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Robust $\mathcal{D}$ Stabilization of Singular Systems with Polytopic Uncertainties

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Abstract— **The robust $\mathcal{D}$ stabilization problem is considered for singular systems with polytopic uncertainties in this paper. Both the derivative matrix E and the state matrix A are with uncertainties, under the assumption that the rank of matrix E is constant. Firstly, with the introduction of some free matrices, a new dilated LMI condition for the singular system to be $\mathcal{D}$ stable is proposed, based on which, the robust $\mathcal{D}$ stable problem is solved, and a sufficient condition for the closed system to be robust $\mathcal{D}$ stabilizable is obtained. The desired state feedback controller is given in an explicit expression. Numerical examples show the efficiency of the obtained approach.**

I. INTRODUCTION

Singular system model is a natural mathematical representation for many practical systems. It provides a description of the dynamic as well as the algebraic relationships between the chosen descriptor variables simultaneously [1]. Due to its direct and general description, singular system has been employed in different areas, e.g., circuit systems, power systems, aerospace engineering and chemical processing [2]. Many basic theories developed for state space models have been generalized to its counterparts for singular systems, for example, controllability and observability [2], H_∞ control [6] etc.

As to the problem of controller design for singular systems, there are usually two ways: one is the regularization problem, about which proportional plus derivative controller is used to make the closed systems nonsingular and stable, see [3], [4] and the references therein. Another is the stabilization problem, and a pure proportional controller is designed such that the closed systems are regular, impulse-free(causal) and stable, see [5], [6], [7], [8], [9]. However, when the problem of $\mathcal{D}$ stabilization is considered, which encompass the stabilization as a special case, few results is available in the literature. $\mathcal{D}$ stabilization is important in the control theory, not only because it can deal the continuous system and the discrete-time system in a unit framework, but also because it can guarantee the performance of the closed system by placing the poles of the system in a certain region $\mathcal{D}$. A mostly studied region is the LMI region which is proposed in [10] and generalized in [11]. For the singular system, the robust $\mathcal{D}$ stability analysis was considered in [14], in which only the state matrix A is with uncertainties.

In this paper, the robust $\mathcal{D}$ stabilization problem is considered. The systems we considered have uncertainties with both the derivative matrix E and state matrix A. Firstly, with the introduction of some free matrices[12], [13], a new necessary and sufficient condition for the singular system to be $\mathcal{D}$ stable is proposed based on the result in [14], and then the result is extended to the system with polytopic uncertainties. Finally, the robust $\mathcal{D}$ stabilization problem is solved and a sufficient condition for the controller design is obtained. The controller is given in an explicit form. Some numerical examples show the applicability of the obtained approach.

Notation: Throughout this paper, R^n denotes the n dimensional real Euclidean space, C denotes the complex plane, I_k is the $k \times k$ identity matrix, the superscripts ' T ' and ' -1 ' stand for the matrix transpose and inverse respectively, $\bar{z}$ denotes the conjugate of z, '*' denotes the symmetric element in a matrix, C^- denotes the the left-hand side of complex plane and $D_{int}(0,1)$ denotes the unitary disk centered at the origin. $W > 0$ ($W \geq 0$) means that W is real, symmetric and positive definite (positive semidefinite), $\otimes$ denotes the Kronecker product, $\delta[\cdot]$ denotes the differential operator for continuous systems (i.e. $\delta[x(t)] = \dot{x}(t)$) and the shift operator for discrete-time systems (i.e. $\delta[x(t)] = x(t+1)$), $\lambda(E,A)$ denotes the set of finite eigenvalues of the (E,A) pair, i.e. $\lambda(E,A) = \{s|\det(sE-A)=0\}$, $Sym(\cdot)$ denotes the matrix plus its transpose, i.e. $Sym(A) = A+A^T$. If not explicitly stated, the matrices are assumed to have compatible dimensions.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the singular system without uncertainties

$$E\delta[x(t)] = Ax(t) \tag{1}$$

The following definitions and lemma are essential for the development of our main results

Definition 1 (Dai [2]):

1. The system (1) is said to be regular if $\det(sE-A)$ is not identically zero.

2. The system (1) is said to be impulse-free(causal) if $\deg(\det(sE-A)) = rank(E)$.

3. The system (1) is said to be stable if $\lambda(E,A) \subset C^-$ for continuous singular systems or $\lambda(E,A) \subset D_{int}(0,1)$ for discrete-time singular systems.

If the singular system is regular, then there exist two nonsingular matrices M_1 and N_1, such that

$$\hat{E} = M_1EN_1 = \begin{bmatrix} I_r & 0 \\ 0 & J \end{bmatrix},\ \hat{A} = M_1AN_1 = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

The pair $(\hat{E},\hat{A})$ is called the *Weierstrass form* of (E,A). J is a nilpotent matrix and r is the number of finite eigenvalues

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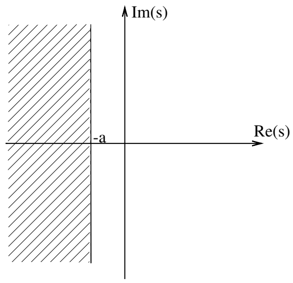

Fig. 1: $\mathcal{D}$ region for continuous time system

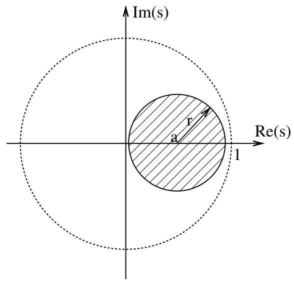

Fig. 2: $\mathcal{D}$ region for discrete time system

of (E, A). It is obvious that the system is impulse-free if and only if $J = 0$.

Definition 2 (Kuo [14]):

The system (1) is called $\mathcal{D}$ stable if it is regular, impulse-free (causal), and $\lambda(E, A) \in \mathcal{D}$.

In this paper, the region we considered is proposed in [11]:

$$\mathcal{D} = \left\{ z \in C : R_1 + R_2 z + R_2^T \bar{z} + R_3 z\bar{z} < 0 \right\}$$

where $R_1 = R_1^T \in R^{d\times d}$ and $0 \leq R_3 = R_3^T \in R^{d\times d}$, for simplicity, written as

$$R = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$$

and d is called the order of the region.

Two typical regions are shown in Fig.1 and Fig.2, which could be formulated by the following choices of R, respectively.

$$R_C = \begin{bmatrix} 2a & 1 \\ 1 & 0 \end{bmatrix},\ R_D = \begin{bmatrix} a^2 - r^2 & -a \\ -a & 1 \end{bmatrix}$$

In Fig.1, when $a = 0$, it becomes the left-hand side of complex plane C^-. In Fig.2, when $a = 0$ and $r = 1$, it becomes the unitary disk centered at the origin $D_{int}(0, 1)$.

A necessary and sufficient condition for the $\mathcal{D}$ stability of the singular systems was proposed in [14], and here we use the dual form to obtain our main result.

Lemma 1: The system (1) is $\mathcal{D}$ stable if and only if there exist matrix $P > 0$ and symmetric matrix Q satisfying

$$M(P, Q, E, A) < 0 \tag{2a}$$

$$EQE^T \geq 0 \tag{2b}$$

with

$$M(P, Q, E, A) = R_1 \otimes (EPE^T) + R_2 \otimes (EPA^T) + R_2^T \otimes (APE^T) + R_3 \otimes (APA^T) + I_d \otimes (AQA^T)$$

III. Main results

Consider the singular system with polytopic uncertainties

$$E(\alpha)\delta[x(t)] = A(\alpha)x(t) + B(\alpha)u(t) \tag{3}$$

matrices $E(\alpha)$, $A(\alpha)$, $B(\alpha)$ are in the following convex sets

$$\mathcal{E} = \left\{ E(\alpha) : E(\alpha) = \sum_{i=1}^{N} \alpha_i E_i; \right\} \tag{4a}$$

$$\mathcal{A} = \left\{ A(\alpha) : A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i; \right\} \tag{4b}$$

$$\mathcal{B} = \left\{ B(\alpha) : B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i; \right\} \tag{4c}$$

$$with \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0, i = 1, 2, \cdots, N$$

where N is the number of the vertices and $rank(E(\alpha)) = rank(E_i), i = 1, \cdots, N$.

Firstly, a new necessary and sufficient condition with some free matrices is given for the system (1) to be $\mathcal{D}$ stable, which plays a key role in this paper.

Theorem 1: The system (1) is $\mathcal{D}$ stable if and only if there exist matrices $P > 0$, Q, U_1, U_2, S_1, S_2, G, F, W and V such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{bmatrix} < 0 \tag{5a}$$

$$\Gamma = \begin{bmatrix} EW + W^T E^T & -W^T + EV \\ * & -V - V^T + Q \end{bmatrix} \geq 0 \tag{5b}$$

with

$\Xi_{11} = U_1^T(I_d \otimes A^T) + (I_d \otimes A)U_1 + U_2^T(I_d \otimes E^T) + (I_d \otimes E)U_2$

$\Xi_{12} = (I_d \otimes E)F - (I_d \otimes A)S_2 - U_2^T$

$\Xi_{13} = (I_d \otimes A)G - (I_d \otimes E)S_1 - U_1^T$

$\Xi_{22} = R_1 \otimes P - F - F^T$

$\Xi_{23} = R_2 \otimes P + S_1 + S_2^T$

$\Xi_{33} = R_3 \otimes P + I_d \otimes Q - G - G^T$

From theorem 1, it is obvious that

Corollary 1: The continuous singular system is $\mathcal{D}$ stable if and only if (5b) holds and there exist matrices $P > 0$, Q, U_1, U_2, S_1, S_2, G and F such that

$$\begin{bmatrix} \Psi_1 & EF - AS_2 - U_2^T & AG - ES_1 - U_1^T \\ * & -F - F^T & P + S_1 + S_2^T \\ * & * & Q - G - G^T \end{bmatrix} < 0$$

with $\Psi_1 = AU_1 + U_1^T A^T + EU_2 + U_2^T E^T$

Corollary 2: The discrete-time singular system is $\mathcal{D}$ stable if and only if (5b) holds and there exist matrices $P > 0$, Q, U_1, U_2, S_1, S_2, G and F such that

$$\begin{bmatrix} \Psi_2 & EF - AS_2 - U_2^T & AG - ES_1 - U_1^T \\ * & -P - F - F^T & S_1 + S_2^T \\ * & * & P + Q - G - G^T \end{bmatrix} < 0$$

with $\Psi_2 = AU_1 + U_1^T A^T + EU_2 + U_2^T E^T$

Now consider the uncertain system (3). The following theorem gives a sufficient condition for the uncertain system to be $\mathcal{D}$ stable.

Theorem 2: The unforced system (3) is robust $\mathcal{D}$ stable if there exist matrices $P(\alpha) > 0$, $Q(\alpha)$, U_1, U_2, S_1, S_2, G, F, W and V such that

$$\Xi(\alpha) = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{bmatrix} < 0 \tag{6a}$$

$$\Gamma(\alpha) = \begin{bmatrix} E(\alpha)W + W^T E^T(\alpha) & -W^T + E(\alpha)V \\ * & -V - V^T + Q(\alpha) \end{bmatrix} \geq 0 \tag{6b}$$

with

$\Xi_{11} = U_1^T(I_d \otimes A^T(\alpha)) + (I_d \otimes A(\alpha))U_1 + U_2^T(I_d \otimes E^T(\alpha)) + (I_d \otimes E(\alpha))U_2$
$\Xi_{12} = (I_d \otimes E(\alpha))F - (I_d \otimes A(\alpha))S_2 - U_2^T$
$\Xi_{13} = (I_d \otimes A(\alpha))G - (I_d \otimes E(\alpha))S_1 - U_1^T$
$\Xi_{22} = R_1 \otimes P(\alpha) - F - F^T$
$\Xi_{23} = R_2 \otimes P(\alpha) + S_1 + S_2^T$
$\Xi_{33} = R_3 \otimes P(\alpha) + I_d \otimes Q(\alpha) - G - G^T$

The result is obvious from theorem 1. As is known to all, the result above is untractable. Considering the convexity of the LMI region, the following theorem gives a sufficient but tractable condition for the $\mathcal{D}$ stability test.

Theorem 3: The unforced system (3) is robust $\mathcal{D}$ stable if there exist matrices $P_i > 0$, Q_i, $i = 1, \ldots, N$ and U_1, U_2, S_1, S_2, G, F, W, V such that

$$\Pi_i = \begin{bmatrix} \Pi_{i11} & \Pi_{i12} & \Pi_{i13} \\ * & \Pi_{i22} & \Pi_{i23} \\ * & * & \Pi_{i33} \end{bmatrix} < 0 \tag{7a}$$

$$\Gamma_i = \begin{bmatrix} E_i W + W^T E_i^T & -W^T + E_i V \\ * & -V - V^T + Q_i \end{bmatrix} \geq 0 \tag{7b}$$

$$i = 1, 2, \ldots, N$$

with

$\Pi_{i11} = U_1^T(I_d \otimes A_i^T) + (I_d \otimes A_i)U_1 + U_2^T(I_d \otimes E_i^T) + (I_d \otimes E_i)U_2$
$\Pi_{i12} = (I_d \otimes E_i)F - (I_d \otimes A_i)S_2 - U_2^T$
$\Pi_{i13} = (I_d \otimes A_i)G - (I_d \otimes E_i)S_1 - U_1^T$
$\Pi_{i22} = R_1 \otimes P_i - F - F^T$
$\Pi_{i23} = R_2 \otimes P_i + S_1 + S_2^T$
$\Pi_{i33} = R_3 \otimes P_i + I_d \otimes Q_i - G - G^T$

Remark 1: The result in theorem 3 is a non-strict LMIs for (7b). Since $rank(E) = rank(E_i), i = 1, \ldots, N$, we can always find matrices $Q_1 > 0$, $Y < 0$ satisfying $Q = Q_1 + UYU^T$, where $U \in R^{n\times(n-r)}$ is of full column rank and satisfies $E(\alpha)U = 0$. Then the non-strict LMI (7b) is satisfied naturally and the condition becomes a strict LMI condition.

The following theorem gives an approach to design a state feedback controller to guarantee the closed system to be $\mathcal{D}$ stable.

Theorem 4: The system (3) is robust $\mathcal{D}$ stabilizable by state feedback if (7b) holds and there exist matrices $P_i > 0$, Q_i, $i = 1, \ldots, N$ and matrices F, $U2$, $S1$, H, V such that

$$\Phi_i = \begin{bmatrix} \Phi_{i11} & \Phi_{i12} & \Phi_{i13} \\ * & \Phi_{i22} & \Phi_{i23} \\ * & * & \Phi_{i33} \end{bmatrix} < 0 \quad i = 1.2. \ldots, N \tag{8}$$

where

$\Phi_{i11} = Sym\,(\alpha_1 I_d \otimes (A_i H + B_i V) + (I_d \otimes E_i)U_2)$
$\Phi_{i12} = -\alpha_2 I_d \otimes (A_i H + B_i V) + (I_d \otimes E_i)F - U_2^T$
$\Phi_{i13} = \alpha_3 I_d \otimes (A_i H + B_i V) - (I_d \otimes E_i)S_1 - \alpha_1 I_d \otimes H^T$
$\Phi_{i22} = R_1 \otimes P_i - F - F^T$
$\Phi_{i23} = R_2 \otimes P_i + S_1 + \alpha_2 I_d \otimes H^T$
$\Phi_{i33} = R_3 \otimes P_i + I_d \otimes Q_i - \alpha_3 I_d \otimes (H + H^T)$

and α_1, α_2, α_3 are tuning parameters and a desired controller is given by

$$K = VH^{-1} \tag{9}$$

Remark 2: The tuning parameters $\beta_i, i = 1, 2, 3$ should be determined before our simulation. In this paper, they are chosen by the searching method. Another way is using the optimal function, such as `fminsearch`, see [16], [17] for details.

IV. Numerical examples

In this section, some examples are presented to demonstrate the applicability of the proposed approach. The system model is as follows

$$E(\alpha)\delta[x(t)] = A(\beta)x(t) + B(\gamma)u(t) \tag{10}$$

Example 1. Consider the continuous singular system with the following parameters, which both E and A are with uncertainties

$$E1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A1 = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A2 = \begin{bmatrix} 2 & 3 & 2 \\ 8 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix}, B = B1 = B2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Obviously, this singular system is irregular. Our objective is to design a controller such that the resultant closed system is regular, casual, and all the finite eigenvalues lying in the left of the line `x=-0.5`, i.e., $R1 = 1$, $R2 = 0$, $R3 = 1$.

To reformulate it in the form of (3), we define the vertices of the system as $(E1, A1, B)$, $(E1, A2, B)$, $(E2, A1, B)$, $(E2, A2, B)$. Using the algorithm in theorem 4, and choosing the tuning parameters as $\alpha_1 = 2$, $\alpha_2 = -2$, $\alpha_3 = 1$, a feasible solution is given as

$$H = \begin{bmatrix} 54.4832 & -13.8122 & 7.7140 \\ -7.1472 & 10.0240 & 8.0223 \\ -81.3291 & 0.4959 & -0.5077 \end{bmatrix},$$
$$V = \begin{bmatrix} 27.3944 & -22.4215 & -87.8714 \end{bmatrix}$$

and the corresponding state feedback control law is given as

$$u(t) = \begin{bmatrix} -3.8346 & -7.4089 & -2.2546 \end{bmatrix} x(t)$$

Using the gridding method, the finite eigenvalues of the closed singular system is shown as in Fig.3, which shows the effectiveness of the proposed algorithm.

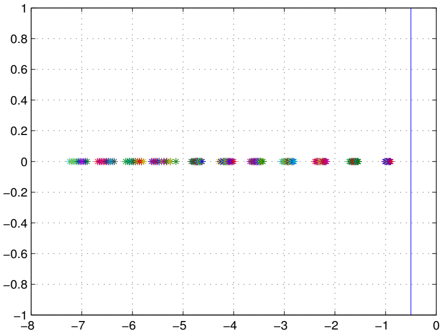

Fig. 3: Roots of the system in example 1

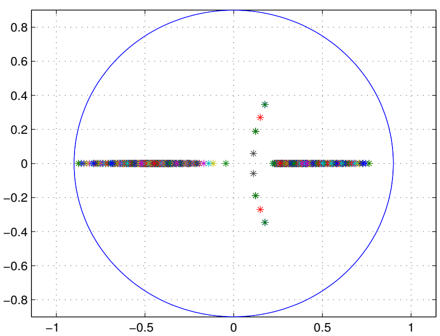

Fig. 4: Roots of the system in example 2

Example 2. Consider the uncertain discrete-time singular system with the following parameters

$$E = E1 = E2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$
$$A2 = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 1 & 5 \\ 0 & 1 & 0 \end{bmatrix}, B1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, B2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The region considered is shown as Fig.2 with $a = 0$ and $r = 0.9$. To reformulate it in accordance with (3), choose the vertices as $(E, A1, B1)$, $(E, A2, B1)$, $(E, A1, B2)$, $(E, A2, B1)$. Using the algorithm in theorem 4, and choosing the tuning parameters as $\alpha_1 = 2$, $\alpha_2 = 1.2$, $\alpha_3 = 1$, a feasible solution is given as

$$H = \begin{bmatrix} 71.2846 & -36.3543 & -2.1594 \\ -18.4597 & 23.1537 & 2.7100 \\ -60.2565 & 28.0543 & 2.1979 \end{bmatrix},$$
$$V = \begin{bmatrix} 10.6018 & -15.7355 & -6.1984 \end{bmatrix}$$

and the corresponding state feedback controller is given by

$$u(t) = \begin{bmatrix} -4.8681 & -1.8002 & -5.3835 \end{bmatrix} x(t).$$

The finite eigenvalues of the closed singular system are shown as in Fig.4, which shows that all the finite eigenvalues are in the region $D(0, 0.9)$.

V. Conclusion

In this paper, a sufficient condition for the robust $\mathcal{D}$ stabilization of the singular system is proposed using the free matrices technique. Both the derivative matrix E and state matrix A are with polytopic uncertainties, and an algorithm for the controller design is given in terms of LMI. Numerical examples show the efficiency of the proposed approach.

VI. Acknowledgement

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