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Fault Estimation for Discrete-Time Delay Systems in Finite Frequency Domain

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***Abstract*— This paper addresses the problem of fault detection(FD) in low frequency domain for linear discrete-time delay systems. It is shown that FD problem in low frequency range can be formulated as looking for an $\mathbf{H}_\infty$ fault detection filter(FDF) and the proposed filter design conditions are given in terms of linear matrix inequalities(LMIs). Finally, a numerical example demonstrates the effectiveness of the present methodology.**

I. INTRODUCTION

Model-based fault detection methods have attracted considerable attention over the decades[1-3].This class of procedures makes use of the plant model to generate additional signals that are compared during the on-line operations with the corresponding measured quantities and generate fault alarms when large discrepancies in their differences, referred to as residuals, arise. So the key of this approach is to generate residual which remains sensitivity to faults while guaranteeing robustness against unknown inputs. There have been a number of results using optimization technique to solve this problem, e.g., the $\mathbf{H}_\infty/\mathbf{H}_\infty$ approach[4], $\mathbf{H}_\infty$[5] or multi-objective $\mathbf{H}_\infty$ approach[6], and recently developed $\mathbf{H}_-/\mathbf{H}_\infty$ approach[7-10].

These achievements, however, heavily rely on the fact that the FD problem is characterized in entire frequency domain, a drawback of it is that full frequency range does not exactly encompass the practical situation. As we all known, for an incipient signal, the fault information is contained within a low frequency band as the fault development is slow[1], and the actuator stuck failures which occur in flight control systems just belong to low frequency domain[11]. The prevailing method for adjusting the discrepancy is the so-called weighting functions[12-15]. However, the design iterations to search for good weighting functions can be time consuming, and the fault detection filter complexity tends to increase with the complexity of the weighting functions.

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Recently, the Kalman-Yakubovic-Popov (KYP) lemma[16] is generalized in [17], and for time-delay systems, [17] converts a certain frequency domain inequality in finite frequency range to a numerically tractable LMI condition, and it gives a sufficient condition for a given transfer function to satisfy a required frequency domain property over a given frequency domain in terms of an LMI condition.

By the aid of [17], unlike those papers[12-15], the method proposed in this paper gives a direct treatment of the finite-frequency fault detection problem, completely avoiding approximations associated with frequency weights. Based on the idea of[5, 18, 19], FD problem is converted to design an $\mathbf{H}_\infty$ filter in low frequency domain, and a filter design method is presented in terms of solutions to a set of LMIs. The effectiveness of this technique is illustrated by an example.

This paper is organized as follows: In Section 2 preliminary results that introduce the results in [17] and the $\mathbf{H}_\infty$ optimization problem are given. The main results are stated in section 3. Section 4 gives a numerical example supporting the effectiveness of the proposed approach and some conclusions end this paper in section 5.

Notation: The following notations are used throughout this paper. For a matrix A, A^T denotes its transpose, and A^*, $A^\perp$ denotes its complex conjugate transpose and orthogonal complement, respectively. The Hermitian part of a square matrix A is denoted by $\mathbf{He}(A) := A + A^*$. The symbol $\mathbf{H_n}$ stands for the set of $n \times n$ Hermitian matrices. The symbol $*$ in matrix represents the symmetric entries. I denotes the identity matrix with an appropriate dimension. $\sigma_{max}G$ denotes maximum singular value of the transfer matrix G. For matrices Φ and $\mathbf{P}$, $\Phi \otimes \mathbf{P}$ means the Kronecker product. For matrices $G \in \mathbf{C^{n\times m}}$ and $\Pi \in \mathbf{H_{n+m}}$, a function σ: $\mathbf{C^{n\times m}} \times \mathbf{H_{n+m}} \rightarrow \mathbf{H_m}$ is defined by

$$\sigma(G,\Pi) := \begin{bmatrix} G \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

II. PROBLEM STATEMENT AND PRELIMINARIES

Let us consider the following linear discrete-time state-delayed system described by

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau) + \begin{bmatrix} B_f & B_d \end{bmatrix} \begin{bmatrix} f(k) \\ d(k) \end{bmatrix}, \\ y(k) &= Cx(k) + \begin{bmatrix} D_f & D_d \end{bmatrix} \begin{bmatrix} f(k) \\ d(k) \end{bmatrix}, \end{aligned} \quad (1)$$

where the initial condition is null, that is, $x(k) = 0, \{k = -\tau, -\tau+1, \cdots 0\}$ and $x(k) \in \mathscr{R}^n$ is the state space vector,

$y(k) \in \mathscr{R}^{n_y}$ is the measurement output vector , $d(k) \in \mathscr{R}^{n_d}$ is the unknown input vector, $f(k) \in \mathscr{R}^{n_f}$ denotes the fault to be detected. Actuator and component faults are modeled by $B_f f(k)$ and sensor faults are modeled by $D_f f(k)$. Here, we denote $B = [B_f \; B_d]$, $D = [D_f \; D_d]$. τ is a unknown constant time delay.

In this paper, the proposed fault detection filter will have the form

$$\begin{aligned} x_f(k+1) &= K_f x_f(k) + L_f y(k) \\ r(k) &= M_f x_f(k) + N_f y(k), \end{aligned} \tag{2}$$

which has the same order as system (1). Here, $r(k) \in \mathscr{R}^{n_r}$ is to estimate the fault vector, and $x_f(k) \in \mathscr{R}^{n_F}$ is the filter state vector, and (K_f, L_f, M_f, N_f) are real matrices of appropriate dimensions to be computed.

If we define $e(k) := r(k) - f(k)$ and $\omega(k) := \begin{bmatrix} f(k) \\ d(k) \end{bmatrix}$, then dynamics (1) and (2) can be rewritten in the following augmented system:

$$\begin{aligned} \xi(k+1) &= \bar{A}\xi(k) + \bar{A_d}\xi(k-\tau) + \bar{B}\omega(k), \\ e(k) &= \bar{C}\xi(k) + \bar{D}\omega(k), \end{aligned} \tag{3}$$

where $\xi(k) = \begin{bmatrix} x(k) \\ x_f(k) \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & 0 \\ L_f C & K_f \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ L_f D \end{bmatrix}$, $\bar{A_d} = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} N_f C & M_f \end{bmatrix}$, $\bar{D} = [N_f D + D_{11}]$, and $D_{11} = \begin{bmatrix} 0 & -I \end{bmatrix}$.

FD relies on the generation of residual, which must be sensitive to faults and as robust as possible to the unknown inputs. Specifically, the design must ensure that the residual is "close" to zero in fault-free situations while suitably deviating from zero in the presence of faults. So for system (3), we can describe the fault detection filter design problem using $\mathbf{H}_\infty$ optimization[5] as follows:

$$\min_{\lambda} \|G_{\omega e}(e^{j\lambda})\|_\infty \tag{4}$$

Here, we add a low frequency constraint $|\lambda| \leq \rho$ into our problem, where ρ is a positive scalar. Then the fault detection filter design problem using $\mathbf{H}_\infty$ optimization is defined as, for system (3), finding K_f, L_f, M_f, N_f such that

$$\min_{|\lambda| \leq \rho} \|G_{\omega e}(e^{j\lambda})\|_\infty \tag{5}$$

Remark 1 (5) can be recast in the satisfaction of the following condition

$$\sup_{|\lambda| \leq \rho} \sigma_{max}(G_{\omega e}(e^{j\lambda})) < \gamma \tag{6}$$

where γ is a given positive number to be minimized.

Remark 2 The disturbance considered in the filter design is assumed to be in the same frequency range as that of the fault since disturbances that belong to the high frequency domain can be decoupled by designing a low-pass filter after the residual outputs.

For system (3), the result in [17] provides an alternative condition to (6), so we introduce the main results about[17]. Given a linear time-delay system

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau) + B\varpi(k), \\ y(k) &= Cx(k) + D\varpi(k), \end{aligned} \tag{7}$$

where $x(k) \in \mathscr{R}^n$ is the state space vector, $y(k) \in \mathscr{R}^{n_y}$ the measurement output vector , $\varpi(k) \in \mathscr{R}^{n\varpi}$ is the disturbance input vector, respectively. A, A_d, B, C, and D are known matrices with appropriate dimensions. τ is constant time delay. The transfer function matrix $G(\lambda)$ from ϖ to y is denoted by

$$G(\lambda) = C(\lambda I - A - \lambda^{-\tau} A_d)^{-1} B + D \tag{8}$$

Given a Hermitian matrix Π, the specification can be described by

$$\sigma(G(\lambda), \Pi) < 0 \quad \forall \lambda \in \overline{\Lambda}(\Phi, \Psi) \tag{9}$$

where

$$\Lambda(\Phi, \Psi) := \{\lambda \in \mathbf{C} | \sigma(\lambda, \Phi) = 0, \sigma(\lambda, \Psi) \geq 0\} \tag{10}$$

and $\overline{\Lambda} := \Lambda$ if Λ is bounded and $\overline{\Lambda} := \Lambda \bigcup \{\infty\}$ if unbounded.

Lemma 1[17]: Let matrices $A \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$, $A_d \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$, $B \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}_{\bar{\omega}}}$, $C \in \mathbf{C}^{\mathbf{n_y}\times\mathbf{n}}$, $D \in \mathbf{C}^{\mathbf{n_y}\times\mathbf{n}_{\bar{\omega}}}$, $\Pi \in H_{n_y+n_{\bar{\omega}}}$, $\Phi, \Psi \in \mathbf{H}_2$ be given and define Λ by (11). Suppose Λ represents curves on the complex plane. then $\sigma(G(\lambda), \Pi) < 0$ holds for all $\lambda \in \overline{\Lambda}(\Phi, \Psi)$ if there exist $P = P^*$, $Q = Q^* > 0$ and $\Theta = \Theta^*$ such that

$$\begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} \Theta & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -\Theta \end{bmatrix} < 0 \tag{11}$$

Remark 3 If we choose $\Phi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Psi = \begin{bmatrix} 0 & 1 \\ 1 & -2\cos\rho \end{bmatrix}$, then $\lambda \in \overline{\Lambda}(\Phi, \Psi)$ is equivalent to $|\lambda| \leq \rho$.[21]

For the later development, we conclude this section with introducing the following lemma:

Lemma 2[20] Let Γ, Λ and $\Theta = \Theta^T$ be given matrices. There exists a matrix F to solve the matrix inequality

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0 \tag{12}$$

if and only if the following conditions are satisfied

$$\Gamma^{\perp} \Theta \Gamma^{\perp^T} < 0 \tag{13}$$

$$\Lambda^{T^\perp} \Theta \Lambda^{T^{\perp^T}} < 0. \tag{14}$$

III. Main Results

A. LMI conditions for performance index (6)

In this section, to design a fault detection filter by solving the optimization problem in (6), we consider the dual system of (3).

Theorem 1 Consider the transfer function $G_{\omega e}(e^{j\lambda})$ of system (3), given $\gamma > 0$, let $\Pi = \begin{bmatrix} I & \\ & -\gamma^2 I \end{bmatrix}$, there exists a filter (2) with $n_F = n$ satisfying the specification

$$\sigma_{max}(G_{\omega e}(e^{j\lambda})) < \gamma, \forall |\lambda| \le \rho \tag{15}$$

if there exist matrices $P = P^*$, $Q = Q^* > 0$ and $\Theta = \Theta^*$ such that

$$\begin{bmatrix} \bar{A} & I \\ \bar{C} & 0 \\ \bar{A}_d & 0 \end{bmatrix} \begin{bmatrix} -P & Q \\ * & P-(2\cos\rho)Q \end{bmatrix} \begin{bmatrix} \bar{A} & I \\ \bar{C} & 0 \\ \bar{A}_d & 0 \end{bmatrix}^* +$$
$$\begin{bmatrix} \begin{bmatrix} \bar{B} & 0 \\ \bar{D} & I \end{bmatrix} \Pi \begin{bmatrix} \bar{B} & 0 \\ \bar{D} & I \end{bmatrix}^* + \begin{bmatrix} \Theta & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -\Theta \end{bmatrix} < 0 \tag{16}$$

Proof:As $\Pi = \begin{bmatrix} I & \\ & -\gamma^2 I \end{bmatrix}$, combing (10) and remark (4), it is easy to to see condition (9) in section 2 becomes

$$G(e^{j\lambda})^* G(e^{j\lambda}) < \gamma^2 I \quad \forall |\lambda| \le \rho \tag{17}$$

which is just (15). Then, combing the dual version of Lemma 1 and the dual system of (3), we can complete this proof.

Note that condition (16) is not convex. Our main purpose is to convert the nonlinear matrix inequality appearing in (16) into LMI.

Before proceeding, we need define the following matrices $\mathbf{J} \in \mathscr{R}^{(6n+n_r)\times 4n}$, $\mathbf{H} \in \mathscr{R}^{(6n+n_r)\times(n_d+n_f+n_r+2n)}$, $\mathbf{Z} \in \mathscr{R}^{(6n+n_r)\times(n_r+4n)}$, $\mathbf{L} \in \mathscr{R}^{(6n+n_r)\times 2n}$ as

$$\mathbf{J} = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ \bar{B} & 0 & 0 \\ \bar{D} & I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \mathbf{L} = \begin{bmatrix} -I \\ \bar{A} \\ \bar{C} \\ \bar{A}_d \end{bmatrix}$$

To provide an alternative condition to (16), we give the following lemma, which follows from [21].

Lemma 3 P, Q are given in Theorem 1, $R \in \mathscr{R}^{2n\times(6n+n_r)}$. Let N be the null space of R. Then the following statements are equivalent.

i) The condition (16) holds and

$$N^T \Xi N < 0, \tag{18}$$

where

$$\Xi := \mathbf{J} \begin{bmatrix} -P & Q \\ * & P-(2\cos\rho)Q \end{bmatrix} \mathbf{J}^T +$$
$$\mathbf{H} \begin{bmatrix} I & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{H}^T + \mathbf{Z} \begin{bmatrix} \Theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Theta \end{bmatrix} \mathbf{Z}^T$$

ii) There exists $W \in \mathscr{R}^{2n\times 2n}$ such that

$$\mathbf{J} \begin{bmatrix} -P & Q \\ * & P-(2\cos\rho)Q \end{bmatrix} \mathbf{J}^T + \mathbf{H} \begin{bmatrix} I & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{H}^T +$$
$$\mathbf{Z} \begin{bmatrix} \Theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Theta \end{bmatrix} \mathbf{Z}^T < \mathbf{He}(\mathbf{L}WR) \tag{19}$$

Proof: Note that the null space of $\mathbf{L}$ is $\begin{bmatrix} \bar{A} & I & 0 & 0 \\ \bar{C} & 0 & I & 0 \\ \bar{A}_d & 0 & 0 & I \end{bmatrix}$, and use lemma 2, we have ii) is equivalent to i).

Remark 4 Condition (19) is still not an LMI as the product terms between the multiplier W and the filter parameters.

To overcome the difficulty, we introduce the change of variable in[22]. Let us partition W and its inverse as

$$W = \begin{bmatrix} X & U \\ * & \hat{X} \end{bmatrix}, W^{-1} = \begin{bmatrix} Y & V \\ * & \hat{Y} \end{bmatrix} \tag{20}$$

where $X, Y \in \mathscr{R}^{n\times n}$, $\hat{X}, \hat{Y} \in \mathscr{R}^{n_F\times n_F}$ are all symmetric matrices. From this partition of matrix W let us introduce the following one-to-one change of variables

$$\begin{bmatrix} K_f & L_f \\ M_f & N_f \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} M & G \\ H & L \end{bmatrix} \begin{bmatrix} U^T X^{-1} & 0 \\ 0 & I \end{bmatrix}^{-1} \tag{21}$$

Denoting $Z = X^{-1}$, and define

$$F = \begin{bmatrix} X^{-1} & Y \\ 0 & V^T \end{bmatrix}, \mathbf{F} := diag(F, F, I, I), \tag{22}$$

then we can obtain

$$\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix} := \begin{bmatrix} F^T \bar{A} W F & F^T \bar{B} \\ \bar{C} W F & \bar{D} \end{bmatrix} =$$
$$\left[\begin{array}{cc|c} ZA & ZA & ZB \\ YA+GC+M & YA+GC & YB+GD \\ \hline LC+H & LC & D_{11}+LD \end{array}\right] \tag{23}$$

and

$$\Delta := \bar{A}_d W F = \begin{bmatrix} A_d & A_d \\ 0 & 0 \end{bmatrix}, \tag{24}$$

$$\mathscr{W} := F^T W F = \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix} \tag{25}$$

Then, the next theorem gives a solution, expressed in terms of LMI, to the $\mathbf{H}_\infty$ fault detection filter problem stated above.

Theorem 2 Let $R \in \mathscr{R}^{2n\times(6n+n_r)}$, given $R = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$, then for system (1) there exists a filter (2) with $n = n_F$ that guarantees

$$\sigma_{max}(G_{\omega e}(e^{j\lambda})) < \gamma, \forall |\lambda| \le \rho \tag{26}$$

if there exists matrices Z,Y,M,G,H,L and symmetric matrices $\Theta,\mathscr{P},\mathscr{Q}>0,\mathscr{X},\Delta$ satisfying the following LMI

$$\begin{bmatrix} -\mathscr{P} & \mathscr{Q}+\mathscr{W} & 0 & 0 & 0 \\ * & \Xi & -\mathscr{C}^T & -\Delta^T & \mathscr{B} \\ * & * & -\gamma^2 I & 0 & \mathscr{D} \\ * & * & * & -\Theta & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{27}$$

where $\Xi = \mathscr{P} - 2(\cos\rho)\mathscr{Q} + \mathscr{X} - \mathscr{A} - \mathscr{A}^T$, $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D},\Delta,\mathscr{W}$ are defined in (23), (24) and (25). If condition (27) holds, the filter parameters (K_f,L_f,M_f,N_f) can be obtained by solving (21).

Proof: First, we proof if $R=\begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$, then (27) will imply the condition (18) in lemma 3. Note the null space of R can be denoted as $\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, then (18) can be convert into

$$\begin{bmatrix} -\mathscr{P} & 0 & 0 \\ 0 & \bar{D}\bar{D}^T-\gamma^2 I & 0 \\ 0 & 0 & -\Theta \end{bmatrix} < 0, \tag{28}$$

note $\bar{D}=\mathscr{D}$, using Schur complement formula, from (27), it is easy to see (28) will be hold. Then, if condition (27) holds, by lemma 3, (19) is equivalent to (16).

It can be verified that the inequality (19) multiplied to the left by the full rank matrix $\mathbf{F}^T$ and to the right by $\mathbf{F}$ provides the following inequality

$$\mathbf{J}\begin{bmatrix} -\mathscr{P} & \mathscr{Q} \\ * & \mathscr{P}-2(\cos\rho)\mathscr{Q} \end{bmatrix}\mathbf{J}^T + \mathscr{H}\begin{bmatrix} I & 0 \\ * & -\gamma^2 I \end{bmatrix}\mathscr{H}^T + \mathscr{K}\begin{bmatrix} \Theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Theta \end{bmatrix}\mathscr{K}^T < \mathbf{He}(\mathscr{L}R), \tag{29}$$

where $\mathscr{P}=F^TPF, \mathscr{Q}=F^TQF$ and

$$\mathscr{H}=\begin{bmatrix} 0 & 0 \\ \mathscr{B} & 0 \\ \mathscr{D} & I \\ 0 & 0 \end{bmatrix}, \mathscr{K}=\begin{bmatrix} 0 & 0 & 0 \\ F^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \mathscr{L}=\begin{bmatrix} -\mathscr{W} \\ \mathscr{A} \\ \mathscr{C} \\ \Delta \end{bmatrix} \tag{30}$$

then from $R=\begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$ and substituting (30) into (29), we get

$$\begin{bmatrix} -\mathscr{P} & \mathscr{Q}+\mathscr{W} & 0 & 0 \\ * & \Xi & -\mathscr{C}^T & -\Delta^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -\Theta \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{B} \\ \mathscr{D} \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ \mathscr{B} \\ \mathscr{D} \\ 0 \end{bmatrix}^T < 0 \tag{31}$$

where $\Xi=\mathscr{P}-2(\cos\rho)\mathscr{Q}+\mathscr{X}-\mathscr{A}-\mathscr{A}^T$ and $\mathscr{X}=F^T\Theta F$.

Using Schur complement, (31) imply (27). So we know (27) is equivalent to (16), which guarantee condition (15) holds, that is (26) will be held. This completes the proof.

B. Stability conditions

Conditions (27) don't ensure a stable filter, so we wish to add an additional constraint to guarantee the stability of the dual system of (3).

Lemma 4 The dual system of (3) is stable, if there exists matrices Z,Y,M,G and symmetric matrices $\mathscr{Q}_s>0,\mathscr{P}_1>0,\mathscr{Q}_1>0$, satisfying the following LMI

$$\begin{bmatrix} \mathscr{P}_1-2\mathscr{W} & \mathscr{A}^T & \Delta^T \\ * & \mathscr{Q}_1-\mathscr{P}_1 & 0 \\ * & * & -\mathscr{Q}_s \end{bmatrix} < 0 \tag{32}$$

where $\mathscr{A},\Delta,\mathscr{W}$ are defined in (23), (24) and (25).

C. Fault detection filter design

So the FDF design problem of (5) can be solved through the following optimization problem

$$\min\gamma$$

$$s.t. \ (27) \ \ (32) \tag{33}$$

Remark 7 Once we solve the LMIs (32) and (27), the filter parameters can be recovered as follows. First let U,V be any factor such that $VU^T=I-YX$ where non-singularity of $I-YX$ can be assumed without loss a generality due to the strictness of the LMIS. Then the filter parameters of filter can be obtained by solving (21) as following

$$\begin{bmatrix} K_f & L_f \\ M_f & N_f \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} M & G \\ H & L \end{bmatrix}\begin{bmatrix} U^TX^{-1} & 0 \\ 0 & I \end{bmatrix}^{-1} \tag{34}$$

D. Threshold design

After designing FDF, the remaining important task is the evaluation of the generated residual. One of the widely adopted approaches is to choose a so-called threshold $J_{th}>0$, and based on this, using the following logical relationship for fault detection:

$$J_r > J_{th} \Rightarrow with faults \Rightarrow alarm,$$

$$J_r \le J_{th} \Rightarrow no faults,$$

where the so-called residual evaluation function J_r is determined by $J_r(n)=\sqrt{\frac{1}{n}\sum_{k=0}^{k=n} r^T(k)r(k)}$, where n denotes the evaluation time steps. Here, we set

$$J_{th} = \sup_{f=0,d\in L_2,|\lambda|\le\rho} J_r.$$

IV. EXAMPLE

In this section, the theory developed in this paper is demonstrated by the following example

$$x(k+1)=Ax(k)+A_dx(k-\tau)+\begin{bmatrix} B_f & B_d \end{bmatrix}\begin{bmatrix} f(k) \\ d(k) \end{bmatrix},$$

$$y(k)=Cx(k)+\begin{bmatrix} D_f & D_d \end{bmatrix}\begin{bmatrix} f(k) \\ d(k) \end{bmatrix}, \tag{35}$$

with the following parameters

$$A = \begin{bmatrix} -0.4830 & -0.2478 \\ 0.4678 & 0.5242 \end{bmatrix}, A_d = \begin{bmatrix} -0.1204 & 0.0544 \\ -0.3946 & -0.3905 \end{bmatrix},$$

$$B_f = \begin{bmatrix} -0.1268 \\ 0.4663 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.4447 \\ -0.6628 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.4466 & -0.1644 \end{bmatrix}, \quad D_f = -0.7272,$$

$$D_d = 0.1724.$$

Assume the frequency range is constricted to be $|\rho| \leq \frac{\pi}{3}$. To achieve optimal estimate of the fault signal, it is formulated as an optimization problem to find an optimal γ subject to the LMI (27) and (32), then we obtain the filter parameter as

$$K_f = \begin{bmatrix} 0.2587 & 0.3092 \\ 0.2080 & 0.2475 \end{bmatrix}, \quad L_f = \begin{bmatrix} 10.1367 \\ 7.6504 \end{bmatrix},$$

$$M_f = \begin{bmatrix} -0.6910 & 0.8369 \end{bmatrix}, \quad N_f = \begin{bmatrix} -1.3020 \end{bmatrix}.$$

The optimal value for the $\mathbf{H}_\infty$ performance index γ is found to be 0.2307.

To illustrated the advantage of our approach, we compare it with the full frequency method. Consider system (3), to achieve condition (5), applying Theorem 1 of [23], we just need to satisfy the following LMI

$$\begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ * & \Phi_4 & 0 \\ * & * & \Phi_5 \end{bmatrix} > 0 \tag{36}$$

where

$$\Phi_1 = diag\left\{ \begin{bmatrix} R_1 & R_1 \\ * & X_1 \end{bmatrix}, I \right\} \tag{37}$$

$$\Phi_2 = \begin{bmatrix} R_1 A & R_1 A \\ X_1 A + Z_1 C + M_1 & X_1 A + Z_1 C \\ -N_f C - N_1 & -N_f C \end{bmatrix} \tag{38}$$

$$\Phi_3 = \begin{bmatrix} R_1 A_d & R_1 B \\ X_1 A_d & X_1 B + Z_1 D \\ 0 & T - N_f D \end{bmatrix} \tag{39}$$

$$\Phi_4 = \begin{bmatrix} R_1 - K_1 & R_1 - K_1 \\ * & X_1 - K_1 \end{bmatrix} \tag{40}$$

$$\Phi_5 = \begin{bmatrix} K_1 & 0 \\ * & \gamma^2 I \end{bmatrix} \tag{41}$$

and $R_1 \in \mathscr{R}^{n\times n}, X_1 \in \mathscr{R}^{n\times n}, K_1 \in \mathscr{R}^{n\times n}$ are symmetric matrices, matrices $M_1 \in \mathscr{R}^{n\times n}, N_1 \in \mathscr{R}^{n_r\times n}, Z_1 \in \mathscr{R}^{n\times n_y}$, furthermore, $K_f = (R_1 - X_1)^{-1} M_1, L_f = (R_1 - X_1)^{-1} Z_1, M_f = N_1$.

Then in full frequency range, given system (35) and the same parameters as above, by solving LMI (36), we obtain the filter parameter as

$$K_f = \begin{bmatrix} -0.0600 & 0.0248 \\ -0.0340 & 0.0192 \end{bmatrix}, \quad L_f = \begin{bmatrix} -0.0660 \\ -0.0541 \end{bmatrix},$$

$$M_f = \begin{bmatrix} 0.3382 & 0.1255 \end{bmatrix}, \quad N_f = \begin{bmatrix} 1.0926 \end{bmatrix}.$$

The optimal value for the $\mathbf{H}_\infty$ performance index γ is found to be 0.5178.

In order to show the effectiveness of our method more clearly, some simulations are also given.

First, the system is simulated with a stuck fault signal f(t) such that $f(t) = 5, t \geq 6s$ and $f(t) = 0$ elsewhere.(See Fig.1)

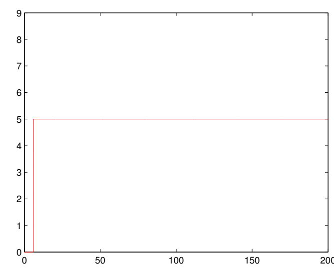

Fig. 1. Fault signal

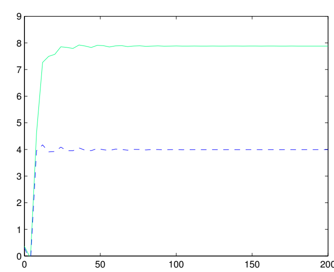

Fig. 2. Residual outputs of the low frequency method(solid lines) and full frequency method(dashed lines) with $d(t) = 0$

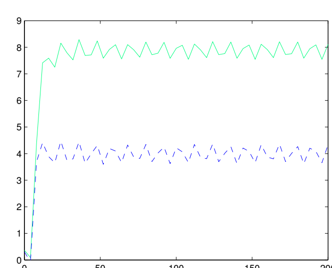

Fig. 3. Residual outputs of the low frequency method(solid lines) and full frequency method(dashed lines) with $d(t) = sin(t)$

Figure (2)-(3) represent the residual behavior when this large fault is introduced. As clear from Figures (2) and (3),

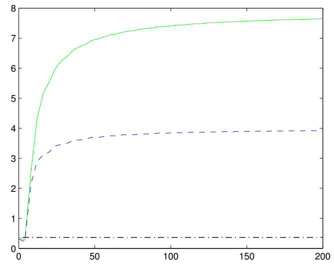

Fig. 4. Residual evaluation of the low frequency method(solid lines) and full frequency method(dashed lines) and the threshold(dash-dot lines)

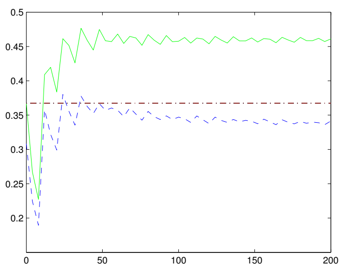

Fig. 5. Residual evaluation of the low frequency method(solid lines) and full frequency method(dashed lines) and the threshold(dash-dot lines)

the response of residual in low frequency domain is much sensitive than that in full range. Although a rigorous analysis has not been present here, it should also be obvious that the proposed finite-frequency approach will receive better results.

The residual evaluation function J_r is reported in Fig.4 for the same disturbance and fault signal in Fig.3. For this example, the threshold obtained through the approach proposed in section III is 0.3674 which is denoted by dash-dot lines.

With $d(t) = sin(t)$, Fig.5 simulates the residual evaluation function J_r for the small fault $f(t) = 0.25, t \geq 6s$ and $f(t) = 0$ elsewhere.

It's evident form Fig.4 and Fig.5, that a large fault can be detected by using finite-frequency or full-frequency approach, however, full frequency scheme can cause the small faults to go undetected. So Fig.5 shows the significance of the proposed finite-frequency FD approach in detecting small and low-frequency faults.

V. Conclusions

In this paper, we have investigated the problem of fault estimations for linear discrete-time delay systems. The main results in [17] have been employed to formulate the fault detection filter design problem in low frequency domain and the filter design has been formulated as an $\mathbf{H}_\infty$ optimization problem. A numerical example has been given to illustrate the effectiveness of the proposed method.

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