2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Output regulation for a class of nonlinear systems using the observer based output feedback control

Dabo Xu and Jie Huang

Abstract— **In this paper we study the global robust output regulation problem for a class of nonlinear systems by output feedback control. The class of systems possesses nonlinear zero-dynamics and, is thus considerably larger than systems studied in the existing literature. As an illustration of our approach, we have applied our approach to the global robust asymptotic tracking problem of the hyperchaotic Lorenz system.**

Index Terms–**adaptive control, output regulation, nonlinear systems.**

I. Introduction

In this paper, we consider the global robust output regulation problem for the following class of uncertain nonlinear systems

$$\begin{aligned}\dot{z} &= f(z, y, v, w)\\ \dot{x}_i &= x_{i+1} + g_i(z, y, v, w),\ i = 1, \cdots, r-1\\ \dot{x}_r &= b_\infty u + g_r(z, y, v, w)\\ y &= x_1\\ e &= x_1 - q(v, w)\end{aligned} \tag{1}$$

where $(z, x) \in \mathbb{R}^n \times \mathbb{R}^r$ with $r \geq 2$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, e represents the tracking error, $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ is an uncertain parameter vector with $\mathbb{W}$ an arbitrarily prescribed subset of $\mathbb{R}^{n_w}$, and $v(t) \in \mathbb{R}^{n_v}$ is an exogenous signal representing both reference input and disturbance. It is assumed that $v(t)$ is generated by a linear system of the following form

$$\dot{v} = A_1 v, \quad v(0) = v_0 \tag{2}$$

where all the eigenvalues of matrix A_1 are simple with zero real part. All functions in (1) are supposed to be globally defined, sufficiently smooth, and satisfy $f(0, 0, 0, w) = 0$, $g_i(0, 0, 0, w) = 0$, and $q(0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$. The quantity b_∞ is called high frequency gain and is assumed to be a nonzero constant with an unknown sign.

The precise statement of our problem is given as follows.

Problem 1.1: Design a dynamic output feedback control law of the form

$$u = u_K(\zeta, e),\ \dot{\zeta} = g_K(\zeta, e) \tag{3}$$

where $\zeta \in \mathbb{R}^{n_\zeta}$ for some integer $n_\zeta > 0$, and u_K and g_K are globally defined sufficiently smooth functions vanishing at the origin, such that, for all initial conditions, and all $w \in \mathbb{W}$, the trajectory of the closed-loop system composed of (1) to (3) exists and is bounded over $[0, +\infty)$, and the error output $e(t)$ asymptotically approaches zero as $t \to +\infty$. ∎

When the exogenous signal $v(t)$ is not present in system (1), or what is the same, the dimension of v is zero, system (1) is the same as the so-called output feedback system. The global robust stabilization problem for such systems has been studied in [6], [8]. For this reason we will also call (1) as output feedback system. It is known that the robust output regulation problem is typically handled by the internal model approach [1], [4]. The internal model approach consists of two steps. In the first step, an appropriate dynamic compensator called internal model is designed. Attachment of the internal model to the given plant leads to an augmented system. The internal model has the property that the stabilization solution of the augmented system will lead to the output regulation solution of the given plant and the exosystem. Thus, the second step is to globally stabilize the augmented system. It is noted that, due to the attachment of the internal model, the augmented system may be more complicated than the original system. Therefore, the stabilization problem of the augmented system can also be more challenging than that of the original system with v set to zero.

A subclass of (1) is given as follows

$$\begin{aligned}\dot{z} &= H(w)z + g_0(y, w)\\ \dot{x}_i &= x_{i+1} + g_i(y, w),\ i = 1, \cdots, r-1\\ \dot{x}_r &= b_\infty u + Q(w)z + g_r(y, w)\\ y &= x_1\end{aligned} \tag{4}$$

where $H(w)$ and $Q(w)$ are matrices of appropriate dimensions, and $H(w)$ is Hurwitz for each constant uncertainty w [11]. A disturbance rejection problem for system (4) has been studied in [2] which can be viewed as a special case of the output regulation. A special feature of (4) is that its zero dynamics $\dot{z} = H(w)z$ is a linear stable system. This special feature lends itself to an effective approach to designing an output feedback control law without employing some type of observer. In contrast, system (1) does not possess this feature and hence the approach in [2] is not applicable here. Therefore, we need to employ a different technique to tackle our problem which involves the use of some observer. Another feature of our problem is that we will not assume the knowledge of the sign of the high frequency gain b_∞.

The organization of this paper is as follows. In Section II, we introduce a set of basic assumptions on system (1) so that the robust output regulation problem of system (1) can be converted into a global robust stabilization problem of an

Dabo Xu and Jie Huang are with the Applied Control and Computing Laboratory, Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong, China. (dbxu@mae.cuhk.edu.hk, jhuang@mae.cuhk.edu.hk)

978-1-4244-4524-0/09/$25.00 ©2009 AACC

augmented system based on the general framework in [4]. Section III presents the main result of this paper. A design example is illustrated in Section IV. Section V concludes this paper with a few remarks.

II. Assumptions and Problem Conversion

It is known from [4] that the output regulation problem of a given plant can be converted into the global robust stabilization problem of an augmented system composed of the given plant and the so-called internal model. To accomplish this conversion, we will first list some standard assumptions as follows.

Assumption 1: There exists a smooth function $\mathbf{z}(v,w): \mathbb{R}^{n_v+n_w} \mapsto \mathbb{R}^n$ with $\mathbf{z}(0,0)=0$ such that

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} A_1 v = f\big(\mathbf{z}(v,w), q(v,w), v, w\big) \tag{5}$$

for all $(v,w) \in \mathbb{R}^{n_v} \times \mathbb{W}$. ∎

Let

$$\mathbf{x}(v,w) = \text{col}(\mathbf{x}_1(v,w), \cdots, \mathbf{x}_r(v,w))$$

with $\mathbf{x}_1(v,w) = q(v,w)$ and for $i = 2, \cdots, r$,

$$\begin{aligned} \mathbf{x}_i(v,w) &= L_{A_1 v}\mathbf{x}_{i-1}(v,w) \\ &\quad - g_{i-1}(\mathbf{z}(v,w), q(v,w), v, w) \\ \mathbf{u}(v,w) &= b_\infty^{-1}\big[L_{A_1 v}\mathbf{x}_r(v,w) \\ &\quad - g_r(\mathbf{z}(v,w), q(v,w), v, w)\big] \end{aligned}$$

where $L_{A_1 v} q(v,w) = \frac{\partial q(v,w)}{\partial v} A_1 v$. Then, under Assumption 1, the solution to the regulator equations associated with system (1) and exosystem (2) is given by $\mathbf{z}(v,w)$, $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$.

Assumption 2: There exist an integer n_s, a sufficiently smooth function $\tau: \mathbb{R}^{n_v+n_w} \mapsto \mathbb{R}^{n_s}$ vanishing at the origin, and a pair of matrices $\Phi \in \mathbb{R}^{n_s \times n_s}$ and $\Psi \in \mathbb{R}^{1 \times n_s}$, such that

$$\frac{d\tau(v,w)}{dt} = \Phi\tau(v,w), \;\; \mathbf{u}(v,w) = \Psi\tau(v,w) \tag{6}$$

for all $(v,w) \in \mathbb{R}^{n_v} \times \mathbb{W}$. Moreover, the pair (Ψ, Φ) is observable and all the eigenvalues of Φ are simple with zero real part. ∎

System (6) is called a steady-state generator in [4]. Assumption 2 guarantees the existence of the internal model. In fact, under Assumption 2, the Sylvester equation

$$T\Phi - MT = N\Psi \tag{7}$$

has a unique nonsingular solution T for a given controllable pair (M,N) with M a Hurwitz matrix and N a vector of appropriate dimensions [12]. Let $\theta(v,w) = T\tau(v,w)$. Then, we have

$$\begin{aligned} \dot{\theta}(v,w) &= (M + N\Psi_o)\theta(v,w) \\ \mathbf{u}(v,w) &= \Psi_o\theta(v,w) \end{aligned} \tag{8}$$

where $\Psi_o = \Psi T^{-1}$. Next, we can define the following dynamics

$$\dot{\eta} = M\eta + Nu \tag{9}$$

as an internal model with output u [4], [12].

Attaching the internal model (9) to system (1) and performing the following coordinate and input transformation

$$\begin{aligned} \bar{z} &= z - \mathbf{z}(v,w), \quad \bar{x} = x - \mathbf{x}(v,w) \\ \bar{\eta} &= \eta - \theta(v,w), \quad \bar{u} = u - \Psi_o\eta \end{aligned} \tag{10}$$

yields

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= (M + N\Psi_o)\bar{\eta} + N\bar{u} \\ \dot{\bar{x}}_i &= \bar{x}_{i+1} + \bar{g}_i(\bar{z}, e, v, w), \;\; i = 1, \cdots, r-1 \\ \dot{\bar{x}}_r &= b_\infty\bar{u} + b_\infty\Psi_o\bar{\eta} + \bar{g}_r(\bar{z}, e, v, w) \end{aligned} \tag{11}$$

where $\bar{x} = \text{col}(\bar{x}_1, \cdots, \bar{x}_r)$ and

$$\begin{aligned} \bar{f}(\bar{z}, e, v, w) &= f(\bar{z} + \mathbf{z}(v,w), e + q(v,w), v, w) \\ &\quad - f\big(\mathbf{z}(v,w), q(v,w), v, w\big) \\ \bar{g}_i(\bar{z}, e, v, w) &= g_i(\bar{z} + \mathbf{z}(v,w), e + q(v,w), v, w) \\ &\quad - g_i(\mathbf{z}(v,w), q(v,w), v, w) \end{aligned} \tag{12}$$

for $i = 1, \cdots, r$.

System (11) is called augmented system and has the following property [4]:

$$\bar{f}(0,0,v,w) = 0, \quad \bar{g}_i(0,0,v,w) = 0, \quad i = 1, \cdots, r \tag{13}$$

for all $(v,w) \in \mathbb{R}^{n_v} \times \mathbb{W}$. Therefore, by Corollary 3.1 in [4], the global robust output regulation problem of system (1) as described in Problem 1.1 will be solved if the following global robust stabilization problem for system (11) is solvable.

Problem 2.1: Design a dynamic output feedback control law of the form

$$\bar{u} = \bar{u}_K(\bar{\zeta}, e), \;\; \dot{\bar{\zeta}} = \bar{g}_K(\bar{\zeta}, e) \tag{14}$$

where $\bar{\zeta} \in \mathbb{R}^{n_{\bar{\zeta}}}$ for some integer $n_{\bar{\zeta}} > 0$. $\bar{u}_K$ and $\bar{g}_K$ are globally defined sufficiently smooth functions vanishing at the origin such that, for any fixed $w \in \mathbb{W}$ and any $v(t)$ generated by (2), the solution of the closed-loop system composed of (11) and (14) is bounded and $\bar{x}_1(t)(= e(t))$ approaches zero asymptotically. ∎

III. Main Result

A specific difficulty with the global robust stabilization problem of system (11) is that it is not in the output feedback form as displayed in (1) due to the presence of the internal model. Moreover, like $\bar{z}$, the state $\bar{\eta}$ is not available for feedback. Nevertheless, performing, as in [7], the following coordinate transformation on (11)

$$\tilde{\eta} = \bar{\eta} - c_r\bar{x}_r - \cdots - c_1\bar{x}_1 \tag{15}$$

where $c_r = b_\infty^{-1}N$, $c_{i-1} = Mc_i$ for $i = 2, \cdots, r$, gives

$$\begin{aligned} \dot{\tilde{\eta}} &= M\tilde{\eta} + Mc_1 e - \sum_{i=1}^{r} c_i\bar{g}_i(\bar{z}, e, v, w) \\ \dot{\bar{x}} &= A_s\bar{x} + b_\infty B\Psi_o\tilde{\eta} + \bar{g}(\bar{z}, e, v, w) + b_\infty B\bar{u} \end{aligned} \tag{16}$$

where $\bar{g}(\bar{z}, e, v, w) = \text{col}(\bar{g}_1(\bar{z}, e, v, w), \cdots, \bar{g}_r(\bar{z}, e, v, w))$

$$A_s = \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline s_r & s_{r-1}, \cdots, s_1 \end{array}\right], \; B = \text{col}(\underbrace{0, \cdots, 0}_{r-1}, 1) \quad (17)$$

and real scalars $s_i = b_\infty \Psi_o c_{r+1-i}$ for $i = 1, \cdots, r$. Further performing another coordinate transformation on $\bar{x}$-system

$$\xi = b_\infty^{-1} U_s \cdot \bar{x} \quad (18)$$

where

$$U_s = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -s_1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ -s_{r-2} & -s_{r-3} & \cdots & 1 & 0 \\ -s_{r-1} & -s_{r-2} & \cdots & -s_1 & 1 \end{bmatrix} \quad (19)$$

gives

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + Mc_1 e - \sum_{i=1}^{r} c_i \bar{g}_i(\bar{z}, e, v, w) \\ \dot{\xi} &= A_c \xi + B\Psi_o \tilde{\eta} + G(\bar{z}, e, v, w) + B\bar{u} \end{aligned} \quad (20)$$

where $A_c = \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline 0 & 0 \end{array}\right]$ and

$$\begin{aligned} G(\bar{z}, e, v, w) &= \text{col}\big(G_1(\bar{z}, e, v, w), \cdots, \\ &\qquad G_r(\bar{z}, e, v, w)\big) \\ G_1(\bar{z}, e, v, w) &= s_1 e + b_\infty^{-1} \bar{g}_1(\bar{z}, e, v, w) \\ G_i(\bar{z}, e, v, w) &= s_i e - b_\infty^{-1} \sum_{j=1}^{i-1} s_j \bar{g}_j(\bar{z}, e, v, w) \\ &\quad + b_\infty^{-1} \bar{g}_i(\bar{z}, e, v, w) \end{aligned} \quad (21)$$

for $i = 2, \cdots, r$. It is noted that U_s is such that

$$U_s A_s U_s^{-1} = \left[\begin{array}{c|c} s_{[r-1]} & I_{r-1} \\ \hline s_r & 0 \end{array}\right] \quad (22)$$

where $s_{[r-1]} = \text{col}(s_1, \cdots, s_{r-1})$ and $e = \bar{x}_1 = b_\infty \xi_1$.

As our purpose is to design an output feedback control law that only relies on $e(t)$, we need to introduce some sort of observer to estimate the state $\xi(t)$. We will adopt a standard observer such as what can be found in [6] as follows:

$$\dot{\hat{\xi}} = A_c \hat{\xi} + \lambda(e - \hat{\xi}_1) + B\bar{u} \quad (23)$$

where $\lambda = \text{col}(\lambda_1, \cdots, \lambda_r)$ is chosen such that the matrix $A_o = \left[\begin{array}{c|c} -\lambda_{[r-1]} & I_{r-1} \\ \hline -\lambda_r & 0 \cdots 0 \end{array}\right]$ is Hurwitz. The observation error $\tilde{\xi} = \xi - \hat{\xi}$ satisfies

$$\dot{\tilde{\xi}} = A_o \tilde{\xi} - \lambda(1 - b_\infty^{-1})e + B\Psi_o \tilde{\eta} + G(\bar{z}, e, v, w). \quad (24)$$

Attaching (24) to (20) and replacing the state variable vector ξ by $(e, \hat{\xi}_2, \cdots, \hat{\xi}_r)$ gives the following system

$$\begin{aligned} \dot{z} &= F(z, e, \mu) \\ \dot{e} &= b_\infty \hat{\xi}_2 + b_\infty \tilde{\xi}_2 + b_\infty G_1(\bar{z}, e, \mu) \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e - \hat{\xi}_1), \; i = 2, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= \bar{u} + \lambda_r(e - \hat{\xi}_1) \end{aligned} \quad (25)$$

where $z = \text{col}(\bar{z}, \tilde{\eta}, \tilde{\xi})$, $\mu = (v, w)$, and

$$F(z, e, \mu) = \begin{bmatrix} \bar{f}(\bar{z}, e, v, w) \\ M\tilde{\eta} + Mc_1 e - \sum_{i=1}^{r} c_i \bar{g}_i \\ A_o \tilde{\xi} - \lambda(1 - b_\infty^{-1})e + B\Psi_o \tilde{\eta} + G \end{bmatrix}.$$

It can be seen that system (25) is in a standard lower triangular form. To guarantee the solvability of the global stabilization problem for such a system, we need one more assumption as follows.

Assumption 3: For any compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathbb{W}$, there exists a C^1 function $V_{\bar{z}}(\bar{z})$ satisfying $\underline{\alpha}_{\bar{z}}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\alpha}_{\bar{z}}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\bar{z}}(\cdot)$ and $\bar{\alpha}_{\bar{z}}(\cdot)$ such that, for any $(v, w) \in \Sigma$, along the trajectory of $\bar{z}$ subsystem

$$\frac{\partial V_{\bar{z}}}{\partial \bar{z}}(\bar{z}) \cdot \bar{f}(\bar{z}, e, v, w) \leq -\alpha_{\bar{z}}(\|\bar{z}\|) + \delta_e \gamma_e(|e|) \quad (26)$$

where δ_e is some unknown constant, $\alpha_{\bar{z}}(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+} \sup(\alpha_{\bar{z}}^{-1}(s^2)/s) < \infty$, and $\gamma_e(\cdot)$ is a known smooth positive definite function. ∎

We are now ready to construct our control law using a recursive method modified from the tuning function approach described in [9]. For this purpose, we introduce the following notation.

$$\begin{aligned} \kappa_1(e, k) &= \mathcal{N}(k)\rho(e)e, \\ \kappa_2(e, k, \hat{b}, \hat{\xi}_1, \hat{\xi}_2) &= -2\omega_1 - \lambda_2(e - \hat{\xi}_1) \\ &\quad -\hat{b}E_1\hat{\xi}_2 - \omega_1 E_1^2 - K_1, \\ \phi_2(e, k, \hat{b}, \hat{\xi}_2) &= \omega_1 E_1 \hat{\xi}_2, \\ \kappa_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i) &= -\omega_{i-2} - \omega_{i-1} - \lambda_i(e - \hat{\xi}_1) \\ &\quad + \sum_{j=1}^{i-1} \frac{\partial \kappa_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j + \frac{\partial \kappa_{i-1}}{\partial \hat{b}}\dot{\hat{b}} \\ &\quad -\hat{b}E_{i-1}\hat{\xi}_2 - \omega_{i-1}E_{i-1}^2 \\ &\quad -K_{i-1}, \; i = 3, \cdots, r, \\ \phi_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i) &= \phi_{i-1} + \omega_{i-1}E_{i-1}\hat{\xi}_2 \\ &\quad i = 3, \cdots, r \end{aligned} \quad (27)$$

where, for $i = 1, \cdots, r$, $E_i = -\frac{\partial \kappa_i}{\partial e}$ and $K_i = -\frac{\partial \kappa_i}{\partial k}\dot{k}$, k is a variable governed by the second equation of (30), $\mathcal{N}(k)$ is a Nussbaum-type function (see [13]), for instance, $\mathcal{N}(k) = \exp(k^2)\cos(0.5\pi k)$ or $\mathcal{N}(k) = k^2\cos(k)$, $\hat{b}$ is an estimate for b_∞ and is governed by the third equation of (30), and $\rho(e) \geq 1$ is a positive continuous function. Also, for $i = 1, \cdots, r-1$, let

$$\omega_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i) = \hat{\xi}_{i+1} - \kappa_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i). \quad (28)$$

For convenience, we let $\omega_r = 0$ and $\hat{\xi}_{r+1} = \bar{u}$. The derivative of ω_i satisfies

$$\begin{aligned} \dot{\omega}_1 &= \dot{\hat{\xi}}_2 - \dot{\kappa}_1 = \omega_2 + \kappa_2 + \lambda_2(e - \hat{\xi}_1) \\ &\quad + E_1(b_\infty \hat{\xi}_2 + b_\infty \tilde{\xi}_2 + b_\infty G_1) + K_1 \\ \dot{\omega}_i &= \dot{\hat{\xi}}_{i+1} - \dot{\kappa}_i = \omega_{i+1} + \kappa_{i+1} + \lambda_{i+1}(e - \hat{\xi}_1) \\ &\quad + E_i \dot{e} - \sum_{j=1}^{i} \frac{\partial \kappa_i}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j - \frac{\partial \kappa_i}{\partial \hat{b}}\dot{\hat{b}} + K_i \end{aligned} \quad (29)$$

for $i = 2, \cdots, r-1$.

Theorem 3.1: Under Assumptions 1 to 3, there exist a sufficiently smooth function $\rho(e) \geq 1$ and a control law of the form

$$\begin{aligned}\bar{u} &= \kappa_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) \\ \dot{k} &= \rho(e)e^2 \\ \dot{\hat{b}} &= \phi_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r)\end{aligned} \quad (30)$$

that solves Problem 2.1. ∎

Outline of the Proof: For any given $v_0 \in \mathbb{R}^{n_v}$ and $w \in \mathbb{W}$, there exists a compact set Σ such that $\mu(t) = (v(t), w) \in \Sigma$ for all $t \geq 0$. Performing the recursive method using the notations of (27), (28) and (29), we can get a C^1 function $V(z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1})$, where $\tilde{b}(t) = b_\infty - \hat{b}(t)$, satisfying $\underline{\alpha}(\| z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1} \|) \leq V \leq \bar{\alpha}(\| z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1} \|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$, such that, along the trajectory of the closed-loop system composed of system (25) and control law (30)

$$\dot{V} \leq \Big(b_\infty \mathcal{N}(k) + p\Big)\dot{k} - \sum_{j=1}^{r-1} \omega_j^2 - \|z\|^2 \quad (31)$$

for some continuous function $\rho(e) \geq 1$ and some constant $p > 0$.

Integrating both sides of (31) over $[0, t)$, $\forall\, t \geq 0$ gives

$$V(t) \leq \int_0^t \Big(b_\infty \mathcal{N}(k(\tau)) + p\Big)\dot{k}(\tau) d\tau + V(0). \quad (32)$$

As in [15], the above inequality implies that $V(t)$ and $k(t)$ are bounded over each time interval $[0, T)$ with $0 < T \leq +\infty$. So the solution of the closed-loop system composed of system (25) and control law (30) is defined on $[0, +\infty)$ and bounded over $[0, +\infty)$.

We now show $e(t)$ will approach the origin as $t \to +\infty$. Since $k(t)$ is bounded over $[0, +\infty)$ and $\dot{k}(t) = \rho(e)e^2$ with $\rho(e) \geq 1$, e is square integrable over $[0, +\infty)$. Furthermore, both $e(t)$ and $\dot{e}(t)$ are bounded over $[0, +\infty)$. Using Barbalat's lemma concludes that $e(t)$ tends to zero as $t \to +\infty$. This completes the proof.

Remark 3.1: As a result of the above theorem, the following control law

$$\begin{aligned}u &= \kappa_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) + \Psi_o \eta \\ \dot{k} &= \rho(e)e^2 \\ \dot{\hat{b}} &= \phi_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) \\ \dot{\eta} &= M\eta + Nu \\ \dot{\hat{\xi}} &= A_c\hat{\xi} + \lambda(e - \hat{\xi}_1) + Bu - B\Psi_o\eta\end{aligned} \quad (33)$$

which is in the form (3) solves the global robust output regulation problem for system (1). ∎

IV. Example

The controlled single-input single-output hyperchaotic Lorenz system [10] is described by the following equations:

$$\begin{aligned}\dot{z}_1 &= a_{11}z_1 + a_{12}x_1 \\ \dot{z}_2 &= a_3 z_2 + z_1 x_1 \\ \dot{x}_1 &= x_2 + a_{21}z_1 + a_{22}x_1 - z_1 z_2 \\ \dot{x}_2 &= b_\infty u + a_4 z_1\end{aligned} \quad (34)$$

where $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4)$ is a constant parameter vector satisfying $a_{11}, a_3 < 0$ and b_∞ is some unknown nonzero constant. A detailed analysis of this system with $u = 0$ has been given in [10] and various types of chaotic behaviors for different values of parameter $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4)$ are exhibited. Also, a full state feedback stabilization of this system is studied in [5]. Here, by designating an output $y = x_1$ and defining a tracking error $e = y - F(t)$ where $F(t) = A_m \sin(\omega t + \phi)$, we will consider a more challenging control problem of designing an error output feedback control law such that all the states of the closed-loop system is bounded and the tracking error e asymptotically approaches zero. To make the problem more interesting, we allow the amplitude A_m to be an arbitrary positive number and the phase angle ϕ an arbitrary real number. We will show that the above problem can be formulated as the global robust output regulation problem described in Section II.

In fact, let $v = \mathrm{col}(v_1, v_2)$ and define a linear autonomous system in the form (2) as follows

$$\begin{bmatrix}\dot{v}_1 \\ \dot{v}_2\end{bmatrix} = \begin{bmatrix}0 & \omega \\ -\omega & 0\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix}, \ v_0 := \begin{bmatrix}v_{10} \\ v_{20}\end{bmatrix}. \quad (35)$$

It can be seen that $v_1(t) = F(t)$ if $(v_{10}, v_{20}) = (A_m \sin\phi, A_m \cos\phi)$. Also, we allow the parameter vector $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4)$ to undergo some perturbation. To be more specific, let

$$a = (\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3, \bar{a}_4) + (w_1, \cdots, w_6)$$

where $(\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3, \bar{a}_4)$ represents the nominal value of a and $(w_1, \cdots, w_6)$ the uncertainty of a. To guarantee $a_{11}, a_3 < 0$, we define $\mathbb{W}$ as $\mathbb{W} = \{w \mid w \in \mathbb{R}^6, \bar{a}_{11} + w_1 < 0, \bar{a}_3 + w_5 < 0\}$.

System (34) is in the form (1) with $r = 2$ and it cannot be transformed into the form (4). Therefore, none of existing results, e.g., the design method in [2], can solve Problem 1.1 for system (34).

It can be easily verified that the regulator equations associated with (34) and (35) are solvable. In fact, from the error equation $e = x_1 - v_1$, we have

$$\mathbf{x}_1(v, w) = v_1. \quad (36)$$

Substituting (36) into the first equation of (34) yields

$$\mathbf{z}_1(v, w) = r_{11}v_1 + r_{12}v_2 \quad (37)$$

where

$$r_{11} = -\frac{a_{11}a_{12}}{\omega^2 + a_{11}^2}, \quad r_{12} = -\frac{a_{12}\omega}{\omega^2 + a_{11}^2}.$$

Substituting (36) and (37) into the second equation of (34) gives

$$\mathbf{z}_2(v, w) = r_{21}v_1^2 + r_{22}v_1v_2 + r_{23}v_2^2 \quad (38)$$

where

$$\begin{aligned}r_{21}(w, \omega) &= -\frac{a_3^2 r_{11} - a_3\omega r_{12} + 2\omega^2 r_{11}}{a_3(a_3^2 + 4\omega^2)} \\ r_{22}(w, \omega) &= -\frac{r_{12}a_3 + 2\omega r_{11}}{a_3^2 + 4\omega^2}, \ r_{23}(w, \omega) = \frac{\omega}{a_3}r_{22}.\end{aligned}$$

Substituting (36) and (38) into the third equation of (34) gives

$$\begin{aligned}\mathbf{x}_2(v,w) &= \omega v_2 - a_{22}v_1 - a_{21}\mathbf{z}_1 + \mathbf{z}_1\mathbf{z}_2 \\ &= r_{31}v_1 + r_{32}v_2 + r_{33}v_1^3 + r_{34}v_1^2 v_2 \\ &\quad + r_{35}v_1 v_2^2 + r_{36}v_2^3 \end{aligned} \tag{39}$$

where

$$\begin{aligned} r_{31} &= -a_{22} - a_{21}r_{11},\ r_{32} = \omega - a_{21}r_{12} \\ r_{33} &= r_{11}r_{21},\ r_{34} = r_{12}r_{21} + r_{11}r_{22} \\ r_{35} &= r_{11}r_{23} + r_{12}r_{22},\ r_{36} = r_{12}r_{23}. \end{aligned}$$

Thus $\mathbf{x}_2(v,w)$ can be put into the following form

$$\mathbf{x}_2(v,w) = \mathcal{X}_{21}(w)v^{[1]} + \mathcal{X}_{23}(w)v^{[3]} \tag{40}$$

where $v^{[1]} = \mathrm{col}(v_1, v_2)$, $v^{[3]} = \mathrm{col}(v_1^3, v_1^2 v_2, v_1 v_2^2, v_2^3)$, and $\mathcal{X}_{21}(w), \mathcal{X}_{23}(w)$ are appropriate row vectors. Finally, substituting (40) into the fourth equation of (34) gives

$$\begin{aligned}\mathbf{u}(v,w) &= \sum_{l=1,3} b_\infty^{-1}\mathcal{X}_{2l}(w)A^{[l]}v^{[l]} \\ &\quad - b_\infty^{-1}a_4\mathbf{z}_1(v,w) \\ &= r_{41}v_1 + r_{42}v_2 + r_{43}v_1^3 + r_{44}v_1^2 v_2 \\ &\quad + r_{45}v_1 v_2^2 + r_{46}v_2^3 \end{aligned} \tag{41}$$

where

$$A^{[1]} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, A^{[3]} = \begin{bmatrix} 0 & 3\omega & 0 & 0 \\ -\omega & 0 & 2\omega & 0 \\ 0 & -2\omega & 0 & \omega \\ 0 & 0 & -3\omega & 0 \end{bmatrix}$$

and

$$\begin{aligned} r_{41}(w,\omega) &= -b_\infty^{-1}(\omega r_{32} + a_4 r_{11}) \\ r_{42}(w,\omega) &= b_\infty^{-1}(\omega r_{31} - a_4 r_{12}) \\ r_{43}(w,\omega) &= -b_\infty^{-1}\omega r_{34} \\ r_{44}(w,\omega) &= b_\infty^{-1}\omega(3r_{33} - 2r_{35}) \\ r_{45}(w,\omega) &= b_\infty^{-1}\omega(2r_{34} - 3r_{36}) \\ r_{46}(w,\omega) &= b_\infty^{-1}\omega r_{35}. \end{aligned}$$

Therefore, Assumption 1 and Assumption 2 are satisfied. The steady-state generator described by (6) can be constructed as follows

$$\begin{aligned} \tau(v,w) &= \mathrm{col}\big(\mathbf{u}, L_{A_1 v}\mathbf{u}, L_{A_1 v}^2\mathbf{u}, L_{A_1 v}^3\mathbf{u}\big) \\ \Phi &= \left[\begin{array}{c|c} 0 & I_3 \\ \hline -9\omega^4 & 0, -10\omega^2, 0 \end{array}\right] \\ \Psi &= [1,0,0,0]. \end{aligned} \tag{42}$$

So we can define the following internal model

$$\dot{\eta} = M\eta + Nu \tag{43}$$

where

$$M = \left[\begin{array}{c|c} 0 & I_3 \\ \hline -m_1 & -m_2, -m_3, -m_4 \end{array}\right],\ N = \mathrm{col}(0,0,0,1)$$

and the parameters $m_i > 0$ are such that M is Hurwitz.

Solving the Sylvester equation (7) to obtain the nonsingular matrix T and performing the transformation (10) gives

$$\begin{aligned} \dot{\bar{z}}_1 &= a_{11}\bar{z}_1 + a_{12}e \\ \dot{\bar{z}}_2 &= a_3\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1 \\ \dot{\bar{\eta}} &= (M + N\Psi_o)\bar{\eta} + N\bar{u} \\ \dot{\bar{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\bar{x} + \begin{bmatrix} \bar{g}_1 \\ \Psi_o\bar{\eta} + \bar{u} + \bar{g}_2 \end{bmatrix} \end{aligned} \tag{44}$$

where $\bar{g}_1 = a_{21}\bar{z}_1 + a_{22}e - (\bar{z}_1 + \mathbf{z}_1)(\bar{z}_2 + \mathbf{z}_2) + \mathbf{z}_1\mathbf{z}_2$ and $\bar{g}_2 = a_4\bar{z}_1$.

To verify Assumption 3, for any fixed compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathbb{W}$, let $V_{\bar{z}} = \frac{\hbar}{2}\bar{z}_1^2 + \frac{\hbar}{4}\bar{z}_1^4 + \frac{1}{2}\bar{z}_2^2$ for some $\hbar > 0$ which satisfies along the trajectory of system (44)

$$\begin{aligned} \dot{V}_{\bar{z}} &= \hbar a_{11}\bar{z}_1^2 + \hbar a_{12}\bar{z}_1 e + \hbar a_{11}\bar{z}_1^4 + \hbar a_{12}\bar{z}_1^3 e \\ &\quad + a_3\bar{z}_2^2 + \bar{z}_2\big((\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1\big) \\ &= \hbar a_{11}\bar{z}_1^2 + \hbar a_{12}\bar{z}_1 e + \hbar a_{11}\bar{z}_1^4 + \hbar a_{12}\bar{z}_1^3 e \\ &\quad + a_3\bar{z}_2^2 + \bar{z}_2\bar{z}_1 e + v_1\bar{z}_2\bar{z}_1 + \mathbf{z}_1\bar{z}_2 e. \end{aligned} \tag{45}$$

Using Young's inequality gives, for any $\varepsilon > 0$,

$$\begin{aligned} \hbar a_{12}\bar{z}_1 e &\le \frac{1}{2}\bar{z}_1^2 + \frac{\hbar^2 a_{12}^2}{2}e^2 \\ \hbar a_{12}\bar{z}_1^3 e &\le \frac{3}{4}\bar{z}_1^4 + \frac{\hbar^4 a_{12}^4}{4}e^4 \\ \bar{z}_2\bar{z}_1 e &\le \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{1}{2\varepsilon}\bar{z}_1^2 e^2 \\ &\le \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{1}{4}\bar{z}_1^4 + \frac{1}{4\varepsilon^2}e^4 \\ v_1\bar{z}_2\bar{z}_1 &\le \frac{1}{2\varepsilon}\bar{z}_1^2 + \frac{\varepsilon v_1^2}{2}\bar{z}_2^2 \\ \mathbf{z}_1\bar{z}_2 e &\le \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{\mathbf{z}_1^2}{2\varepsilon}e^2. \end{aligned} \tag{46}$$

Substituting (46) into (45) gives

$$\begin{aligned} \dot{V}_{\bar{z}} &\le \Big(\hbar a_{11} + \frac{1}{2} + \frac{1}{2\varepsilon}\Big)\bar{z}_1^2 + \Big(\hbar a_{11} + 1\Big)\bar{z}_1^4 \\ &\quad + \Big(a_3 + \varepsilon + \frac{\varepsilon v_1^2}{2}\Big)\bar{z}_2^2 + \Big(\frac{\hbar^2 a_{12}^2}{2} + \frac{\mathbf{z}_1^2}{2\varepsilon}\Big)e^2 \\ &\quad + \Big(\frac{\hbar^4 a_{12}^4}{4} + \frac{1}{4\varepsilon^2}\Big)e^4. \end{aligned} \tag{47}$$

Since Σ is compact, for all $(v,w) \in \Sigma$, there exist a sufficiently small $\varepsilon > 0$, a sufficiently large $\hbar > 0$, and constant $\ell_i > 0$, $i = 1, \cdots, 5$, such that

$$a_3 + \varepsilon + \frac{\varepsilon v_1^2}{2} < -\ell_1 \tag{48}$$

$$\hbar a_{11} + \frac{1}{2} + \frac{1}{2\varepsilon} < -\ell_2 < 0, \quad \hbar a_{11} + 1 < -\ell_3 < 0 \tag{49}$$

and

$$\frac{\hbar^2 a_{12}^2}{2} + \frac{\mathbf{z}_1^2}{2\varepsilon} < \ell_4, \quad \frac{\hbar^4 a_{12}^4}{4} + \frac{1}{4\varepsilon^2} < \ell_5. \tag{50}$$

It follows from (47) to (50) that

$$\dot{V}_{\bar{z}} \le -\ell_2\bar{z}_1^2 - \ell_3\bar{z}_1^4 - \ell_1\bar{z}_2^2 + \ell_4 e^2 + \ell_5 e^4. \tag{51}$$

Thus Assumption 3 is satisfied for $(\bar{z}_1, \bar{z}_2)$ subsystem. By Theorem 3.1, the global robust output regulation problem for

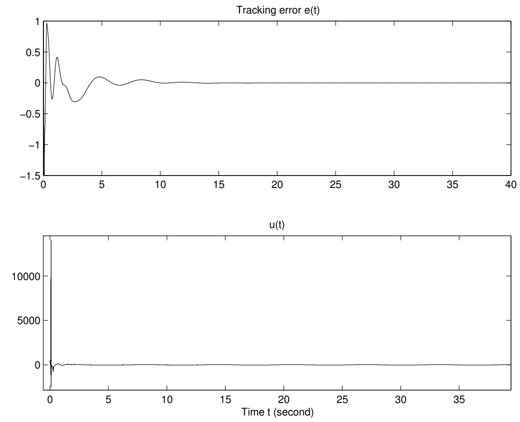

Fig. 1. Profile of tracking error $e(t)$ and control input $u(t)$.

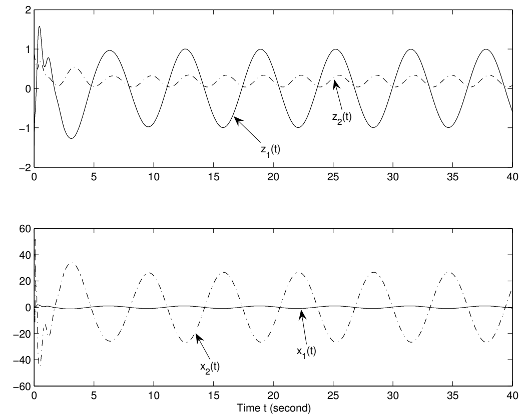

Fig. 2. Plant state responses of (z_1, z_2, x_1, x_2).

this system is solvable. In fact, by choosing

$$c_1 = b_\infty^{-1} MN,\ c_2 = b_\infty^{-1} N,\ s_1 = \Psi_o N,\ s_2 = \Psi_o MN$$

performing the transformations (15) and (18), and incorporating the observer (23), we can obtain system (25) as follows

$$\begin{aligned}
\dot{\bar{z}}_1 &= a_{11}\bar{z}_1 + a_{12}e \\
\dot{\bar{z}}_2 &= a_3\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1 \\
\dot{\tilde{\eta}} &= M\tilde{\eta} + (Mc_1 e - c_1\bar{g}_1 - c_2\bar{g}_2) \\
\dot{\tilde{\xi}} &= \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 0 \end{bmatrix}\tilde{\xi} + \begin{bmatrix} -\lambda_1(1 - b_\infty^{-1})e + G_1 \\ -\lambda_2(1 - b_\infty^{-1})e + \Psi_o\tilde{\eta} + G_2 \end{bmatrix} \\
\dot{e} &= b_\infty\hat{\xi}_2 + b_\infty\tilde{\xi}_2 + b_\infty G_1 \\
\dot{\hat{\xi}}_2 &= u - \Psi_o\eta + \lambda_2(e - \hat{\xi}_1) \qquad (52)
\end{aligned}$$

where $G_1 = s_1 e + b_\infty^{-1}\bar{g}_1$ and $G_2 = s_2 e - b_\infty^{-1} s_1\bar{g}_1 + b_\infty^{-1}\bar{g}_1$. According to the design procedure detailed in Section 3, we can obtain a specific control law in the form (33) with $\rho(e) = 3(e^6 + 1)$.

Simulations are performed for the closed-loop system composed of system (34) and a controller in the form (33). Various parameters are chosen as follows. $\lambda = \mathrm{col}(2, 3)$; $(m_1, m_2, m_3, m_4) = (4, 12, 13, 6)$; $\omega = 1$; $b_\infty = 1$; $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4) = (-10, 10, 28, -1, -8/3, -1)$. The initial conditions are $(z_1(0), z_2(0), x_1(0), x_2(0)) = (-2, 1, 2, 1)$, $v_0 = \mathrm{col}(1, 0)$, $\eta(0) = 0$, $\hat{\xi}(0) = 0$, $\hat{b}(0) = 0$, and $k(0) = 1$. The responses of the tracking error, control input, and the plant state variables are shown in Figures 1 and 2.

V. Conclusion

In this paper, we have presented the solvability conditions for the global robust output regulation problem for nonlinear system (1) by output feedback control. Since the zero dynamics of our system is not linear, the existing approach as used in [2] is not applicable here. Moreover, our approach does not assume the sign of high frequency gain is known. It should be noted that when $r = 1$, there is no need to design an observer to estimate e. Therefore, a simpler approach can be used to handle this case.

To illustrate the effectiveness of our approach, we have applied our approach to the global robust asymptotic tracking problem of the well known hyperchaotic Lorenz system.

Acknowledgement

The work was substantially supported by the Research Grants Council of the Hong Kong Special Administration Region under grant No. CUHK412006.

References

[1] C. I. Byrnes, F. Delli Priscoli, and A. Isidori. *Output Regulation of Uncertain Nonlinear Systems*. Birkhäuser-Boston, 1997.

[2] Z. Ding. Universal disturbance rejection for nonlinear systems in output feedback form. *IEEE Trans. Automat. Contr.*, vol. 48, no. 7, pp. 1222-1227, 2003.

[3] J. Huang. *Nonlinear Output Regulation: Theory and Applications*. Philadelphia, PA: SIAM, 2004.

[4] J. Huang and Z. Chen. A general framework for tackling the output regulation problem. *IEEE Trans. Automat. Contr.*, vol. 49, no. 12, pp. 2203-2218, 2004.

[5] Q. Jia. Hyperchaos generated from the lorenz chaotic system and its control. *Physics Letters A*, vol. 366, no. 3, pp. 217–222, 2007.

[6] Z.-P. Jiang. A combined backstepping and small-gain approach to adaptive output feedback control. *Automatica*, vol. 35, no. 6, pp. 1131-1139, 1999.

[7] Z.-P. Jiang and M. Arcak. Robust global stabilization with ignored input dynamics: an input-to-state stability (ISS) small-gain approach. *IEEE Trans. Automat. Contr.*, vol. 46, no. 8, pp. 1336-1342, 2001.

[8] Z.-P. Jiang, I. M. Y. Mareels, D. J. Hill, and J. Huang. A unifying framework for global regulation via nonlinear output feedback: from ISS to iISS. *IEEE Trans. Automat. Contr.*, Vol. 49, No. 4, pp. 549-561, 2004.

[9] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear & Adaptive Control Design*. New York: Wiley, 1995.

[10] Y. Li, W. K. S. Tang, and G. Chen. Hyperchaos evolved from the generalized lorenz equation. *Int. J. Circ. Theor. Appl.*, vol. 33, pp. 235–251, 2005.

[11] R. Marino and P. Tomei. *Nonlinear Control Design: Geometric, Adaptive and Robust*. Prentice Hall International (UK) Limited, London, 1995.

[12] V. O. Nikiforov. Adaptive non-linear tracking with complete compensation of unknown disturbances. *European Journal of Control*, vol. 4, pp. 132-139, 1998.

[13] R. D. Nussbaum. Some remarks on a conjecture in parameter adaptive control. *Systems & Control Letters*, vol. 3, no.5, pp. 243-246, 1983.

[14] E. D. Sontag and A. R. Teel. Changing supply functions in input/state stable systems. *IEEE Trans. Automat. Contr.*, vol. 40, no. 8, pp. 1476-1478, 1995.

[15] X. Ye. Asymptotic regulation of time-varying uncertain nonlinear systems with unknown control directions, *Automatica*, vol. 35, no.5, pp. 929-935, 1999.