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A Finite Frequency Approach to Reliable $\mathcal{H}_\infty$ Filtering for Linear Continuous-Time Systems with Sensor Faults

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Abstract— **The paper deals with the reliable $\mathcal{H}_\infty$ filtering problem for linear uncertain continuous-time systems with bounded disturbances. Different from the existing approaches, the filter is designed in certain finite frequency ranges, which is important in practice, since the full frequency approaches are conservative to some extent for the case when the frequency ranges of disturbances are known beforehand. With the aid of the Generalized Kalman-Yakubovich-Popov (GKYP) lemma, the filter design problem is formulated into a set of linear matrix inequalities (LMIs). A numerical example is given to illustrate the effectiveness of the proposed methods.**

I. INTRODUCTION

Recently, the $\mathcal{H}_\infty$ filtering approach, has received considerable attention due to its wide applicability when robustness is imposed, where the main objective is to minimize the $\mathcal{H}_\infty$ norm from disturbances to the estimation error [1]-[3]. On the other hand, LMI techniques have been applied to filtering problems [4]-[7], which can be solved effectively using the LMI control toolbox. In [8]-[13], the parameter dependent Lyapunov method is adopted, and through introducing appropriate slack matrix variables, the conservatism is reduced greatly. In [14]-[16], the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem is studied, where the trade-off between the minimization of the filtering error variance and the $\mathcal{H}_\infty$ disturbance attenuation is established.

Note that all the above filtering approaches are based on the assumption that the sensors can provide uninterrupted signal measurement. In practice, however, contingent faults are possible for all sensors in a system, which may result in a large degree of filter performance degradation and, more importantly, possible hazard. Therefore, the need for "reliable" filter that ensure performance despite the presence of sensor faults is fairly evident. Similar to the general notation of "reliable" controllers in [17]-[19], the "reliable" filter has been designed in [20].

In this paper, the reliable filter design problem is revisited, different from the existing reliable filtering approach [20], here the reliable $\mathcal{H}_\infty$ filter is designed in finite frequency frequency ranges, which is important in practice, since sometimes the frequency ranges of disturbances are known beforehand, for these cases, designing a filter in the full frequency domain may introduce conservatism to some extent [21]. A set of performance indexes are derived to reflect the design constraints on the transfer functions from disturbances and faults to the estimation error. By satisfying these performance indexes simultaneously, the disturbance and fault effects on estimation error are minimized. The recently proposed GKYP lemma [22] is used in this paper to formulate these performance indexes into a set of LMIs.

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II. PROBLEM FORMULATION

A. System model

Consider a stable linear uncertain system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bd(t) \\ y(t) &= Cx(t) + Dd(t) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state, $x(0) = x_0$, $d(t) \in \mathbf{R}^{n_d}$ is the energy bounded disturbance input satisfying $d(t)^T d(t) \leq \bar{d}^2$, $y(t) \in \mathbf{R}^m$ denotes the measured output, $z(t) \in \mathbf{R}^q$ is the vector to be estimated. All matrices are of compatible dimensions. Assume that All matrices A, B, C, D, L are unknown but

$$\bar{M} := \begin{bmatrix} A & B \\ C & D \\ L & 0 \end{bmatrix} \in D_c \tag{2}$$

where D_c is a given convex bounded polyhedral domain described by z_p vertices. That is each uncertain matrix in this domain may be written as an unknown convex combination of z_p given extreme matrices $\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_{z_p}$ such that

$$D_c := \{\bar{M}(\lambda) : \bar{M}(\lambda) = \sum_{l=1}^{z_p} \lambda_l \bar{M}_l, \lambda_l \geq 0, \sum_{l=1}^{z_p} \lambda_l = 1\}$$

where $\bar{M}_l = \begin{bmatrix} A_l & B_l \\ C_l & D_l \\ L_l & 0 \end{bmatrix}$.

B. Fault model

In this paper, it is assumed that at least one sensor is fault-free, and the following type sensor fault model is adopted.

Definition 1 (Sensor fault model): when sensor faults occur, the sensor signals of systems are given by

$$y_{si}(t) = F_i y(t) + (I - F_i) f_i, i = 0, 1, \ldots, N_s \tag{3}$$

TABLE I
DIFFERENT FREQUENCY RANGES

	LF	MF	HF
Ω	$\lvert\omega\rvert \leq \varpi_\ell$	$\varpi_1 \leq \omega \leq \varpi_2$	$\lvert\omega\rvert \geq \varpi_h$

where N_s is the number of the total possible fault modes, and

$$f_i = \begin{bmatrix} f_{i1} & \dots & f_{ik} & \dots & f_{im} \end{bmatrix}^T$$

with $\underline{f}_{ik} \leq f_{ik} \leq \bar{f}_{ik}$ being zero or nonzero constant, $k = 1, \dots, m$, and m is the dimension of $y(t)$. The diagonal matrices $F_i^{'}s$ are defined as

$$F_i = diag \begin{bmatrix} F_{i1} & \dots & F_{ik} & \dots & F_{im} \end{bmatrix} \quad (4)$$

where $F_{ik} = 1$ if the kth sensor is fault-free, and $F_{ik} = 0$ if the kth sensor gets faulty. Without loss of generality, assume that $F_0 = I$ which means that all the sensors are fault-free.
Remark 1: In the above sensor fault model, if $F_{ik} = 0$ and $f_{ik} = 0$, it means that the kth sensor is of outage. If $F_{ik} = 0$ and f_{ik} equals to certain nonzero constant, the kth sensor is called to be "stuck" here which is similar to the stuck-actuator fault as stated in [19][23]. If $F_i = I$, $y_{si}(t) = y(t)$, which corresponds to the fault-free case. N_s denotes the total number of the possible fault modes.

C. Filter design

The filter is of the form:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f y(t) \\ \hat{z}(t) &= C_f \hat{x}(t) \end{aligned} \quad (5)$$

where the vector $\hat{x}(t)$ is the filter state vector, and A_f, B_f, and C_f are real matrices of appropriate dimensions to be determined. The order of the filter n_f is restricted to be equal to the system order n.
The dynamics of (1) and (5) can be rewritten as the following augmented system considering the faulty cases as stated in subsection 2.2:

$$\dot{\xi}(t) = \bar{A}_{F_i} \xi(t) + \bar{B}_{F_i} d(t) + \bar{B}_{f_i} \quad (6)$$
$$e(t) = \bar{C} \xi(t) \quad (7)$$

where $e(t) = z(t) - \hat{z}(t)$ is the estimation error, $\xi(t) = \begin{bmatrix} x(t)^T & \hat{x}(t)^T \end{bmatrix}^T$, and

$$\begin{bmatrix} \bar{A}_{F_i} & \bar{B}_{F_i} & \bar{B}_{f_i} \\ \bar{C} & 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|c|c} A & 0 & B & 0 \\ B_f F_i C & A_f & B_f F_i D & B_f (I - F_i) f_i \\ \hline L & -C_f & 0 & 0 \end{array}\right]$$

Finite frequency reliable $\mathcal{H}_\infty$ filtering problem: The finite frequency reliable $\mathcal{H}_\infty$ filtering problem can be formulated as follows: Given a prescribed scalar $\gamma > 0$, $\beta > 0$, design a filter of form (5) such that for all $d(t) \in \mathcal{L}_2[0, \infty)$, the augmented error system (6)-(7) is stable and satisfies performance indexes

$$\|G_{ed_i}(j\omega)\|_\infty = \sup_{\omega \in \mathbf{\Omega}} \sigma_{max}(G_{ed_i}(j\omega)) < \gamma, \quad (8)$$
$$\|G_{f_i}(j\omega)\|_\infty = \sup_{\omega = 0} \sigma_{max}(G_{f_i}(j\omega)) < \beta \quad (9)$$

for $i = 0, 1, \dots, N_s$, respectively, where $G_{ed_i}(j\omega) = \bar{C}(j\omega I - \bar{A}_{F_i})^{-1} \bar{B}_{F_i}, G_{f_i}(j\omega) = \bar{C}(j\omega I - \bar{A}_{F_i})^{-1} \bar{B}_{f_i}$ and $\omega \in \mathbf{R}$, Ω is defined in Table I, where LF, MF, and HF stand for low, middle, and high frequency ranges, respectively.
Remark 2: Here performance index (8) is used to minimize the disturbance effects on estimation error in certain finite frequency ranges. Performance index (9) is used to minimize the fault effects on estimation error, where the frequency is zero.

D. Preliminaries

The following lemmas are essential for later developments.
Lemma 1: (Generalized KYP Lemma [22]) Given system (A, B, C, D), let a symmetric matrix $\Pi \in \mathbf{R}^{(n+n_z)\times(n+n_z)}$ be given, the following statements are equivalent:
i) The finite frequency inequality

$$\begin{bmatrix} G(j\omega) & I \end{bmatrix} \Pi \begin{bmatrix} G(j\omega)^* \\ I \end{bmatrix} < 0, \ \forall \ \omega \in \Omega \quad (10)$$

where $G(j\omega) = C(j\omega I - A)^{-1} B + D$ is the transfer function, ω is defined in Table I.
ii) There exist Hermitian matrices $P, Q \in \mathbf{H}_n$ satisfying $Q > 0$, and

$$\begin{bmatrix} \bar{A} & I \\ \bar{C} & 0 \end{bmatrix} \Xi \begin{bmatrix} \bar{A} & I \\ \bar{C} & 0 \end{bmatrix}^* + \begin{bmatrix} \bar{B} & 0 \\ 0 & I \end{bmatrix} \Pi \begin{bmatrix} \bar{B} & 0 \\ 0 & I \end{bmatrix}^* < 0 \quad (11)$$

where $\Xi = \begin{bmatrix} -Q & P \\ P & \varpi_\ell^2 Q \end{bmatrix}$ for low frequency range $|\omega| \leq \varpi_\ell$, $\Xi = \begin{bmatrix} -Q & P + j\varpi_c Q \\ P - j\varpi_c Q & -\varpi_1 \varpi_2 Q \end{bmatrix}, \varpi_c = (\varpi_1 + \varpi_2)/2$ for middle frequency range $\varpi_1 \leq \omega \leq \varpi_2$, and $\Xi = \begin{bmatrix} Q & P \\ P & -\varpi_h^2 Q \end{bmatrix}$ for high frequency range $|\omega| \geq \varpi_h$.
Lemma 2(*Projection Lemma* [24]): Let Γ, Λ, Θ be given. There exists a matrix F satisfying $\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0$ if and only if the following two conditions hold

$$\Gamma^\perp \Theta \Gamma^{\perp^T} < 0, \quad \Lambda^{T^\perp} \Theta \Lambda^{T^{\perp^T}} < 0$$

III. FINITE FREQUENCY RELIABLE FILTER DESIGN

A. LMI Conditions for performance index (8)

In this subsection, LMI conditions for performance index (8) in different frequency ranges are formulated. Firstly, two lemmas based on the GKYP lemma are formulated which are essential for the main theorem of this paper.
Lemma 3: Let real matrices $\bar{A}_{F_i} \in \mathbf{R}^{n\times n}$, $\bar{B}_{F_i} \in \mathbf{R}^{n\times n_d}$, $\bar{C} \in \mathbf{R}^{n_z \times n}$, symmetric matrix $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \in \mathbf{R}^{(n+n_z)\times(n+n_z)}$ be given, the system is given in (6)-(7),

then the following statements are equivalent:
i) The finite frequency performance index

$$\sigma_{max}(G_{ed_i}(j\omega)) < \gamma,\ \forall\ \omega \in \Omega \tag{12}$$

is satisfied, where Ω is defined in Table I.
ii) There exist Hermitian matrices $P, Q \in \mathbf{H}_n$ satisfying $Q > 0$, and

$$\begin{bmatrix} \bar{A}_{F_i} & I \\ \bar{C} & 0 \end{bmatrix} \Xi \begin{bmatrix} \bar{A}_{F_i} & I \\ \bar{C} & 0 \end{bmatrix}^* + \begin{bmatrix} \bar{B}_{F_i} & 0 \\ 0 & I \end{bmatrix} \Pi \begin{bmatrix} \bar{B}_{F_i} & 0 \\ 0 & I \end{bmatrix}^* < 0 \tag{13}$$

where Ξ is the same as defined in Lemma 1.

Proof. As $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, and the frequency range is $\omega \in \Omega$, applying Lemma 1, it is immediate. □

Remark 3: Lemma 3 gives an inequality condition to the performance index (8). Note that if we set $Q = 0$ of (13), using Schur complement and after some matrix manipulation, (13) becomes

$$\begin{bmatrix} \bar{A}P + P\bar{A}^* + \bar{B}\bar{B}^* & P\bar{C}^* \\ \star & -\gamma^2 I \end{bmatrix} < 0 \tag{14}$$

which is equivalent to the full frequency H_∞ norm condition.

The following lemma provides an alternative condition to (13) by introducing a multiplier R through the projection lemma, which is similar to that of [25]. Firstly, define $J \in \mathbf{R}^{(2n+n_z)\times 2n}, \bar{H} \in \mathbf{R}^{(2n+n_z)\times(n_d+n_z)}$, and $\bar{L} \in \mathbf{R}^{(2n+n_z)\times n}$ as

$$J := \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \bar{H} := \begin{bmatrix} 0 & 0 \\ \bar{B}_{F_i} & 0 \\ 0 & I \end{bmatrix}, \bar{L} := \begin{bmatrix} -I \\ \bar{A}_{F_i} \\ \bar{C} \end{bmatrix}$$

Lemma 4: Let Hermitian matrix variables $P, Q \in \mathbf{H}_n$ and $Q > 0, R \in \mathbf{R}^{n\times(2n+n_z)}$. Let N be the null space of R. The following statements are equivalent:
i) The condition in (13) holds and

$$N^*(J\Xi J^* + \bar{H} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \bar{H}^*)N < 0 \tag{15}$$

ii) There exists $W \in \mathbf{R}^{n\times n}$ such that

$$J\Xi J^* + \bar{H} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \bar{H}^* + \mathrm{He}(\bar{L}WR) < 0 \tag{16}$$

Proof. Notice that the null space of $\bar{L}$ is $\begin{bmatrix} \bar{A}_{F_i} & I & 0 \\ \bar{C} & 0 & I \end{bmatrix}$, and using Lemma 2, we have that ii) is equivalent to i). □

To convert inequality (16) into convex, we introduce the change of variables proposed in [7]. Let X, Y, U, V be defined by

$$W = \begin{bmatrix} X & U \\ \star & \hat{X} \end{bmatrix}, W^{-1} = \begin{bmatrix} Y & V \\ \star & \hat{Y} \end{bmatrix}$$

$X, Y \in \mathbf{R}^{n_p\times n_p}$, $\hat{X}, \hat{Y} \in \mathbf{R}^{n_z\times n_z}$ are all symmetric matrices. Multiplying the first row of W by the first column of W^{-1}, we have $XY + UV^* = I$, and then define the new variables

$$\begin{bmatrix} M & G \\ H & 0 \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} \begin{bmatrix} U^*X^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{17}$$

denoting $Z := X^{-1}$, and define

$$F := \begin{bmatrix} X^{-1} & Y \\ 0 & V^* \end{bmatrix}, \bar{F} := diag(F, F, I), \tag{18}$$

we have

$$\begin{bmatrix} \mathcal{A}^i & \mathcal{B}^i \\ \mathcal{C} & 0 \end{bmatrix} := \begin{bmatrix} F^*\bar{A}_{F_i}WF & F^*\bar{B}_{F_i} \\ \bar{C}WF & 0 \end{bmatrix}$$
$$= \left[\begin{array}{cc|c} ZA & ZA & ZB \\ YA + GF_iC + M & YA + GF_iC & YB + GF_iD \\ \hline L - H & L & 0 \end{array}\right] \tag{19}$$

$$\mathcal{W} := F^*WF = \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix} \tag{20}$$

Considering the vertexes of system matrices A, B, C, D, define $\mathcal{M}^i = \begin{bmatrix} \mathcal{A}^i & \mathcal{B}^i \\ \mathcal{C} & 0 \end{bmatrix}$, then we have that $\mathcal{M}^i$ belongs to a given convex bounded polyhedral domain $\mathcal{D}_c$, which is defined as

$$\mathcal{D}_c := \{\mathcal{M}^i(\lambda) : \mathcal{M}^i(\lambda) = \sum_{l=1}^{N} \lambda_l \mathcal{M}_l^i, \lambda_l \geq 0, \sum_{l=1}^{N} \lambda_l = 1\}$$

$$\mathcal{M}_l^i = \begin{bmatrix} \mathcal{A}_l^i & \mathcal{B}_l^i \\ \mathcal{C}_l^i & 0 \end{bmatrix} =$$
$$\left[\begin{array}{cc|c} ZA_l & ZA_l & ZB_l \\ YA_l + GF_iC_l + M & YA_l + GF_iC_l & YB_l + GF_iD_l \\ \hline L - H & L & 0 \end{array}\right]$$

Theorem 1: Consider the error system (6) with (A, B, C, D, L) being uncertain, let symmetric matrix $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$ and a non-negative scalar ϖ_ℓ be given. Suppose $R = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \in \mathbf{R}^{n\times(2n+n_z)}$, and $n = 2n_p$, then there exists a filter (5) with $n_f = n_p$ satisfying the specification

$$\sigma_{max}(G_{ed_i}(j\omega)) < \gamma,\ \forall\ |\omega| \leq \varpi_\ell \tag{21}$$

if there exist matrix variables Z, Y, M, G, H, and Hermitian matrix variables $\mathcal{P}_l = \begin{bmatrix} P_{1l} & P_{2l} \\ \star & P_{3l} \end{bmatrix}, \mathcal{Q}_l = \begin{bmatrix} Q_{1l} & Q_{2l} \\ \star & Q_{3l} \end{bmatrix} > 0$ satisfying the following LMIs

$$\begin{bmatrix} -Q_{1l} & -Q_{2l} & P_{1l} - Z & P_{2l} - Z & 0 & 0 \\ \star & -Q_{3l} & P_{2l}^* - Z & P_{3l} - Y & 0 & 0 \\ \star & \star & \Phi_{1l} & \Phi_{2l} & (L-H)^* & -ZB_l \\ \star & \star & \star & \Phi_{3l} & L^* & -YB_l - GF_iD_l \\ \star & \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, l = 1, \ldots, N \tag{22}$$

where $\Phi_{1l} = \varpi_\ell^2 Q_{1l} + ZA_l + (ZA_l)^*, \Phi_{2l} = \varpi_\ell^2 Q_{2l} + ZA_l + (YA_l + GF_iC_l + M)^*, \Phi_{3l} = \varpi_\ell^2 Q_{3l} + (YA_l + GF_iC_l) + (YA_l + GF_iC_l)^*$.

Proof. Applying Lemma 3 and Lemma 4, it follows that inequality (16) gives a sufficient condition to performance index (21). Since the frequency belongs to the low frequency range, inequality (16) multiplied to the left by the full rank matrix $\bar{F}^*$ and to the right by $\bar{F}$ provides the following inequality $J\begin{bmatrix}-\mathcal{Q} & \mathcal{P}\\ \mathcal{P} & \varpi_l^2\mathcal{Q}\end{bmatrix}J^* + \mathcal{H}\begin{bmatrix}I & 0\\ 0 & -\gamma^2 I\end{bmatrix}\mathcal{H}^* + \mathrm{He}(\mathcal{L}R) < 0$ where $\mathcal{P} := F^*PF, \mathcal{Q} := F^*QF, \mathcal{H} := \begin{bmatrix}0 & 0\\ \mathcal{B}^i & 0\\ 0 & I\end{bmatrix}, \mathcal{L} := \begin{bmatrix}-\mathcal{W}\\ \mathcal{A}^i\\ \mathcal{C}\end{bmatrix}$, then we have

$$\begin{bmatrix}-\mathcal{Q} & \mathcal{P}-\mathcal{W} & 0\\ \star & \varpi_\ell^2\mathcal{Q}+\mathcal{A}^i+\mathcal{A}^{i^*} & \mathcal{C}^*\\ \star & \star & -\gamma^2 I\end{bmatrix} + \begin{bmatrix}0\\ \mathcal{B}^i\\ 0\end{bmatrix}\begin{bmatrix}0\\ \mathcal{B}^i\\ 0\end{bmatrix}^* < 0 \tag{23}$$

Using Schur complement and after some substitution of the corresponding matrices, it follows that (23) is equivalent to

$$\begin{bmatrix}-Q_1 & -Q_2 & P_1-Z & P_2-Z & 0 & 0\\ \star & -Q_3 & P_2^*-Z & P_3-Y & 0 & 0\\ \star & \star & \Phi_1 & \Phi_2 & (L-H)^* & -ZB\\ \star & \star & \star & \Phi_3 & L^* & -YB-GF_iD\\ \star & \star & \star & \star & -\gamma^2 I & 0\\ \star & \star & \star & \star & \star & -I\end{bmatrix} < 0 \tag{24}$$

where

$$\Phi_1 = \varpi_\ell^2 Q_1 + ZA + (ZA)^*,$$
$$\Phi_2 = \varpi_\ell^2 Q_2 + ZA + (YA + GF_iC + M)^*,$$
$$\Phi_3 = \varpi_\ell^2 Q_3 + (YA + GF_iC) + (YA + GF_iC)^*$$

which means that inequality (24) gives a sufficient condition to performance index (21).

Note that inequality (22) is linear on P_{1l}, P_{2l}, P_{3l}, Q_{1l}, Q_{2l}, Q_{3l}, A_l, B_l, C_l, D_l, multiply each inequality in (22) by the uncertain parameter λ_l and then evaluate the sum from $l = 1, \ldots, N$, we have

$$\begin{bmatrix}-Q_1(\lambda) & -Q_2(\lambda) & P_1(\lambda)-Z & P_2(\lambda)-Z & 0 & 0\\ \star & -Q_3(\lambda) & P_2(\lambda)^*-Z & P_3(\lambda)-Y & 0 & 0\\ \star & \star & \Phi_1(\lambda) & \Phi_2(\lambda) & (L-H)^* & -ZB(\lambda)\\ \star & \star & \star & \Phi_3(\lambda) & L^* & -YB(\lambda)-GF_iD(\lambda)\\ \star & \star & \star & \star & -\gamma^2 I & 0\\ \star & \star & \star & \star & \star & -I\end{bmatrix} < 0 \tag{25}$$

with

$$P_1(\lambda) := \sum_{l=1}^{N}\lambda_l P_{1l},\ P_2(\lambda) := \sum_{l=1}^{N}\lambda_l P_{2l},$$
$$P_3(\lambda) := \sum_{l=1}^{N}\lambda_l P_{3l},\ Q_1(\lambda) := \sum_{l=1}^{N}\lambda_l Q_{1l},$$
$$Q_2(\lambda) := \sum_{l=1}^{N}\lambda_l Q_{2l},\ Q_3(\lambda) := \sum_{l=1}^{N}\lambda_l P_{3l},$$
$$\Phi_1(\lambda) = \varpi_\ell^2 Q_1(\lambda) + ZA + (ZA)^*,$$
$$\Phi_2(\lambda) = \varpi_\ell^2 Q_2(\lambda) + ZA + (YA + GF_iC + M)^*,$$
$$\Phi_3(\lambda) = \varpi_\ell^2 Q_3(\lambda) + (YA + GF_iC) + (YA + GF_iC)^*$$

Since inequality (25) is nothing but (24), we can conclude that if each inequality in (22) holds, condition (24) is then satisfied, which completes the proof. □

Similar to Theorem 1, the following three corollaries provide LMI conditions for performance index (8) in middle/high frequency range and for performance index (9), respectively.

Corollary 1: Consider the error system (6) with (A, B, C, D, L) being uncertain, let symmetric matrix $\Pi = \begin{bmatrix}I & 0\\ 0 & -\gamma^2 I\end{bmatrix}$ and scalars ϖ_1, ϖ_2 be given. Suppose $R = \begin{bmatrix}0 & I & 0\end{bmatrix} \in \mathbf{R}^{n\times(2n+n_z)}$, and $n = 2n_p$, then there exists a filter (5) with $n_f = n_p$ satisfying the specification

$$\sigma_{max}(G_{ed_i}(j\omega)) < \gamma,\ \forall\ \varpi_1 \le \omega \le \varpi_2 \tag{26}$$

if there exist matrices Z, Y, M, G, H, and Hermitian matrices $\mathcal{P}_l = \begin{bmatrix}P_{1l} & P_{2l}\\ \star & P_{3l}\end{bmatrix}, \mathcal{Q}_l = \begin{bmatrix}Q_{1l} & Q_{2l}\\ \star & Q_{3l}\end{bmatrix} > 0$ satisfying the following LMIs

$$\begin{bmatrix}-Q_{1l} & -Q_{2l} & P_{1l}+j\varpi_c Q_{1l}-Z & P_{2l}+j\varpi_c Q_{2l}-Z & 0 & 0\\ \star & -Q_{3l} & P_{2l}+j\varpi_c Q_{2l}-Z & P_{3l}+j\varpi_c Q_{3l}-Y & 0 & 0\\ \star & \star & \Phi_{1l} & \Phi_{2l} & (L-H)^* & -ZB_i\\ \star & \star & \star & \Phi_{3l} & L^* & -YB_i-GD_i\\ \star & \star & \star & \star & -\gamma^2 I & 0\\ \star & \star & \star & \star & \star & -I\end{bmatrix} < 0 \tag{27}$$

for $l = 1, \ldots, N$, where $\varpi_c = (\varpi_1 + \varpi_2)/2, \Phi_{1l} = -\varpi_1\varpi_2 Q_{1l} + ZA_l + (ZA_l)^*, \Phi_{2l} = -\varpi_1\varpi_2 Q_{2l} + ZA_l + (YA_l + GC_l + M)^*, \Phi_{3l} = -\varpi_1\varpi_2 Q_{3l} + (YA_l + GC_l) + (YA_l + GC_l)^*$.

Proof. Following the same lines for that of Theorem 1, it is immediate. □

Corollary 2: Consider the error system (6) with (A, B, C, D, L) being uncertain, let symmetric matrix $\Pi = \begin{bmatrix}I & 0\\ 0 & -\gamma^2 I\end{bmatrix}$ and a positive scalar ϖ_h be given. Suppose $R = \begin{bmatrix}I & -I & 0\end{bmatrix} \in \mathbf{R}^{n\times(2n+n_z)}$, and $n = 2n_p$, then there exists a filter (5) with $n_f = n_p$ satisfying the specification

$$\sigma_{max}(G_{ed_i}(j\omega)) < \gamma,\ \forall\ |\omega| \geq \varpi_h \tag{28}$$

if there exist matrices Z, Y, M, G, H, and Hermitian matrices $\mathcal{P}_l = \begin{bmatrix} P_{1l} & P_{2l} \\ \star & P_{3l} \end{bmatrix}, \mathcal{Q}_l = \begin{bmatrix} Q_{1l} & Q_{2l} \\ \star & Q_{3l} \end{bmatrix} > 0$ satisfying the following LMIs

$$\begin{bmatrix} Q_{1l} - Z - Z^* & Q_{2l} - Z - Z^* & P_{1l} + Z + (ZA_l)^* & \Phi_{2l} & (L-H)^* & 0 \\ \star & Q_{3l} - Y - Y^* & P_{2l}^* + Z + (ZA_l) & \Phi_{3l} & L^* & 0 \\ \star & \star & \Phi_{1l} & \Phi_{4l} & -(L-H)^* & -ZB_l \\ \star & \star & \star & \Phi_{5l} & -L^* & -YB_l - GD_l \\ \star & \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, l = 1, \ldots, N \tag{29}$$

where $\Phi_{1l} = -\varpi_h^2 Q_{1l} - ZA_l - (ZA_l)^*, \Phi_{2l} = P_{2l} + Z + (YA_l + GC_l + M)^*, \Phi_{3l} = P_{3l} + Y + (YA_l + GC_l)^*, \Phi_{4l} = -\varpi_h^2 Q_{2l} - ZA_l - (YA_l + GC_l + M)^*, \Phi_{5l} = -\varpi_h^2 Q_{3l} - (YA_l + GC_l) - (YA_l + GC_l)^*$.

Proof. Following the same lines for that of Theorem 1, it is immediate. □

Corollary 3: Consider the error system (6) with (A, B, C, D, L) being uncertain, let symmetric matrix $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\beta^2 I \end{bmatrix}$ be given. Suppose $R = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \in \mathbf{R}^{n \times (2n+n_z)}$, and $n = 2n_p$, then there exists a filter (5) with $n_f = n_p$ satisfying the specification

$$\sigma_{max}(G_{f_i}(j\omega)) < \beta,\ \text{for}\ \omega = 0 \tag{30}$$

if there exist matrix variables Z, Y, M, G, H, and Hermitian matrix variables $\mathcal{P}_l = \begin{bmatrix} P_{1l} & P_{2l} \\ \star & P_{3l} \end{bmatrix}, \mathcal{Q}_l = \begin{bmatrix} Q_{1l} & Q_{2l} \\ \star & Q_{3l} \end{bmatrix} > 0$ satisfying $f_i \in \{\underline{f}_{ik}, \bar{f}_{ik}\}$ and the following LMIs

$$\begin{bmatrix} -Q_{1l} & -Q_{2l} & P_{1l} - Z & P_{2l} - Z & 0 & 0 \\ \star & -Q_{3l} & P_{2l}^* - Z & P_{3l} - Y & 0 & 0 \\ \star & \star & \Phi_{1l} & \Phi_{2l} & (L-H)^* & 0 \\ \star & \star & \star & \Phi_{3l} & L^* & -G(I-F_i)f_i \\ \star & \star & \star & \star & -\beta^2 I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0 \tag{31}$$

for $l = 1, \ldots, N$, where $\Phi_{1l} = ZA_l + (ZA_l)^*, \Phi_{2l} = ZA_l + (YA_l + GF_iC_l + M)^*, \Phi_{3l} = (YA_l + GF_iC_l) + (YA_l + GF_iC_l)^*$.

Proof. Following the same lines for that of Theorem 1, it can readily be formulated that (31) provides a sufficient condition for performance index (30), note that inequality (31) is linear dependent on f_i, only vertices of f_i(i.e., $\underline{f}_{ik}, \bar{f}_{ik}$) need to be checked, this completes the proof. □

B. Stability conditions

In this subsection, conditions for the stability of the augmented error system (6) are formulated.

Lemma 5: Given system (6) with matrices (A, B, C, D, L) being uncertain, $\bar{A}_{F_i}$ is Hurwitz if and only if there exist matrix variables $\mathcal{W} = \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix}$ and $\mathcal{P}_{sl} = \begin{bmatrix} P_{s1l} & P_{s2l} \\ \star & P_{s3l} \end{bmatrix} > 0$ such that, for $l = 1, \ldots, N$

$$\begin{bmatrix} -2Z & -2Z & P_{1l} + (ZA_l)^* & P_{2l} + (YA_l + GF_iC_l + M)^* & Z & Z \\ \star & -2Y & P_{2l}^* + (ZA_l)^* & P_{3l} + (YA_l + GF_iC_l)^* & Z & Y \\ \star & \star & -P_{s1l} & -P_{s2l} & 0 & 0 \\ \star & \star & \star & -P_{s3l} & 0 & 0 \\ \star & \star & \star & \star & -P_{s1l} & -P_{s2l} \\ \star & \star & \star & \star & \star & -P_{s3l} \end{bmatrix} < 0 \tag{32}$$

Proof. Applying Theorem 3.1 of [26], following the same lines for that of Theorem 1, it is immediate. □

Combining Theorem 1, Corollaries 1-3 and Lemma 5, the matrix variables that needed for determining the finite frequency filter can be obtained as follows. Given β, solve the following optimization problem:

$$\begin{aligned} \min \quad & \gamma \\ s.t. \quad & (22), (31), (32),\ i = 0, 1, \ldots, N_s,\ l = 1, \ldots, N \\ \text{or} \quad & (27), (31), (32),\ i = 0, 1, \ldots, N_s,\ l = 1, \ldots, N \\ \text{or} \quad & (29), (31), (32),\ i = 0, 1, \ldots, N_s,\ l = 1, \ldots, N \end{aligned} \tag{33}$$

for low/middle/high frequency ranges, respectively.

Then the filter parameters can be obtained as follows. Let U and V be any factor such that $VU = I - YX$ where non-singularity of $I - YX$ can be assumed without loss of generality due to the strictness of the LMIs. Then the filter parameters (A_f, B_f, C_f) can then be obtained by solving (17) as following

$$\begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} M & G \\ H & 0 \end{bmatrix} \begin{bmatrix} U^* X^{-1} & 0 \\ 0 & I \end{bmatrix}^{-1} \tag{34}$$

IV. NUMERICAL EXAMPLE

This section gives a numerical example to illustrate the effectiveness of our approach. Consider the following system

TABLE II

COMPARISON OF FILTERING PERFORMANCES OBTAINED BY DIFFERENT METHODS

Method	Given $\|\omega\| \leq 0.32$	Given $\|\omega\| \leq 0.55$	Given $\|\omega\| \leq 1$	Full frequency approach
γ	0.4553	0.4601	0.48	0.6116

model

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 & 0.4 \\ 0 & -0.8 & 0.2+\rho \\ -0.1 & -0.6 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d(t) \\ y(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(t) \\ z(t) &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t) \end{aligned} \tag{35}$$

where ρ is the uncertain parameter satisfying $0 \leq \rho \leq 0.4$, and the sensor fault f_i as defined in (3) is assumed to be $0 \leq f_i \leq 0.3$, $i = 1, 2$.

Without loss of generality, consider the low frequency filtering problem, the middle/high frequency filtering problems are similar. Assume that the frequency range of disturbance is $|\omega| \leq 0.55$, given $\beta = 0.63$, solve the optimization problem (33), the disturbance attenuation performance index γ is obtained as $\gamma = 0.46$.

Applying the existing full frequency approach, given the same $\beta = 0.63$, we get $\gamma = 0.61$.

In Table II, the finite frequency filtering approach is compared with the the existing full frequency approach, which shows that the finite frequency approach proposed in this paper can receive better results.

V. CONCLUSIONS

In this paper, the problem of finite frequency reliable $\mathcal{H}_\infty$ filtering problem has been investigated, which is important in practice, since the full frequency approaches are conservative to some extent for the case when the frequency ranges of disturbances are known beforehand. Additionally, our filtering approach is valid for both normal and sensor faulty cases. Through solving a set of LMIs, the parameters of the $\mathcal{H}_\infty$ filter can readily be obtained. The numerical example has illustrated the effectiveness of the proposed approach.

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