2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Stability and Convergence of Perturbed Switched Linear Time-Delay Systems

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***Abstract*— We address the issue of stability and convergence of perturbed switched linear time-delay systems. By introducing the Variation-of-constants formula, the conditions of the stability and convergence of perturbed switched linear systems with time-delay are established, and the difficulties caused by the interaction between the switchings and time-delay are conquered. Based on the general result of perturbed switched linear time-delay systems, under two different switching schemes, new delay dependent and independent stability criteria for switched linear systems with time-delay are developed. The numerical examples show feasibility and validity of the results.**

Notations

$\mathbb{R}^n$	n dimensional Euclidean space.
A^T	Transpose of matrix A.
$\underline{N}$	Set $\{1, 2, \cdots n\}$.
$\|\cdot\|$	The usual 2-norm.
$L_2[0,\infty)$	The space of square integrable functionson $[0,\infty)$.
$\mathscr{L}_1^{\text{loc}}([\varrho,\infty),\mathbb{R}^n)$	The space of locally Lebesgue integrable functions on $[\varrho,\infty)$.
$\lambda_{\max}(P)\,(\lambda_{\min})$	Maximum (minimum) eigenvalue of P.
$P>0\,(<0)$	Positive (negative) definite matrix P.
$x_t(\theta)$	$x(t+\theta), \theta\in[-\tau,0]$.
$C([-\tau,0],\mathbb{R}_n)$	Banach space of continuous mapping from$([-\tau,0],\mathbb{R}_n)$ to $\mathbb{R}^n$ with topology of uniform convergence.
$\sigma(t):[0,\infty)\rightarrow\underline{N}$	The right continuous function denotes the switching signal which can be characterized by the switching sequence $\Sigma=\{x_0;(i_0,t_0),(i_1,t_1),\cdots,$ $(i_j,t_j),\cdots\vert i_j\in\underline{N}, j=0,1,\cdots\}$.
$\sigma\triangleq\sigma(t)=i$	Mean that the ith subsystem is active.
$\|x_t\|_{cl}$	$\sup_{-\tau\le\theta\le0}\{\|x(t+\theta)\|,\|\dot{x}(t+\theta)\|\}$ or $\sup_{-\tau\le\theta\le0}\{\|x(t+\theta)\|\}$.

This work was supported in part by Dogus University Fund for Science and Ministry of Education and Science of the Republic of Macedonia, NSF of China under Grants 90816028 and 60874024, and supported by Program for New Century Excellent Talents in University (NCET-06-0207).

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I. Introduction

For a perturbed switched linear system

$$\dot{x}(t) = A_{\sigma}(t)x(t) + f_{\sigma}(t),\ x(t_0) = x_0, \tag{1}$$

where $f_{\sigma(t)}(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is piecewise continuous vector function representing system perturbations, paper [7] had derived that under certain conditions, the state of system (1) has the same convergence properties as the perturbations $f_i(t),\ i \in \underline{N}$, i.e.,

(i) the state of the perturbed system (1) is bounded if the perturbations are bounded;
(ii) the state of the perturbed system (1) is practically convergent if the perturbations are convergent.

On the other hand, with the Variation-of-constants formula [2], for $(\varrho, \phi) \in \mathbb{R} \times C$, one can easily show that under certain conditions, the state convergence properties of the linear perturbed time-delay system

$$\begin{aligned}\dot{x}(t) &= L(t, x_t) + f(t),\ t \ge \varrho,\\ x_{\varrho} &= \phi,\end{aligned} \tag{2}$$

are consistent with the perturbation $f(t)$.

Inspired by above works, it is natural to consider the stability and convergence of perturbed switched linear systems with time-delay. The importance of the study of the issue arises from the hot topics, switched systems with time-delay which have been attracting considerable attention due to the significance both in theory development and practical applications (see e.g., [4],[5],[6],[9],[10],[12]). It is well known that perturbations and uncertainties are inevitably encountered in many control systems, including switched time-delay systems and time-delay systems using switching techniques. However, the stability and convergence of the perturbed switched systems with time-delay is not easy to handle and challenging due to the interactions between the switchings and time-delay, a simple combination of the existing results of switched systems and time-delay systems can not give the solution of the problem addressed. Some efforts are undertaken to dispose of the perturbations for switched systems (see, e.g., [13]) or time-delay systems (see, e.g., [11]), it is often assumed that the perturbations to satisfy the linear growth condition, i.e., $\|f_i(t)\| \le \eta_i \|x(t)\|,\ i \in \underline{N}$, generally speaking, such disposal is neither practical nor reasonable.

In this paper, we address the issue of stability and convergence of perturbed switched linear time-delay systems. The novelties of this paper are as follows. In the first place, by introducing the Variation-of-constants formula, the conditions of the stability and convergence of perturbed switched linear systems with time-delay are established. In the next place, under average dwell time switching scheme, delay dependent stability and convergence criteria for perturbed switched linear time-delay systems are developed based on the general results, alternatively, under multiple Lyapunov functional switching scheme, delay independent stability and convergence criteria are given. As for deriving the delay dependent criteria, by introducing Jensen integral inequality and choosing a special Lyapunov functional which does not include the time varying term, the computation complexities and analysis difficulties are decreased, whereas the conservativeness is not increased compare to existing results (see, e.g., [3]). On the other hand, as for delay independent criteria, some subsystems are allowed to be unstable. It is highly desirable that a non-switched time-delay system can not earn such property.

II. Preliminaries

Before developing the conditions for the stability and convergence of switched linear time-delay systems, a preliminary result is presented. The following state constitutes a generalization of Hale's results (see in [2]).

Consider the perturbed linear time-delay system

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + f(t),\ t \geq \varrho, \\ x_\varrho &= \phi, \end{aligned} \tag{3}$$

and its homogeneous system

$$\dot{x}_t = L(t, x_t),\ t \geq \varrho. \tag{4}$$

Hypothesis 1. The operator $L(t, \phi)$ in (3) is linear in ϕ and, there are $n \times n$ matrix functions $\eta(t, \theta)$ measurable in $(t, \theta) \in \mathbb{R} \times \mathbb{R}$, normalized so that

$$\eta(t, \theta) = 0 \quad \text{for } \theta > 0, \quad \eta(t, \theta) = \eta(t, -\tau) \quad \text{for } \theta < -\tau,$$

$\eta(t, \theta)$ is continuous from the left in θ on $(-\tau, 0)$, $\eta(t, \theta)$ has bounded variation in θ on $[-\tau, 0]$ for each t, and there are $m(t) \in \mathscr{L}_1^{\text{loc}}((-\infty, \infty), \mathbb{R})$ such that

$$\begin{aligned} L(t, \phi) &= \int_{-\tau}^{0} [d_\theta \eta(t, \theta)] \phi(\theta), \\ |L(t, \phi)| &\leq m(t) |\phi| \end{aligned} \tag{5}$$

for all $t \in (-\infty, \infty),\ \phi \in C$.

Remark 1. According to Hale's Theory, if above hypotheses on L are satisfied, a unique function $x(\varrho, \phi)$ defined and continuous on $[\varrho - \tau, \infty)$ which satisfies system (3) on $[\varrho, \infty)$ can be guaranteed (c.f. [2]). Hypothesis 1 is basic assumption for linear time-delay system and is easily checked to be held.

Example 1. Consider a LTI time-delay system

$$\dot{x}(t) = Ax(t) + Dx(t - \tau).$$

We can let $L(t, \phi) = A\phi(0) + D\phi(-\tau)$, in which $\phi(\theta) = x(t + \theta), \theta \in (-\tau, 0)$, and there has

$$L(t, \phi) = \int_{-\tau}^{0} [d_\theta \eta(t, \theta)] \phi(\theta)$$

where

$$\eta(t, \theta) = \begin{cases} 0, & \theta = 0, \\ -A, & -\tau < \theta < 0, \\ -A - D, & \theta = -\tau. \end{cases}$$

Lemma 1 (*Variation-of-constants* [2]). If L satisfies the hypotheses 1, $x(\varrho, \phi, f)$ denotes the solution of system (3), and $x(\varrho, \phi, 0)$ is the solution of the corresponding homogeneous system (4), then

$$\begin{aligned} x(\varrho, \phi, f)(t) &= x(\varrho, \phi, 0)(t) + \int_{\varrho}^{t} U(t, s) f(s) ds, \\ x_\varrho &= \phi, \quad t \geq \varrho, \end{aligned} \tag{6}$$

where $U(t, s)$ is the solution of the equation

$$U(t, s) = \begin{cases} \int_s^t L(u, U_u(\cdot, s)) du + I \ a.e. \text{ in } s, t \geq s \\ 0. \qquad s - r \leq t < s \end{cases}$$

in which $U_t(\cdot, s)(\theta) = U(t + \theta, s), -\tau \leq \theta \leq 0$.

For the convenience of using the variation-of-constants formula, some notations are introduced to rewrite the formula.

Denote $x(\varrho, \phi, 0)(t + \theta)$ as $x_t(\varrho, \phi, 0)$, and if $x(\varrho, \phi, 0) \triangleq T(t, \varrho)\phi$, then the $T(t, \varrho)$ is a continuous linear operator. Therefore

$$U_t(\cdot, s) = T(t, s) X_0, \quad X_0(\theta) = \begin{cases} 0, -\tau \leq \theta < 0, \\ I, \theta = 0. \end{cases}$$

With the above notations, the variation-of-constants formula becomes

$$x_t(t, \varrho, \phi, f) = T(t, \varrho)\phi + \int_{\varrho}^{t} T(t, s) X_0 f(s) ds,\ t \geq \varrho.$$

Before concluding this section, we recall another lemma.

Lemma 2 (*Jensen integral inequality* [1]). For any constant matrix $M \in \mathbb{R}^{m \times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^m$ such that the integrations in the following are well defined, then

$$\gamma \int_0^\gamma \omega^T(\beta) M \omega(\beta) d\beta \geq \Big(\int_0^\gamma \omega(\beta) d\beta \Big)^T M \Big(\int_0^\gamma \omega(\beta) d\beta \Big).$$

III. Main results

For $(\varrho, \phi_\sigma) \in \mathbb{R} \times C$, consider the switched linear system with time-delay

$$\begin{cases} \dot{x}(t) = L_\sigma(t, x_t) + f_\sigma(t),\ t \geq t_0, \\ \phi_i(\theta) = x(t + \theta),\ \theta \in [t_j - \tau, t_j], \\ x(0) = \phi_0(0) = 0,\ j = 0, 1, \cdots. \end{cases} \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the state, $f_i(t) \in \mathscr{L}_1^{\text{loc}}([t_0, \infty), \mathbb{R}^n)$, $i \in \underline{N}$ the perturbations; $\phi_i(t)$ the continuous vector valued function specifying the initial state of each subsystem.

Remark 2. If $L_i(t, x_t)$ in each subsystem of (7) satisfies the Hypothesis 1, with the stepping method in finite interval $[t_j, t_{j+1})$ for each subsystem of (7) and well-defined switching law, the existence and uniqueness of the solution with initial condition for switched linear time-delay system (7) can be guaranteed.

Definition 1[8]. Let $M = \sup_{t \geq \varrho,\, k \in \underline{N}} \{\|f_i(t)\|\}$. The perturbations $f_i(t)$, $i \in \underline{N}$ are said to be

(i) bounded, if $M < \infty$;
(ii) convergent (to the origin), if $\|f_i(t)\| \to 0$ as $t \to \infty$ for all $i \in \underline{N}$;
(iii) exponentially convergent (to the origin), if $\|f_i(t)\| \leq \kappa e^{-\lambda(t-t_0)}$, $\forall t \geq t_0$, for some constants $\kappa \geq 0$ and $\lambda > 0$.

A. General result on perturbed switched linear time-delay systems

We have the following general result on perturbed switched linear time-delay systems.

Theorem 1. For the perturbed switched linear time-delay system (7), suppose that Hypothesis 1 hold. If under certain switching law, the solution $x = 0$ of nominal system of (7), i.e., the homogeneous system

$$\dot{x}(t) = L_\sigma(t, x_t), \tag{8}$$

is exponentially stable, then the system state of (7) is

(i) bounded if the perturbation $f_\sigma(t)$ is bounded;
(ii) asymptotically convergent (to the origin) if the perturbation $f_\sigma(t)$ is bounded and asymptotically convergent (to the origin);
(iii) exponentially convergent (to the origin) if the perturbation $f_\sigma(t)$ is exponentially convergent (to the origin).

Proof. According to variation-of-constants formula, since $f_\sigma(t) \in \mathscr{L}_1^{\text{loc}}([t_0, \infty), R^n)$, when $t \in [t_j, t_{j+1})$, the solution of the i_jth subsystem through (t_j, ϕ_{i_j}) of (7) can be expressed as follows

$$x_t(t, t_j, \phi_{i_j}, f_{i_j}) = T_{i_j}(t, t_j)\phi_{i_j} + \int_{t_j}^{t} T_{i_j}(t, s) X_0(\theta) f_{i_j}(s) ds, \quad t \geq t_0,$$

where $T_{i_j}(t, t_0)$, $i \in \underline{N}$ are continuous linear operators with relations $x(t_j, \phi_{i_j}, 0) \triangleq T_{i_j}(t, t_j)\phi_{i_j}$. By the stepping method in finite interval $[t_j, t_{j+1})$ for each subsystem of (7) and well-defined switching law, the existence and uniqueness of the solution with initial condition for system (7) can be obtained, and there has

$$x_t(t, t_0, \phi_0, f_\sigma) = T(t, t_0)\phi_0 + \int_{t_0}^{t} T(t, s) X_0(\theta) f_\sigma(s) ds, \quad t \geq t_0, \tag{9}$$

where $T(t, t_0) = T_{i_j}(t, t_j) \cdot T_{i_{j-1}}(t_j, t_{j-1}) \cdots T_{i_0}(t_1, t_0)$ is a continuous operator with relation $x(t_0, \phi_0, 0) \triangleq T(t, t_0)\phi_0$ $(\phi_0 = \phi_{i0})$.

Since the solution $x = 0$ of nominal system of (7), i.e., $\dot{x}(t) = L_\sigma(t, x_t)$ is exponentially stable, this implies that there exist constants $\alpha > 0$, $\kappa > 0$, such that the solution operator $T(t_0, t)$ satisfy

$$\|T(t, t_0)\| \leq \kappa e^{-\alpha(t-t_0)}, \ \|T(t, t_0) X_0\| \leq \kappa e^{-\alpha(t-t_0)}.$$

Firstly, suppose that $f_\sigma(t)$ is bounded, that is, there exists $M > 0$, such that $\|f_\sigma(t)\| \leq M$, (9) gives that

$$\begin{aligned} \|x_t(t, t_0, \phi_0, f_\sigma)\| &\leq \kappa e^{-\alpha(t-t_0)}\|\phi_0\| + \kappa M \cdot \int_{t_0}^{t} e^{-\alpha(t-s)} ds \\ &\leq \kappa\|\phi_0\| + \frac{\kappa M}{\alpha}. \end{aligned}$$

Hence, the state is bounded.

Secondly, suppose that the perturbation $f_\sigma(t)$ is bounded and asymptotically convergent, that is, for any given number $\epsilon > 0$, $\exists \mathcal{T}_1 > 0$, when $t > \mathcal{T}_1$, there holds $\|f_i(t)\| < \epsilon$. The boundedness of $f_i(t)$ gives $\|f_i(t)\| \leq B_0\,(i \in \underline{N})$. When $t \in [t_j, t_{j+1}) \subset [\mathcal{T}_1, \infty)$, (9) gives rise to

$$\begin{aligned} &\|x_t(t, t_0, \phi_0, f_\sigma)\| \\ &\leq \kappa e^{-\alpha(t-t_0)}\|\phi_0\| + \int_{t_0}^{t} \kappa e^{-\alpha(t-s)} \|f_{\sigma(t)}(s)\| ds \\ &\leq \kappa e^{-\alpha(t-t_0)}\|\phi_0\| + \kappa B_0 \int_{t_0}^{T_1} e^{-\alpha(t-s)} ds \\ &\quad + \kappa \int_{T_1}^{t} e^{-\alpha(t-s)} ds \cdot \epsilon \\ &\leq \left(\kappa\|\phi_0\| + \frac{\kappa B_0}{\alpha}\right) e^{-\alpha(t-T_1)} + \frac{\kappa}{\alpha}\epsilon. \end{aligned} \tag{10}$$

For any given $\varepsilon > 0$, choose $\mathcal{T} = \mathcal{T}_1 + \frac{\ln \varepsilon^{-1}}{\alpha}$ and $\epsilon = \varepsilon$. When $t \in [t_j, t_{j+1}) \subset [\mathcal{T}, \infty)$, it follows from (10) that

$$\|x_t(t, t_0, \phi_0, f_\sigma)\| \leq \left(\kappa\|\phi_0\| + \frac{\kappa B_0}{\alpha} + \frac{\kappa}{\alpha}\right)\varepsilon.$$

From the arbitrariness of ε, the asymptotic convergence of the state follows.

Thirdly, suppose that for $t > t_0$, $\|f_\sigma(t)\| \leq \varrho\, e^{-\gamma(t-t_0)}$. Then, we have

$$\begin{aligned} &\|x_t(t, t_0, \phi_0, f_\sigma)\| \\ &\leq \kappa e^{-\alpha(t-t_0)}\|\phi_0\| + \int_{t_0}^{t} \kappa e^{-\alpha(t-s)} \cdot \varrho\, e^{-\gamma(s-t_0)} ds \\ &\leq \begin{cases} \left(\kappa\|\phi_0\| + \frac{\kappa\varrho}{\gamma-\alpha}\right) e^{-\alpha(t-t_0)}, & \text{if } \gamma > \alpha \\ \left(\kappa\|\phi_0\| + \frac{\kappa\varrho}{\gamma-\alpha}\right) e^{-\gamma(t-t_0)}, & \text{if } \gamma < \alpha \\ \left(\kappa\|\phi_0\| + \frac{\kappa\varrho}{\gamma-\alpha}\right) e^{-(\alpha-\varepsilon)(t-t_0)}, & \text{if } \gamma = \alpha \end{cases} \end{aligned} \tag{11}$$

where ε is any sufficient small positive number. Thus, the state is exponentially convergent. The proof is complete. □

In the following subsections, we will give delay dependent exponential stability criteria based on time-dependent switching law and delay independent exponential stability criteria based on state-dependent switching law, respectively.

B. Delay dependent convergence criteria: time-dependent switching law

In the perturbed switched linear time-delay systems (7), if $L_\sigma(t, \phi) = A_\sigma\phi(0) + D_\sigma\phi(-d_\sigma(t))$, where $d_i(t)$, $i \in \underline{N}$ is time-varying term satisfies $0 < d_i(t) \leq \tau$, and $\phi(\theta) = x(t + \theta), \theta \in (-\tau, 0)$, note that the initial state of each subsystem is related with the switching instant, we denote $\phi_\sigma(\theta) \triangleq \phi(\theta)$.

We have the familiar switched time-delay system with the form

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + D_\sigma x(t - d_\sigma(t)) + f_\sigma(t), \\ \phi_\sigma(\theta) = x(t+\theta),\ \theta \in [t_j - \tau, t_j], \\ \ \ x(0) = \phi_\sigma(0) = 0,\ j = 0, 1, \cdots. \end{cases} \tag{12}$$

We are in a position to present the delay dependent stability and convergence criteria.

Theorem 2. For system (12), suppose that the time-varying delay $d_i(t)$ satisfies $0 < d_i(t) \le \tau$ ($\tau > 0$, $i \in \underline{N}$). For given positive constant η, if there exist positive definite matrices P_i, S_i, with appropriate dimensions, such that the following linear matrices inequalities

$$\Theta_i := \begin{bmatrix} \varphi_{11}^i & \varphi_{12}^i \\ * & \varphi_{22}^i \end{bmatrix} < 0, \qquad i \in \underline{N} \tag{13}$$

hold, then, for any switching signal with average dwell time satisfying

$$T_\eta > T_\eta^* = \frac{\ln \mu}{\eta}, \tag{14}$$

where $\mu \ge 1$ satisfies

$$\begin{aligned} & P_i \le \mu P_j,\ S_i \le \mu S_j,\ \forall i, j \in \underline{N}, \qquad (15) \\ & \varphi_{11}^i = A_i^T P_i + P_i A_i + \eta P_i + \tau A_i^T S_i A_i - \tau^{-1} e^{-\eta\tau} S_i, \\ & \varphi_{12}^i = P_i D_i + \tau A_i^T S_i D_i + \tau^{-1} e^{-\eta\tau} S_i, \\ & \varphi_{22}^i = \tau D_i^T S_i D_i - \tau^{-1} e^{-\eta\tau} S_i. \end{aligned}$$

Then the system state of (12) is

(i) bounded if the perturbation $f_\sigma(t)$ is bounded;

(ii) asymptotically convergent (to the origin) if the perturbation $f_\sigma(t)$ is bounded and asymptotically convergent (to the origin);

(iii) exponentially convergent (to the origin) if the perturbation $f_\sigma(t)$ is exponentially convergent (to the origin).

Proof. According to Theorem 1, we need only to prove that under the average dwell time switching law (14), the nominal system of (12), i.e., the switched linear time-delay system

$$\dot{x}(t) = A_\sigma x(t) + D_\sigma x(t - d_\sigma(t)) \tag{16}$$

is exponentially stable.

Define the piecewise Lyapunov functional candidate

$$\begin{aligned} V(x(t)) = & V_{\sigma(t)}(x_t) = x^T(t) P_{\sigma(t)} x(t) \\ & + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) e^{-\eta(t-s)} S_{\sigma(t)} \dot{x}(s) ds d\theta, \end{aligned} \tag{17}$$

which is positive definite since P_i and S_i ($i \in \underline{N}$) are positive definite matrices.

When $t \in [t_k, t_{k+1})$, for the simplicity of notations, suppose that the ith subsystem is active, i.e., $\sigma(t) = i$. Differentiating (17) along the trajectory of (16) and noticing $d_i(t) \le \tau$, we obtain

$$\begin{aligned} \dot{V}_i(x_t) \le & 2x^T(t) P_i (A_i x(t) + D_i x(t - d_i(t))) \\ & + \tau \dot{x}^T(t) S_i \dot{x}(t) - \int_{t-d_i(t)}^{t} \dot{x}^T(s) e^{-\eta\tau} S_i \dot{x}(s) ds \\ & - \eta \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) e^{-\eta(t-s)} S_i \dot{x}(s) ds d\theta. \end{aligned} \tag{18}$$

Note that

$$\begin{aligned} \tau \dot{x}^T(t) S_i \dot{x}(t) = & x^T(t) \tau A_i^T S_i A_i x(t) \\ & + 2x^T(t) \tau A_i^T S_i D_i x(t - d_i(t)) \\ & + x^T(t - d_i(t)) \tau D_i^T S_i D_i x(t - d_i(t)). \end{aligned} \tag{19}$$

From the Jensen integral inequality, we obtain

$$\begin{aligned} & - \int_{t-d_i(t)}^{t} \dot{x}^T(s) e^{-\eta\tau} S_i \dot{x}(s) ds \\ & \le - \frac{1}{\tau} \left(\int_{t-d_i(t)}^{t} \dot{x}(s) ds \right)^T e^{-\eta\tau} S_i \int_{t-d_i(t)}^{t} \dot{x}(s) ds \\ & = - \frac{1}{\tau} [x(t) - x(t - d_i(t))]^T e^{-\eta\tau} S_i [x(t) - x(t - d_i(t))] \\ & = - \frac{1}{\tau} x^T(t) e^{-\eta\tau} S_i x(t) + \frac{2}{\tau} x^T(t) e^{-\eta\tau} S_i x(t - d_i(t)) \\ & \quad - \frac{1}{\tau} x^T(t - d_i(t)) e^{-\eta\tau} S_i x(t - d_i(t)). \end{aligned} \tag{20}$$

Substituting (19) and (20) into (18), yields

$$\begin{aligned} & \dot{V}_i(x_t) + \eta V_i(x_t) \\ & \le \begin{bmatrix} x(t) \\ x(t - d_i(t)) \end{bmatrix}^T \begin{bmatrix} \varphi_{11}^i & \varphi_{12}^i \\ * & \varphi_{22}^i \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - d_i(t)) \end{bmatrix}. \end{aligned}$$

Imposing condition (13) produces that

$$\dot{V}_i(x_t) + \eta V_i(x_t) < 0, \quad i \in \underline{N}. \tag{21}$$

When $t \in [t_k, t_{k+1})$, integrating (21) from t_k to t gives

$$V(x_t) = V_{\sigma(t)}(x_t) \le e^{-\eta(t-t_k)} V_{\sigma(t_k)}(x_{t_k}). \tag{22}$$

Using (15) and (17), at the switching instant t_i, we have

$$V_{\sigma(t_i)}(x_{t_i}) \le \mu V_{\sigma(t_i^-)}(x_{t_i^-}),\ i = 1, 2, \cdots. \tag{23}$$

Therefore, it follows from (22), (23) and the relation $k = N_\sigma(t_0, t) \le N_0 + \frac{t - t_0}{T_\eta}$, noticing $N_0 > 0$, that

$$\begin{aligned} V(x_t) \le & e^{-\eta(t-t_k)} \mu V_{\sigma(t_k^-)}(x_{t_k^-}) \\ \le & e^{-\eta(t-t_{k-1})} \mu V_{\sigma(t_{k-1})}(x_{t_{k-1}}) \le \cdots \\ \le & e^{-\eta(t-t_0)} \mu^k V_{\sigma(t_0)}(x_{t_0}) \\ \le & \mu^{N_0} \cdot e^{-(\eta - \frac{\ln \mu}{T_\eta})(t-t_0)} V_{\sigma(t_0)}(x_{t_0}). \end{aligned} \tag{24}$$

In view of (17) again, it holds that

$$a \|x(t)\|^2 \le V(x_t),\ V_{\sigma(t_0)}(x_{t_0}) \le b \|x_{t_0}\|_{cl}^2, \tag{25}$$

where $a = \min_{i \in \underline{N}} \lambda_{\min}(P_i)$, $b = \max_{i \in \underline{N}} \lambda_{\max}(P_i) + \tau^2 \max_{i \in \underline{N}} \lambda_{\max}(S_i)$. Let $\lambda = \frac{1}{2}(\eta - \frac{\ln \mu}{T_\eta})$. Combining (24) and (25) gives rise to

$$\|x(t)\|^2 \le \frac{1}{a} V(x_t) \le \frac{b}{a} \mu^{N_0} \cdot e^{-(\eta - \frac{\ln \mu}{T_\eta})(t-t_0)} \|x_{t_0}\|_{cl}^2.$$

Therefore $\|x(t)\| \le \sqrt{\frac{b}{a}} \mu^{\frac{N_0}{2}} \cdot e^{-\lambda(t-t_0)} \|x_{t_0}\|_{cl}$, which means that system (16) is exponentially stable. The remainder of the proof follows the procedures as in theorem 1. □

Remark 4. We introduce Jensen integral inequality in our proof, the computation complexities and analysis difficulties are decreased, whereas the conservativeness is not increased compare to existing results, such as free weighting matrix method (see, e.g., [3]), the correlative reference can be seen [14].

Example 2. Consider the perturbed switched linear time-varying delay system (12) with

$$A_1 = \begin{bmatrix} -4 & -2.5 \\ 1.2 & -1.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2 & 0.5 \\ -3.2 & -3.5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0 \end{bmatrix},$$

and $d_\sigma(t) = 0.6 + 0.6 \sin t$. For $\eta = 0.6,\ \tau = 1.2$, solving (13) gives piecewise Lyapunov functional (17) with

$$P_1 = \begin{bmatrix} 1.7368 & -0.0128 \\ -0.0128 & 1.9422 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1.8561 & -0.0567 \\ -0.0567 & 1.7216 \end{bmatrix},$$

and

$$S_1 = \begin{bmatrix} 0.3536 & -0.1162 \\ -0.1162 & 1.15345 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1.0467 & -0.2930 \\ -0.2930 & 0.5838 \end{bmatrix}.$$

Solving (15) gives $\mu = 2.9634$, and according (14), we have $\tau_\alpha^* = \frac{\ln \mu}{\alpha} = 1.8106$. By using average dwell time method provided by Theorem 2, with the convergent perturbation $f_\sigma(t) = \frac{\sin t}{t}$ and the bounded perturbation $f_\sigma(t) = \sin t$, the stability and convergence of the system (12) can be guaranteed, the simulation results are depicted in Fig.1.

C. Delay independent convergence criteria: state-dependent switching law

In the perturbed switched linear time-delay systems (7), if $L_\sigma(t, \phi) = A_\sigma \phi(0) + D_\sigma \phi(-\tau(t))$, where τ is constant dealy, and $\phi(\theta) = x(t+\theta), \theta \in (-\tau, 0)$. We have the following switched time-delay system with the form

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-\tau) + f_\sigma(t), \\ \phi_\sigma(\theta) = x(t+\theta),\ \theta \in [t_j - \tau, t_j], \\ x(0) = \phi_\sigma(0) = 0,\ j = 0, 1, \cdots . \end{cases} \quad (26)$$

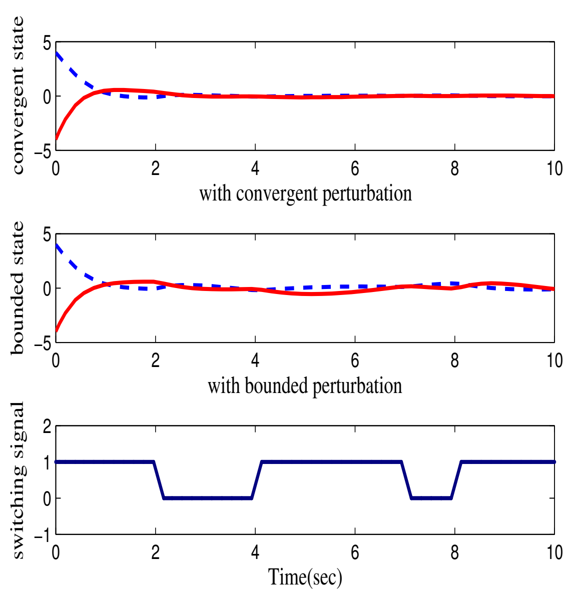

Fig. 1. The state of the perturbed switched time-delay system: average dwell time method.

We have the following delay independent stability and convergence criteria.

Theorem 3. For given positive constant $\eta > 0$, if there exist positive definite matrices P_i, S, and scalars $\alpha_{ij} > 0$, $(i, j \in \underline{N})$, such that the following matrices inequalities

$$\begin{bmatrix} \Xi_i & P_i D_i \\ * & -e^{-\eta\tau} S \end{bmatrix} < 0 \quad (27)$$

hold, where

$$\Xi_i := A_i^T P_i + P_i A_i + \eta P_i + S + \sum_{j \neq i, j \in \underline{N}} \alpha_{ij}(P_j - P_i),$$

if the system (26) is with the following switching law

$$\sigma(t) = arg \min_{i \in \underline{N}} \{x^T(t) P_i x(t)\}. \quad (28)$$

Then the system state of (26) is

(i) bounded if the perturbation $f_\sigma(t)$ is bounded;

(ii) asymptotically convergent if the perturbation $f_\sigma(t)$ is bounded and asymptotically convergent;

(iii) exponentially convergent if the perturbation $f_\sigma(t)$ is exponentially convergent.

Proof. The result follows from Theorem 1 if we can prove that under the multiple Lyapunov functional switching law (28), the nominal system of (26), i.e., the switched linear time-delay system

$$\dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-\tau) \quad (29)$$

is exponentially stable.

Design Lyapunov-Krasovskii functional candidate as

$$V(x(t)) = x^T(t) P_{\sigma(t)} x(t) + \int_{t-\tau}^{t} x^T(s) e^{-\eta(t-s)} S x(s) ds. \quad (30)$$

Obviously, the Lyapunov-Krasovskii functional candidate is positive definite.

For any $t > 0$, the jth switching instant is denoted by t_j $(j \geq 0)$. During any time interval $[t_j, t_{j+1})$, suppose that the ith subsystem is active. Let $\xi(t) = \left[x^T(t),\ x^T(t-\tau)\right]^T$. Consider the time derivative of $V(x(t))$ along the trajectory of (29), we have

$$\begin{aligned} &\dot{V}(x(t)) + \eta V(x(t)) \\ &= \xi^T(t) \begin{bmatrix} A_i^T P_i + P_i A_i + \eta P_i + S & P_i D_i \\ * & -e^{-\eta\tau} S \end{bmatrix} \xi(t). \end{aligned} \quad (31)$$

According to the condition (27), we have

$$\begin{bmatrix} A_i^T P_i + P_i A_i + \eta P_i + S & P_i D_i \\ * & -e^{-\eta\tau} S \end{bmatrix} < \begin{bmatrix} \Pi_i & 0 \\ 0 & 0 \end{bmatrix}, \quad (32)$$

where $\Pi_i := -\sum_{j \neq i, j \in \underline{N}} \alpha_{i,j}(P_j - P_i)$.

By virtue of the designed switching law (28), it follows

$$x^T(t)\Big(\sum_{j \neq i, j \in \underline{N}} \alpha_{ij}(P_j - P_i)\Big)x(t) \geq 0,\ \forall x(t) \in \mathbb{R}^n.$$

During $[t_j, t_{j+1})$, when $\xi(t) \neq 0$, we easily get

$$\dot{V}(x(t)) + \eta V(x(t)) < \xi^T(t) \begin{bmatrix} \Pi_i & 0 \\ 0 & 0 \end{bmatrix} \xi(t) \leq 0.$$

Thus, there holds

$$\dot{V}(x(t)) \leq -\eta V(x(t)), \tag{33}$$

During any $[t_j, t_{j+1})$, (33) gives rise that

$$V(x(t)) \leq e^{-\eta(t-t_j)} V(x(t_j)). \tag{34}$$

In addition, by the switching law (28), at the switching instant t_j, we have

$$x^T(t_j) P_{\sigma(t_j)} x(t_j) \leq \lim_{t \to t_j^-} x^T(t) P_{\sigma(t)} x(t),$$

which implies $V(x^T(t_j)) \leq \lim_{t \to t_j^-} V(x^T(t))$, by induction on $t_0, t_1, \cdots, t_j$, from (34) we get

$$V(x(t)) \leq e^{-\eta(t-t_0)} V(x(t_0)).$$

Then we have $\|x(t)\|^2 \leq \frac{\lambda_M(P_i)+\tau\lambda_M(S)}{\lambda_m(P_i)} e^{-\eta(t-t_0)} \|x(t_0)\|_{cl}^2$, which implies exponential stability of the nominal systems of (29), i.e., the system (26). The remainder of the proof is omitted since it follows the iterative steps of Theorem 1. □

Remark 5. The delay independent criteria we obtained are based on the multiple Lyapunov functional switching scheme. It is highly desirable that some subsystems are allowed to be unstable, comparatively, time-delay systems without switching scheme can not earn such property.

Example 3. Consider the perturbed switched linear time-delay system (26) with

$$A_1 = \begin{bmatrix} -1.5 & -1.2 \\ -1.2 & 1 \end{bmatrix}, \; D_1 = \begin{bmatrix} -0.5 & 0.5 \\ -0.1 & -0.4 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2.5 & 1 \\ 1 & -2.3 \end{bmatrix}, \; D_2 = \begin{bmatrix} -0.3 & -0.2 \\ 0.1 & -0.3 \end{bmatrix},$$

and $\tau = 0.9$. For $\eta = 0.4$, solving (27) gives Lyapunov functional (30) with

$$P_1 = \begin{bmatrix} 17.9344 & 1.6235 \\ 1.6235 & 33.9704 \end{bmatrix}, \; P_2 = \begin{bmatrix} 16.9512 & 6.4649 \\ 6.4649 & 23.0416 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 34.5285 & 0.6258 \\ 0.6258 & 32.0362 \end{bmatrix}.$$

By multiple Lyapunov functional method provided by Theorem 3, with the convergent perturbation $f_\sigma(t) = e^{-3t}$ and the bounded perturbation $f_\sigma(t) = \sin t$, the stability and convergence of the system (26) can be guaranteed, the simulation results are depicted in Fig.2.

IV. Conclusion

In this paper, we investigate the issue of stability and convergence of perturbed switched linear time-delay systems. By introducing the Variation-of-constants formula, the conditions of the stability and convergence of perturbed switched linear systems with time-delay are established. Based on the general result of perturbed switched linear time-delay systems, under two different switching schemes, new delay dependent and independent stability criteria for switched linear systems with time-delay are developed. The numerical examples show feasibility and validity of the results.

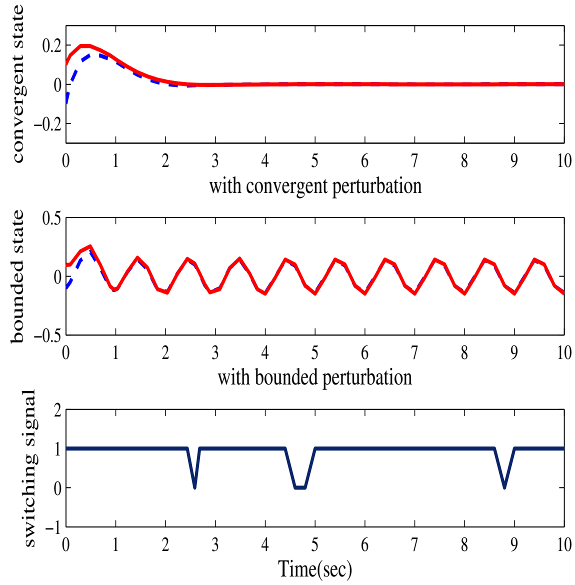

Fig. 2. The state of the perturbed switched time-delay system: multiple Lyapunov functional method.

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