \begin{bmatrix}\mathbf{R}(t) & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}(t) & -\mathbf{R}(t)\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix}.$$

Notice that (10) is a Lyapunov coordinate transformation as

- $\mathbf{T}_2(t)$ is continuously differentiable for all t;
- Both $\mathbf{T}_2(t)$ and $\dot{\mathbf{T}}_2(t)$ are bounded for all t, where

$$\dot{\mathbf{T}}_2(t) = \begin{bmatrix}\mathbf{R}(t)\mathbf{S}(t) & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}(t)\mathbf{S}(t) & -\mathbf{R}(t)\mathbf{S}(t)\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix};$$

- $\det[\mathbf{T}_2(t)] = 1$.

The fact that (10) is a Lyapunov transformation establishes the equivalence of observability properties between $\mathbf{x}_2$ and $\mathbf{x}_{\bar{2}}$.

The dynamics of $\mathbf{x}_{\bar{2}}$ are given by

$$\begin{cases}\dot{\mathbf{x}}_{\bar{2}}(t) = \mathbf{A}_{\bar{2}}(t)\mathbf{x}_{\bar{2}}(t) + \mathbf{B}_{\bar{2}}(t)\mathbf{u}_2(t)\\ \mathbf{y}_2(t) = \mathbf{C}_{\bar{2}}(t)\mathbf{x}_{\bar{2}}(t)\end{cases},$$

where

$$\mathbf{A}_{\bar{2}}(t) = \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{R}(t)\mathbf{S}(t)\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\ \mathbf{B}_{\bar{2}}(t) = \begin{bmatrix}\mathbf{R}(t)\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix},$$

and $\mathbf{C}_{\bar{2}}(t) = \left[\mathbf{R}^T(t)\ \mathbf{0}\ \mathbf{0}\right]$. The observability Gramian associated with the pair $(\mathbf{A}_{\bar{2}}(t),\ \mathbf{C}_{\bar{2}}(t))$ can be written as

$$\mathcal{W}_{\bar{2}}(t_f,\ t_0) = \begin{bmatrix}(t_f-t_0)\mathbf{I} & \frac{(t_f-t_0)^2}{2}\mathbf{I} & \mathcal{W}_{\bar{2}}^{(1,3)}(t_f,\ t_0)\\ * & \frac{(t_f-t_0)^3}{3}\mathbf{I} & \mathcal{W}_{\bar{2}}^{(2,3)}(t_f,\ t_0)\\ * & * & \mathcal{W}_{\bar{2}}^{(3,3)}(t_f,\ t_0)\end{bmatrix},$$

where

$$\mathcal{W}_{\bar{2}}^{(1,3)}(t_f,\ t_0) = \frac{(t_f-t_0)^2}{2}\mathbf{R}(t_0) - \mathbf{R}^{[2]}(t_f,t_0),$$

$$\begin{aligned}\mathcal{W}_{\bar{2}}^{(2,3)}(t_f,\ t_0) =& \frac{(t_f-t_0)^3}{3}\mathbf{R}(t_0) - (t_f-t_0)\mathbf{R}^{[2]}(t_f,t_0)\\ &+\mathbf{R}^{[3]}(t_f,t_0),\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}_{\bar{2}}^{(3,3)}(t_f,\ t_0) =& \tfrac{(t_f-t_0)^3}{3}\mathbf{I} + \mathbf{R}^{[2,2]}(t_f,t_0)\\ &-(t_f-t_0)\left(\mathbf{R}(t_0)^T\mathbf{R}^{[2]}(t_f,t_0) + \left[\mathbf{R}(t_0)^T\mathbf{R}^{[2]}(t_f,t_0)\right]^T\right)\\ &+\mathbf{R}(t_0)^T\mathbf{R}^{[3]}(t_f,t_0) + \left[\mathbf{R}(t_0)^T\mathbf{R}^{[3]}(t_f,t_0)\right]^T.\end{aligned}$$

It is a simple matter of computation to show that $\mathcal{W}_{\bar{2}}(t_f,\ t_0)$ is invertible if and only if (9) is positive definite since it corresponds to the Schur complement of the observability Gramian $\mathcal{W}_{\bar{2}}(t_f,\ t_0)$. ■

The following theorem is the main result of this section.

Theorem 2: The dynamic system (8) is observable on $[t_0,\ t_f]$ if and only if

$$\underset{t_0 \le t_i \le t_f}{\exists}\ \underset{\alpha \in \mathbb{R}}{\forall} : \boldsymbol{\omega}(t_i) \neq \mathbf{0} \wedge \dot{\boldsymbol{\omega}}(t_i) \neq \alpha\boldsymbol{\omega}(t_i). \quad (11)$$

Proof:

a) Necessity: Suppose that (8) is observable and (11) is not satisfied. Then, either

$$\underset{t\in[t_0,\,t_f]}{\forall}\ :\ \boldsymbol{\omega}(t)=\mathbf{0}$$

or

$$\underset{t\in[t_0,\,t_f]}{\forall}\ \underset{\alpha(t)\in\mathbb{R}}{\exists}\ :\ \dot{\boldsymbol{\omega}}(t)=\alpha(t)\boldsymbol{\omega}(t) \qquad (12)$$

are satisfied (or both). If the angular velocity is null for all t, it has been shown that the system is not observable. If $\boldsymbol{\omega}(t_0)\neq\mathbf{0}$, then it follows, from (12), that either

$$\boldsymbol{\omega}(t)=\boldsymbol{\omega}(t_0),\ \forall t\in[t_0,\ t_f],$$

i.e., the angular velocity remains constant on $[t_0,\ t_f]$, or

$$\boldsymbol{\omega}(t)=e^{\int_{t_0}^{t}\alpha(\tau)d\tau}\boldsymbol{\omega}(t_0). \qquad (13)$$

It has already been proved that if the angular velocity remains constant the system is not observable. Thus, to prove that (11) is a necessary condition, it remains to demonstrate that if (13) is satisfied the system is not observable. Suppose then that (13) is satisfied. That is equivalent to say that the body-fixed coordinate frame rotates around a fixed axis of rotation, possibly with time-varying angular velocity. Thus, it is possible to write $\mathbf{R}(t)$ as

$$\mathbf{R}(t)=\mathbf{R}(t_0)\,\mathbf{R}_u(t) \qquad (14)$$

where $\mathbf{R}_u(t)$ is a rotation about the angular velocity vector, whose axis does not change. Next, it is shown that for $\mathbf{R}(t)$ given by (14), (9) is not positive definite and thus the system is not observable. Multiplying (9) on the left and on the right by $\boldsymbol{\omega}^T(t_0)$ and $\boldsymbol{\omega}(t_0)$, respectively, gives

$$\begin{aligned}\boldsymbol{\omega}^T(t_0)\,\boldsymbol{\mathcal{S}}_{\boldsymbol{\mathcal{W}}_{\bar{2}_\mathbf{A}}}(t_f,t_0)\,\boldsymbol{\omega}(t_0)&=\boldsymbol{\omega}^T(t_0)\,\mathbf{R}^{[2,2]}\boldsymbol{\omega}(t_0)\\ &-\tfrac{4}{T}\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[2]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[2]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)\\ &+\tfrac{6}{T^2}\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[2]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[3]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)\\ &+\tfrac{6}{T^2}\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[3]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[2]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)\\ &-\tfrac{12}{T^3}\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[3]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[3]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0). \end{aligned} \qquad (15)$$

Expanding, for instance, the first term of (15), gives

$$\begin{aligned}&\boldsymbol{\omega}^T(t_0)\,\mathbf{R}^{[2,2]}\boldsymbol{\omega}(t_0)=\\ &=\int_{t_0}^{t_f}\left[\int_{t_0}^{\sigma}\boldsymbol{\omega}^T(t_0)\,\mathbf{R}^T(\sigma_1)\,d\sigma_1\right]\left[\int_{t_0}^{\sigma}\mathbf{R}(\sigma_2)\,\boldsymbol{\omega}(t_0)\,d\sigma_2\right]d\sigma.\end{aligned} \qquad (16)$$

Now, note that

$$\mathbf{R}(t)\,\boldsymbol{\omega}(t_0)=\mathbf{R}(t_0)\,\boldsymbol{\omega}(t_0),\ \forall t\in[t_0,\ t_f] \qquad (17)$$

since $\mathbf{R}_u(t)$ is a rotation about the angular velocity vector, whose axis remains constant for all time. Substituting (17) in (16) it follows that

$$\boldsymbol{\omega}^T(t_0)\,\mathbf{R}^{[2,2]}\boldsymbol{\omega}(t_0)=\frac{T^3}{3}\left\|\boldsymbol{\omega}(t_0)\right\|^2. \qquad (18)$$

Similar procedures yield

$$\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[2]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[2]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)=\frac{T^4}{4}\left\|\boldsymbol{\omega}(t_0)\right\|^2, \qquad (19)$$

$$\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[2]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[3]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)=\frac{T^5}{12}\left\|\boldsymbol{\omega}(t_0)\right\|^2, \qquad (20)$$

$$\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[3]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[2]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)=\frac{T^5}{12}\left\|\boldsymbol{\omega}(t_0)\right\|^2, \qquad (21)$$

and

$$\boldsymbol{\omega}^T(t_0)\left[\mathbf{R}^{[3]}(t_f,t_0)\right]^T\left[\mathbf{R}^{[3]}(t_f,t_0)\right]\boldsymbol{\omega}(t_0)=\frac{T^6}{36}\left\|\boldsymbol{\omega}(t_0)\right\|^2. \qquad (22)$$

Substituting (18)-(22) in (15) gives

$$\boldsymbol{\omega}^T(t_0)\,\boldsymbol{\mathcal{S}}_{\boldsymbol{\mathcal{W}}_{\bar{2}_\mathbf{A}}}(t_f,t_0)\,\boldsymbol{\omega}(t_0)=0.$$

Thus, the Schur complement (9) is not positive definite, and, from Proposition 1, it follows that the system is not observable. This completes the proof of necessity.

b) Sufficiency: Suppose that (11) is satisfied, i.e., there exists $t_i\in[t_0,\ t_f]$ such that

$$\boldsymbol{\omega}(t_i)\neq\mathbf{0}\wedge\left(\underset{\alpha\in\mathbb{R}}{\forall}\ \dot{\boldsymbol{\omega}}(t_i)\neq\alpha\boldsymbol{\omega}(t_i)\right). \qquad (23)$$

Consider the matrix defined as

$$\boldsymbol{\mathcal{L}}_{\bar{2}}(t)=\left[\boldsymbol{\mathcal{L}}_{\bar{2}0}^T(t)\ \ldots\ \boldsymbol{\mathcal{L}}_{\bar{2}q}^T(t)\right]^T,$$

where

$$\begin{cases}\boldsymbol{\mathcal{L}}_{\bar{2}0}(t)=\mathbf{C}_{\bar{2}}(t)\\ \boldsymbol{\mathcal{L}}_{\bar{2}i}(t)=\boldsymbol{\mathcal{L}}_{\bar{2}(i-1)}(t)\mathbf{A}_{\bar{2}}(t)+\dot{\boldsymbol{\mathcal{L}}}_{\bar{2}(i-1)}(t),\ i=1,2,\ldots,q\end{cases},$$

and such that $\mathbf{C}_{\bar{2}}(t)$ is q times continuously differentiable and $\mathbf{A}_{\bar{2}}(t)$ is $q-1$ times continuously differentiable. Then, if there exists $t_a\in[t_0,\ t_f]$ such that

$$\operatorname{rank}\left[\boldsymbol{\mathcal{L}}_{\bar{2}}(\mathrm{t_a})\right]=9,$$

the system is observable on $[t_0,\ t_f]$, see [13]. Let $q=3$. Then, straightforward computations yield

$$\boldsymbol{\mathcal{L}}_{\bar{2}}(t_i)=\begin{bmatrix}\mathbf{R}^T(t_i)&\mathbf{0}&\mathbf{0}\\ -\mathbf{S}(t_i)\,\mathbf{R}^T(t_i)&\mathbf{R}^T(t_i)&\mathbf{0}\\ *&*&-\mathbf{S}(t_i)\\ *&*&2\mathbf{S}^2(t_i)-\dot{\mathbf{S}}(t_i)\end{bmatrix}.$$

Assuming (23), it follows

$$\operatorname{rank}\left[\boldsymbol{\mathcal{L}}_{\bar{2}}(\mathrm{t_i})\right]=9,$$

which concludes the proof. ∎

The technical condition stated in Theorem 2 is equivalent to say that the body-fixed frame must rotate and the axis of rotation must change so that observability is attained. This behavior is usually known as coning and confirms the suspicions previously stated.

C. Navigation with known gravity

This section examines the observability of the dynamic system (6). Before presenting the main result, notice that, for constant angular velocity the system is always observable. Thus, one can expect the system to be observable for all $\boldsymbol{\omega}$. The following theorem establishes this property.

Theorem 3: The dynamic system (6) is observable.

Proof: In compact form, the dynamic system (6) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}_3(t) = \mathbf{A}_3(t)\mathbf{x}_3(t) + \mathbf{B}_3\mathbf{u}_3(t) \\ \mathbf{y}_3(t) = \mathbf{C}_3\mathbf{x}_3(t) \end{cases}, \tag{24}$$

where $\mathbf{x}_3(t) = \left[\mathbf{p}^T(t)\,\mathbf{v}^T(t)\,\mathbf{b}^T(t)\right]^T \in \mathbb{R}^9$ is the vector of states of the system, $\mathbf{u}_3(t) = \left[\mathbf{a}^T(t)\,\mathbf{g}^T(t)\right]^T \in \mathbb{R}^6$ is the input of the system,

$$\mathbf{A}_3(t) = \begin{bmatrix} -\mathbf{S}\left[\boldsymbol{\omega}(t)\right] & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}\left[\boldsymbol{\omega}(t)\right] & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathbf{B}_3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_3 = [\mathbf{I}\ \mathbf{0}\ \mathbf{0}]$. Consider the Lyapunov transformation

$$\mathbf{x}_{\overline{3}}(t) := \mathbf{T}_3(t)\mathbf{x}_3(t),$$

with

$$\mathbf{T}_3(t) := \operatorname{diag}\left(\mathbf{R}(\mathrm{t}),\, \mathbf{R}(\mathrm{t}),\, \mathbf{I}\right).$$

Then, the new system dynamics can be written as

$$\begin{cases} \dot{\mathbf{x}}_{\overline{3}}(t) = \mathbf{A}_{\overline{3}}(t)\mathbf{x}_{\overline{3}}(t) + \mathbf{B}_{\overline{3}}(t)\mathbf{u}_3(t) \\ \mathbf{y}_{\overline{3}}(t) = \mathbf{C}_{\overline{3}}(t)\mathbf{x}_{\overline{3}}(t) \end{cases},$$

where

$$\mathbf{A}_{\overline{3}}(t) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathbf{B}_{\overline{3}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{R}(t) & \mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_{\overline{3}}(t) = \left[\mathbf{R}^T(t)\ \mathbf{0}\ \mathbf{0}\right]$. Consider the matrix defined as

$$\boldsymbol{\mathcal{L}}_{\overline{3}}(t) = \left[\boldsymbol{\mathcal{L}}_{\overline{3}0}(t)\ \ldots\ \boldsymbol{\mathcal{L}}_{\overline{3}q}(t)\right],$$

where

$$\begin{cases} \boldsymbol{\mathcal{L}}_{\overline{3}0}(t) = \mathbf{C}_{\overline{3}}(t) \\ \boldsymbol{\mathcal{L}}_{\overline{3}i}(t) = \boldsymbol{\mathcal{L}}_{\overline{3}(i-1)}(t)\mathbf{A}_{\overline{3}}(t) + \dot{\boldsymbol{\mathcal{L}}}_{\overline{3}(i-1)}(t),\ i = 1, 2, \ldots, q \end{cases},$$

and such that $\mathbf{C}_{\overline{3}}(t)$ is q times continuously differentiable and $\mathbf{A}_{\overline{3}}(t)$ is $q-1$ times continuously differentiable. Let $q = 2$. Then, straightforward computations yield

$$\boldsymbol{\mathcal{L}}_{\overline{3}}(t) = \begin{bmatrix} \mathbf{R}^T(t) & \mathbf{0} & \mathbf{0} \\ * & -\mathbf{R}^T(t) & \mathbf{0} \\ * & * & \mathbf{I} \end{bmatrix},$$

which has rank 9. Thus, the system (24) is observable [13]. ■

D. Navigation with dynamic accelerometer bias determination in the presence of unknown gravity

This section presents the last result of the paper, which assesses the observability of a navigation system with dynamic accelerometer bias estimation. This result is closely related to the one presented in Section III-B, since the nominal dynamics for navigation with dynamic accelerometer bias determination can be regarded as an extension of the dynamics for dynamic accelerometer bias estimation.

Theorem 4: The dynamic system (7) is observable on $[t_0,\ t_f]$ if and only if

$$\underset{t_0 \le t_i \le t_f}{\exists} \quad \underset{\alpha \in \mathbb{R}}{\forall} \quad : \boldsymbol{\omega}\left(t_i\right) \neq \mathbf{0} \wedge \dot{\boldsymbol{\omega}}\left(t_i\right) \neq \alpha\boldsymbol{\omega}\left(t_i\right).$$

Proof: The proof follows the same steps as the proof of Theorem 2 and therefore it is omited. ■

IV. CONCLUSIONS

This paper provided observability results regarding systems related to the estimation of linear motion quantities of mobile platforms (position, linear velocity, and linear acceleration), in 3-D, assuming exact angular measurements. Four different cases were studied: i) in the first a simple calibrated sensor suite consisting of an IMU and a positioning sensor was considered and it was shown that the system is always observable, even without the knowledge of the acceleration of gravity; ii) in the second case the problem of dynamic accelerometer bias estimation was studied and it was shown that not all trajectories yield observability of the system state. In particular, it was shown that the trajectories should be rich enough in what concerns the evolution of the attitude of the body-fixed frame, namely, the body-fixed coordinate frame should describe trajectories with coning; iii) the third situation described in the paper considers a triad of accelerometers with unknown biases but the gravity is assumed to be known. It was shown that, in this situation, the state of the system is always observable; iv) the last case addressed the most general setup where the triad of accelerometers may have an unknown bias and the gravity is also supposed to be unknown. It was shown that the system is observable if and only if the attitude evolution is sufficiently rich, in the same sense as the one presented for dynamic accelerometer bias estimation.

REFERENCES

[1] M. Bryson and S. Sukkarieh, "Vehicle Model Aided Inertial Navigation for a UAV using Low-Cost Sensors," in *Proc. Australasian Conference on Robotics and Automation*, Canberra, Australia, Dec. 2004.

[2] I. Bar-Itzhack and N. Berman, "Control Theoretic Approach to Inertial Navigation Systems," *Journal of Guidance, Control, and Dynamics*, vol. 11, no. 3, pp. 237–245, May 1988.

[3] Y. Jiang and Y. Lin, "Error Estimation of INS Ground Alignment Through Observability Analysis," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 28, no. 1, pp. 92–97, Jan. 1992.

[4] D. Goshen-Meskin and I. Bar-Itzhack, "Observability Analysis of Piece-Wise Constant Systems - Part II: Application to Inertial Navigation In-Flight Alignment," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 28, no. 4, pp. 1068–1075, Oct. 1992.

[5] D. Chung, C. Park, and J. Lee, "Observability Analysis of Strapdown Inertial Navigation System Using Lyapunov Transformation," in *Proc. 35th IEEE Conference on Decision and Control*, Kobe, Japan, Dec. 1996, pp. 23–28.

[6] I. Rhee, M. Abdel-Hafez, and J. Speyer, "Observability of an Integrated GPS/INS During Maneuvers," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 40, no. 2, pp. 526–535, Apr. 2004.

[7] S. Hong, M. Lee, H.-H. Chun, S.-H. Kwon, and J. Speyer, "Observability of Error States in GPS/INS Integration," *IEEE Trans. on Vehicular Technology*, vol. 54, no. 2, pp. 731–743, Mar. 2005.

[8] C. Chen, *Linear System Theory and Design*. Holt Reinhart and Winston, Inc., 1984.

[9] J. Kinsey, R. Eustice, and L. Whitcomb, "A Survey of Underwater Vehicle Navigation: Recent Advances and New Challenges," in *Proc. MCMC2006 - 7th IFAC Conference on Manoeuvring and Control of Marine Vehicles*, Lisboa, Portugal, Sep. 2006.

[10] A. Kelly, *Modern Inertial and Satellite Navigation Systems*. Carnegie Mellon University, 1994.

[11] P. Batista, C. Silvestre, and P. Oliveira, "Position and Velocity Navigation Filters for Marine Vehicles," in *Proc. 17th IFAC World Congress*, Seoul, Korea, Jul. 2008.

[12] R. Brockett, *Finite Dimensional Linear Systems*. Wiley, 1970.

[13] W. Rugh, *Linear system theory (2nd ed.)*. Prentice-Hall, Inc., 1996.