2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Analysis and Control of Hybrid Systems with Parameter Uncertainty Based on Interval Methods

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***Abstract*— In this paper, by applying interval methods, a new framework for analysis and control of discrete-time hybrid systems with parameter uncertainty is proposed. In analysis and control of hybrid systems, there are problem formulations such that convex polyhedra are computed, but for high-dimensional systems, it is difficult to solve these problems within the practical computation time. In this paper, instead of computing convex polyhedra, an interval method, which is one of classical methods in verified numerical computation, is applied to analysis and control of hybrid systems. By applying an interval method, discrete-time piecewise systems with parameter uncertainty can be approximately transformed into a mixed logical dynamical model.**

I. Introduction

In many cases of analysis and control of hybrid systems, one of the technical difficulties is that the computation time to solve the analysis/control problems is too long. For example, in some of the verification problems and the controllability problems of hybrid systems, it is necessary to compute convex polyhedra [3], [4], [5]. However, the computation of convex polyhedra is difficult for high-dimensional systems, and it will be desirable from the practical viewpoint to compute an approximation of convex polyhedra.

On the other hand, as well as linear systems and nonlinear systems, it is important to consider hybrid systems with parameter uncertainty. For instance, mechanical systems with friction phenomena are well-known as one of typical examples of hybrid systems, but it is difficult to precisely identify friction phenomena. Recent literatures in robust control of hybrid systems are presented in [13], but these results are complicate from the theoretical viewpoint, and it will be desirable to consider a simpler approach from the practical viewpoint.

In this paper, based on interval methods [9], a new framework for analysis and control of discrete-time hybrid systems with parameter uncertainty is proposed. An interval method is based on interval arithmetic, and is well-known as one of classical techniques in verified numerical computation. By applying an interval method, a convex polyhedron is approximated as a box (an interval). So an over-approximation of a convex polyhedron is obtained. Obviously, an approximation via an interval is conservative, but the computation using an interval method is relatively easier than the computation of convex polyhedra. As applications of an interval method to control theory and theoretical computer science, the trajectory generation problem [12] and the reachability problem [10], [11] have been considered. However, in these works, an interval of the state at each time is computed by recursively solving some algorithm. So it is difficult to extend these approaches to the control problem. In this paper, by approximately expressing interval arithmetic on matrix computations, discrete-time piecewise affine systems with parameter uncertainty is transformed into a mixed logical dynamical (MLD) model [6]. By using the obtained MLD model, a kind of the optimal control problem can be solved. Therefore, by using the proposed method, for example, a design of controller satisfying state/input constraints for discrete-time hybrid systems with parameter uncertainty can be realized. Furthermore, from the result in this paper, we will give a new potentiality for the MLD model framework.

This paper is organized as follows. In Section II, some basics of interval arithmetic are explained. In Section III, we propose the method to approximately transform discrete-time linear systems with parameter uncertainty into the MLD model. In Section IV, the result on linear systems is extend to piecewise affine systems. In Section V, as applications of the obtained model, the trajectory generation problem, the controllability problem and the optimal control problem are considered. In Section VI, numerical examples on a piecewise linear system are shown. In Section VII, we conclude this paper.

Notation: Let $\mathcal{R}$ express the set of real numbers. Let $\{0,1\}^{m\times n}$ express the set of $m\times n$ matrices, which consists of elements 0 and 1. Let I_n, $0_{m\times n}$ express the $n\times n$ identity matrix, the $m\times n$ zero matrix, respectively. The matrix inequality $X \le Y$ denotes that $X_{ij} - Y_{ij}$ is nonpositive, where X_{ij}, Y_{ij} is the (i,j)-th element of X, Y, respectively. For a vector x, let $x^{(i)}$ express the i-th element of x. For a matrix/vector M, the matrix/vector $|M|$ denotes that each element of $|M|$ is given by an absolute value of each element of M. For simplicity of notation, we sometimes use the symbol 0 instead of $0_{m\times n}$, and the symbol I instead of I_n.

II. Interval Arithmetic

In this section, some basics of interval arithmetic are explained. See [9] for further details.

First, an *interval* is defined as the following bounded set of real numbers

$$[\underline{x},\overline{x}] := \{x\in\mathcal{R} \mid \underline{x}\le x\le\overline{x}\}$$

where $\underline{x}\le\overline{x}\in\mathcal{R}$ holds, and $\underline{x}$, $\overline{x}$ are the infimum and the supremum of the interval, respectively. For simplicity of notation, we may denote $[\underline{x},\overline{x}]$ as $[x]$.

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Suppose that two intervals $[x]$ and $[y]$ are given. Then four operations, addition $+$, multiplication $\times$, subtraction $-$, and division $\div$ of $[x]$ and $[y]$ are given as follows:

$$
\begin{aligned}
&[x]+[y]=[\underline{x}+\underline{y},\overline{x}+\overline{y}], \qquad (1)\\
&[x]-[y]=[\underline{x}-\overline{y},\overline{x}-\underline{y}],\\
&[x]\times[y]=\left[\min\{\underline{x}\underline{y},\overline{x}\overline{y},\underline{x}\overline{y},\overline{x}\underline{y}\},\max\{\underline{x}\underline{y},\overline{x}\overline{y},\underline{x}\overline{y},\overline{x}\underline{y}\}\right],\\
&[x]\div[y]=[x]\times\left[\frac{1}{\overline{y}},\frac{1}{\underline{y}}\right],\quad 0\notin[y].
\end{aligned}
$$

Next, an interval is extended to an interval matrix (vector). An interval matrix is defined as

$$[X]=[\underline{X},\overline{X}]:=\{X\in\mathcal{R}^{m\times n}\mid \underline{X}\le X\le\overline{X}\}$$

where $\underline{X},\overline{X}\in\mathcal{R}^{m\times n}$. Also, the center $c([X])$ and the radius $r([X])$ of an interval matrix $[X]$ are defined as

$$c([X]):=(\overline{X}+\underline{X})/2,\quad r([X]):=(\overline{X}-\underline{X})/2,$$

respectively. From the definitions of $c([X])$ and $r([X])$,

$$[\underline{X},\overline{X}]=[c([X])-r([X]),c([X])+r([X])]$$

holds. Then we introduce the result on the multiplication of two interval matrices $[X]$ and $[Y]$ [9].

Lemma 1: Suppose that interval matrices $[X]$ and $[Y]$ are given. Then the following condition holds:

$$
\begin{aligned}
[X]\times[Y] &= [c([X])-r([X]),c([X])+r([X])]\\
&\quad\times[c([Y])-r([Y]),c([Y])+r([Y])]\\
&\subseteq [c([X])c([Y])-r([X])|c([Y])|\\
&\quad -|c([X])|r([Y])-r([X])r([Y]),\\
&\quad c([X])c([Y])+r([X])|c([Y])|\\
&\quad +|c([X])|r([Y])+r([X])r([Y])]. \qquad (2)
\end{aligned}
$$

Note here that in Lemma 1, if $[X]$ or $[Y]$ is given as some point ($r([X])=0$ or $r([Y])=0$), then the equality in (2) holds. Furthermore, (2) is an over-approximation of $[X]\times[Y]$, but the size of the obtained over-approximation is less than about 1.5 times of the accurate interval.

By Lemma 1, we can approximately express discrete-time linear systems with parameter uncertainty as a mixed logical dynamical (MLD) systems [6]. In Section III, this fact will be shown. After that, in Section IV, we will extend discrete-time linear systems to discrete-time piecewise affine (DT-PWA) systems.

III. Modeling of Discrete-time Linear Systems with Parameter Uncertainty

In this section, based on Lemma 1, we consider to express discrete-time linear systems with parameter uncertainty as the MLD model.

Consider the following discrete-time linear system with parameter uncertainty

$$
\begin{aligned}
&x(k+1)=Ax(k)+Bu(k)+a, \qquad (3)\\
&x(k)\in[\underline{x}(k),\overline{x}(k)],\\
&A\in[\underline{A},\overline{A}],\quad B\in[\underline{B},\overline{B}],\quad a\in[\underline{a},\overline{a}]
\end{aligned}
$$

where $\underline{x}(k),\overline{x}(k)\in\mathcal{R}^n$, $u(k)\in\mathcal{R}^m$, and a is an affine term. It is remarked that the state $x(k)$ is given by an interval, because the system (3) has parameter uncertainty, and even if an initial state $x(0)$ is given by some point ($\underline{x}(0)=\overline{x}(0)$), $x(k)$ becomes some region. Furthermore, for simplicity of notation, we denote $c([\underline{x}(k),\overline{x}(k)])$ and $r([\underline{x}(k),\overline{x}(k)])$ as $x_c(k)$ and $x_r(k)$, respectively. Then the center $x_c(k)$ and the radius $x_r(k)$ of $[\underline{x}(k),\overline{x}(k)]$ are given by

$$\begin{bmatrix} x_c(k)\\ x_r(k)\end{bmatrix}=\frac{1}{2}\begin{bmatrix} I_n & I_n\\ -I_n & I_n\end{bmatrix}\begin{bmatrix}\underline{x}(k)\\ \overline{x}(k)\end{bmatrix}, \qquad (4)$$

respectively. Similarly, the center A_c and the radius A_r of $[\underline{A},\overline{A}]$ and the center B_c and the radius B_r of $[\underline{B},\overline{B}]$ are given by $A_c=(\overline{A}+\underline{A})/2$, $A_r=(\overline{A}-\underline{A})/2$, and $B_c=(\overline{B}+\underline{B})/2$, $B_r=(\overline{B}-\underline{B})/2$, respectively.

By using Lemma 1, we can approximately calculate the interval $[\underline{x}(k+1),\overline{x}(k+1)]$ of the state at time $k+1$. The result is shown by the following lemma.

Lemma 2: Suppose that the system (3) is given. Then the interval $[\underline{x}(k+1),\overline{x}(k+1)]$ of the state at time $k+1$ is approximately derived by

$$
\begin{aligned}
&[\underline{x}(k+1),\overline{x}(k+1)]\\
&=[A_c-A_r,A_c+A_r][x_c(k)-x_r(k),x_c(k)+x_r(k)]\\
&\quad +[B_c-B_r,B_c+B_r]u(k)+[\underline{a},\overline{a}]\\
&\subseteq [A_cx_c(k)-A_r|x_c(k)|-|A_c|x_r(k)-A_rx_r(k)\\
&\quad +B_cu(k)-B_r|u(k)|+\underline{a},\\
&\quad A_cx_c(k)+A_r|x_c(k)|+|A_c|x_r(k)+A_rx_r(k)\\
&\quad +B_cu(k)+B_r|u(k)|+\overline{a}]\\
&=:[\underline{x}'(k+1),\overline{x}'(k+1)]. \qquad (5)
\end{aligned}
$$

Proof: From Lemma 1 and (1), we can obtain Lemma 2 straightforwardly. ■

By using Lemma 2, for a given interval of initial state $[\underline{x}(0),\overline{x}(0)]$ and a given input sequence $u(0),u(1),\ldots$, an approximate interval of the state at each time can be calculated as $[\underline{x}'(1),\overline{x}'(1)],[\underline{x}'(2),\overline{x}'(2)],\ldots$. In this paper, we use $[\underline{x}'(k+1),\overline{x}'(k+1)]$ of (5) as an approximate interval of $[\underline{x}(k+1),\overline{x}(k+1)]$, and consider the relation between $[\underline{x}'(k),\overline{x}'(k)]$ and $[\underline{x}'(k+1),\overline{x}'(k+1)]$ for a given $[\underline{x}(0),\overline{x}(0)]$. For simplicity of discussion, we omit "$'$" (dash) in $[\underline{x}'(k),\overline{x}'(k)]$ hereafter. Then from (5), we obtain

$$
\begin{aligned}
\underline{x}(k+1) &= -A_r|x_c(k)|-(|A_c|+A_r)x_r(k)\\
&\quad -B_r|u(k)|+A_cx_c(k)+B_cu(k)+\underline{a}, \qquad (6)\\
\overline{x}(k+1) &= A_r|x_c(k)|+(|A_c|+A_r)x_r(k)\\
&\quad +B_r|u(k)|+A_cx_c(k)+B_cu(k)+\overline{a}. \qquad (7)
\end{aligned}
$$

$|x_c(k)|$ and $|u(k)|$ is transformed into a linear form with continuous variables and binary variables by applying the following lemma.

Lemma 3: For a given vector $w\in\mathcal{R}^n$, $|w|$ is rewritten as

$$
\begin{aligned}
&|w|=2z-w,\\
&z^{(i)}=\delta^{(i)}w^{(i)},\quad i=1,2,\ldots,n, \qquad (8)\\
&\left[\delta^{(i)}=1\right]\leftrightarrow\left[w^{(i)}\ge 0\right],\quad i=1,2,\ldots,n \qquad (9)
\end{aligned}
$$

where $z \in \mathcal{R}^n$, $\delta \in \{0,1\}^n$ are auxiliary continuous variables and auxiliary binary variables, respectively. Further, (9) is a logical formula, and "$\leftrightarrow$" denotes logical equivalence.

Proof: From (8) and (9), if $w^{(i)} \geq 0$ then, $z^{(i)} = w^{(i)}$ and $2z^{(i)} - w^{(i)} = w^{(i)}$ hold, and if $w^{(i)} < 0$ then, $z^{(i)} = 0$ and $2z^{(i)} - w^{(i)} = -w^{(i)}$ hold. So $2z^{(i)} - w^{(i)} = |w^{(i)}|$ holds. ■

Note that (8) and (9) can be expressed as linear inequalities. See [6] for further details. Therefore, $|x_c(k)|$ and $|u(k)|$ can be transformed into some linear form with continuous variables and binary variables.

Thus we obtain the following theorem.

Theorem 1: Suppose that the discrete-time linear system with parameter uncertainty (3) is given. Then (3) is approximately expressed by the following representation

$$\left\{ \begin{array}{l} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{v}(k) + \hat{a}, \\ \hat{C}\hat{x}(k) + \hat{D}\hat{v}(k) \leq \hat{E} \end{array} \right. \tag{10}$$

where

$$\hat{x}(k) = \left[\begin{array}{c} \underline{x}(k) \\ \overline{x}(k) \end{array} \right], \quad \hat{v}(k) = \left[\begin{array}{c} u(k) \\ \hat{z}(k) \\ \hat{\delta}(k) \end{array} \right]$$

and $\hat{z}(k) \in \mathcal{R}^{n+m}$, $\hat{\delta}(k) \in \{0,1\}^{n+m}$. $\hat{A}$, $\hat{B}$, $\hat{a}$, $\hat{C}$, $\hat{D}$ and $\hat{E}$ are some matrices/vectors.

Proof: By applying Lemma 3 to (6) and (7), and by transforming $x_c(k)$, $x_r(k)$ into $\underline{x}(k)$, $\overline{x}(k)$ via (4), we obtain (10). ■

Since from Theorem 1, (10) is equivalent to the MLD model, we see that discrete-time linear systems with parameter uncertainty can be approximately represented as a kind of hybrid systems. Also, for (10), suppose that $\hat{x}(0)$ and the input sequence $u(0), u(1), \ldots, u(f-1)$ are given. Then the problem to find the state sequence $\hat{x}(1), \hat{x}(2), \ldots, \hat{x}(f)$ can be rewritten as a mixed integer feasibility test (MIFT) problem with continuous variables $\hat{z}(k)$ and binary variables $\hat{\delta}(k)$. The MIFT problem can be solved by using a suitable solver, e.g. ILOG CPLEX [14].

IV. Modeling of Discrete-time Piecewise Affine Systems with Parameter Uncertainty

In this section, the result on discrete-time linear systems is extended to DT-PWA systems, and as well as discrete-time linear systems, DT-PWA system with parameter uncertainty is transformed into the MLD model.

Consider the following DT-PWA system with parameter uncertainty

$$\left\{ \begin{array}{l} x(k+1) = A_{I(k)}x(k) + B_{I(k)}u(k) + a_{I(k)}, \\ I(k+1) = I_+, \quad \text{if } x(k+1) \in \mathcal{S}_{I_+} \end{array} \right. \tag{11}$$

where

$$\begin{array}{l} x(k) \in [\underline{x}(k), \bar{x}(k)], \quad \underline{x}(k), \bar{x}(k) \in \mathcal{X} \subset \mathcal{R}^n, \\ u(k) \in \mathcal{U} \subset \mathcal{R}^m, \\ A_{I(k)} \in \left[\underline{A}_{I(k)}, \overline{A}_{I(k)}\right], \quad B_{I(k)} \in \left[\underline{B}_{I(k)}, \overline{B}_{I(k)}\right], \\ a_{I(k)} \in \left[\underline{a}_{I(k)}, \overline{a}_{I(k)}\right] \end{array}$$

and $I(k) \in \mathcal{M} := \{1, 2, \ldots, M\}$ is the mode of system, M is the number of modes, $\mathcal{X}, \mathcal{U}$ are closed and bounded convex sets. Also, $\mathcal{S}_I$, $I = 1, 2, \ldots, M$ is the bounded convex polyhedron satisfying $\bigcup_{I \in \mathcal{M}} \mathcal{S}_I = \mathcal{X}$ and $\mathcal{S}_I \bigcap \mathcal{S}_J = \emptyset$ for all $I \neq J \in \mathcal{M}$. For simplicity of discussion, the following assumption is made for $\mathcal{X}$ and $\mathcal{S}_I$:

Assumption 1: $\mathcal{X}$ and $\mathcal{S}_I$, $I = 1, 2, \ldots, M$ are expressed by an interval.

Consider to express the system (11) as the MLD model. First, a binary variable $\delta_i(k) \in \{0,1\}$, $i = 1, 2, \ldots, M$ is assigned to each mode, i.e., we assign $\delta_i(k)$ such as $[\delta_i(k) = 1] \leftrightarrow [x(k) \in \mathcal{S}_i]$. By using auxiliary binary variables, this condition can be expressed as a set of linear inequalities [6]. In the standard DT-PWA systems without parameter uncertainty, the equality constraint $\sum_{i=1}^{M} \delta_i(k) = 1$ is imposed for guaranteeing the uniqueness of a mode sequence. However, since the DT-PWA system (11) has parameter uncertainty, more than one mode may be active. So in this paper, $\sum_{i=1}^{M} \delta_i(k) = 1$ is not imposed. By using (10) and $\delta_i(k)$, as an expression to approximately represent (11), we obtain

$$\left\{ \begin{array}{l} \hat{x}(k+1) = \displaystyle\sum_{I=1}^{M+N} \delta_I(k) \left\{ \hat{A}_I \hat{x}(k) + \hat{B}_I \hat{v}_I(k) + \hat{a}_I \right\}, \\ \displaystyle\sum_{I=1}^{M+N} \delta_I(k) \left\{ \hat{C}_I \hat{x}(k) + \hat{D}_I \hat{v}_I(k) \right\} \leq \sum_{I=1}^{M+N} \delta_I(k) \hat{E}_I \end{array} \right. \tag{12}$$

where $\hat{v}_1(k) = \hat{v}_2(k) = \cdots = \hat{v}_M(k) (= \hat{v}(k))$ and $\hat{v}_I(k) = [\ \hat{v}^T(k) \ \ \hat{w}^T(k)\ ]^T$, $I = M+1, M+2, \ldots, M+N$, $\hat{w}(k)$ is auxiliary continuous and binary variables. In addition, $\hat{x}(k+1) = \hat{A}_I \hat{x}(k) + \hat{B}_I \hat{v}_I(k) + \hat{a}_I$, $\hat{C}_I \hat{x}(k) + \hat{D}_I \hat{v}_I(k) \leq \hat{E}_I$, $I = M+1, M+2, \ldots, N$ is the state equation in the case that multiple modes are active simultaneously, and can be derived by simple calculations (The state interval for each mode is used). Since $[\underline{x}(k), \overline{x}(k)]$ is in general included in more than one $\mathcal{S}_{I(k)}$, it is necessary to derive such a state equation.

Furthermore, since (12) can be expressed by a linear state equation and a linear inequality [6], DT-PWA system with parameter uncertainty of (11) can be approximately expressed as the following form, i.e., the MLD model

$$\left\{ \begin{array}{l} x(k+1) = Ax(k) + Bv(k), \\ Cx(k) + Dv(k) \leq E \end{array} \right. \tag{13}$$

where $x(k) = [\underline{x}^T(k) \ \ \overline{x}^T(k)]^T \in \mathcal{R}^{2n}$ is a vector consisting the infimum and the supremum of an interval of the state $x(k)$, and $v(k)$ is given by $v(k) = [\ u^T(k) \ \ z^T(k) \ \ \delta^T(k)\ ]^T$, $u(k) \in \mathcal{R}^m$ is the control input, and $z(k) \in \mathcal{R}^{m_1}$, $\delta(k) \in \{0,1\}^{m_2}$ are auxiliary continuous and binary variables, respectively. A, B, C, D, and E are some vector/matrices.

From the above discussion, we see that DT-PWA systems with parameter uncertainty can be expressed by the MLD model as well as the standard DT-PWA systems without parameter uncertainty. Therefore, the controllability problem and the optimal control problem can be solved in the framework of the MLD model.

Remark 1: In [12], continuous-time piecewise affine systems with parameter uncertainty are discretized with respect to time, using mode transitions in each interval between sampling points. In this paper, we consider the DT-PWA system (11) at first. It is one of future works to consider the case of time-discretized systems.

V. Application to Analysis and Control

In this section, using the obtained model of (13), the trajectory generation problem, the controllability problem, and the optimal control problem are discussed.

A. Preliminaries

As a preparation, some matrices are defined. Suppose that the MLD model (13) and the finite time f are given. First, a state sequence and an input sequence are denoted by

$$\bar{x} := [\ x^T(0)\ \ x^T(1)\ \ \cdots\ \ x^T(f)\ ]^T,$$
$$\bar{v} := [\ v^T(0)\ \ v^T(1)\ \ \cdots\ \ v^T(f-1)\ ]^T.$$

Then from the state equation of (13), we obtain

$$\bar{x} = \bar{A}x(0) + \bar{B}\bar{v} \tag{14}$$

where

$$\bar{A} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^f \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{f-1}B & \cdots & AB & B \end{bmatrix}.$$

Note here that these matrices are different to $\overline{A}$ and $\overline{B}$ of (3). Also, $\tilde{B} := \left[\ A^{f-1}B\ \ A^{f-2}\ \ \cdots\ \ B\ \right]$ are defined. Next, from the linear inequality of (13), we obtain

$$\bar{C}\bar{x} + \bar{D}\bar{v} \leq \bar{E}$$

where

$$\bar{C} = \begin{bmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} D & & 0 \\ & \ddots & \\ 0 & & D \\ 0 & \cdots & 0 \end{bmatrix},$$
$$\bar{E} = \left[\ E^T\ \ \cdots\ \ E^T\ \right]^T.$$

B. Trajectory Generation Problem

Consider the following trajectory generation problem.

Problem 1: Consider the DT-PWA system with parameter uncertainty (11). Suppose that the terminal time f, the interval of the initial state $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0] \subseteq \mathcal{X}$, and the input sequence $u(0), u(1), \ldots, u(f-1)$ are given. Then find an over-approximation of the interval of the state at each time $[\underline{x}(k), \overline{x}(k)]$, $k = 1, 2, \ldots, f$.

This problem can be solved by using the MLD model (13), i.e., this problem can be transformed into the following MIFT problem

$$\begin{aligned} \text{given}\quad & x_0, u(0), u(1), \ldots, u(f-1) \\ \text{find}\quad & \bar{v} \\ \text{subject to}\quad & (\bar{D} + \bar{C}\bar{B})\bar{v} \leq \bar{E} - \bar{C}\bar{A}x_0 \end{aligned}$$

where $x_0 := [\ \underline{x}_0^T\ \ \overline{x}_0^T\ ]^T$. By solving this MIFT problem, $z(k), \delta(k)$, $k = 0, 1, \ldots, f-1$ are obtained. Furthermore, from (14), we obtain $\bar{x}$ as an over-approximation of the interval of the state.

C. Controllability Problem

Based on [5], [7], we give the definition of controllability.

Definition 1: Suppose that for the system of (11), the terminal time f, the interval of the initial state $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0] \subseteq \mathcal{X}$, and the interval of the terminal state $\mathcal{X}_f = [\underline{x}_f, \overline{x}_f] \subseteq \mathcal{X}$ are given. Then the system (11) is said to be $(f, \mathcal{X}_0, \mathcal{X}_f)$-*controllable*, if for every $x(0) \in \mathcal{X}_0$, there exists an input sequence $u(0), u(1), \ldots, u(f-1)$ such that $[\underline{x}(f), \overline{x}(f)] \subseteq [\underline{x}_f, \overline{x}_f]$.

In general, $\mathcal{X}_0, \mathcal{X}_f$ are given as convex polyhedra, but in this paper, for simplicity of discussion, $\mathcal{X}_0, \mathcal{X}_f$ are given as intervals.

By using the MLD model (13), we can derive a sufficient condition for the system (11) to be $(f, \mathcal{X}_0, \mathcal{X}_f)$-controllable. The result is shown as the following theorem.

Theorem 2: Suppose that the MLD model (13), which approximately expresses the system (11), is given. Then the system (11) is $(f, \mathcal{X}_0, \mathcal{X}_f)$-controllable, if the following MIFT problem

$$\begin{aligned} \text{find}\quad & \bar{v} \\ \text{subject to}\quad & (\bar{D} + \bar{C}\bar{B})\bar{v} \leq \bar{E} - \bar{C}\bar{A}x_0 \\ & \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tilde{B}\bar{v} \leq \begin{bmatrix} -\underline{x}_f \\ \overline{x}_f \end{bmatrix} \\ & \qquad\qquad - \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} A^f x_0 \end{aligned}$$

is feasible, where $x_0 := [\ \underline{x}_0^T\ \ \overline{x}_0^T\ ]^T$.

Proof: In Definition 1, the condition $[\underline{x}(f), \overline{x}(f)] \subseteq [\underline{x}_f, \overline{x}_f]$ is equivalent to $\underline{x}_f \leq \underline{x}(f)$ and $\overline{x}(f) \leq \overline{x}_f$. Since

$$x(f) = \begin{bmatrix} \underline{x}(f) \\ \overline{x}(f) \end{bmatrix} = A^f x(0) + \tilde{B}\bar{v}$$

holds, the second inequality condition of the MIFT problem is obtained. The first inequality condition corresponds to the inequality of the MLD model (13). Thus we obtain the MIFT problem. If the MIFT problem is infeasible, then there does not exist an input sequence $u(0), u(1), \ldots, u(f-1)$ satisfying $[\underline{x}(f), \overline{x}(f)] \subseteq [\underline{x}_f, \overline{x}_f]$, i.e., the system (11) is not $(f, \mathcal{X}_0, \mathcal{X}_f)$-controllable. This completes the proof. ■

By solving the MIFT problem in Theorem 2, we can check the controllability of the system (11).

D. Optimal Control Problem

Since the DT-PWA system with parameter uncertainty (11) can be expressed as the MLD model (13), we can consider control problems. For example, we can derive a controller satisfying a kind of temporal logic constraints, e.g. time-varying state/input constraints. Such constraints can be embedded in the MLD model (13). The obtained MLD model is also time-varying, but this complexity does not produce any difficulty. On the other hand, in this paper, we consider a kind of the optimal control problem with time-invariant state/input constraints as one of simple control problems.

Consider the following problem.

Problem 2: Consider the MLD model of (13), which approximately represents the DT-PWA system with parameter uncertainty (11). Suppose that the initial state $x(0) = x_0$ is given. Then find $v^*(k)$, $k = 0, 1, \ldots, f-1$, minimizing the cost function

$$J = \sum_{i=0}^{f-1} \left\{ x^T(i) Q x(i) + v^T(i) R v(i) \right\} + x^T(f) Q_f x(f)$$

where Q, Q_f are semi-positive matrices, and R is a positive matrix.

In Problem 2, as one of methods to give weight matrices Q, Q_f and R, we can consider to minimize a weighted sum of $x_c(k)$ and $x_r(k)$, which are the center and the radius of the interval of the state (see (4)). Then the cost function is given by

$$\begin{aligned} J = & \sum_{i=0}^{f-1} \left\{ x_c^T(i) Q_c x_c(i) + x_r^T(i) Q_r x_r(i) + v^T(i) R v(i) \right\} \\ & + x_c^T(f) Q_{cf} x_c(f) + x_r^T(f) Q_{rf} x_r(f) \qquad (15) \\ = & \sum_{i=0}^{f-1} \left\{ x^T(i) \begin{bmatrix} (Q_c + Q_r)/4 & (Q_c - Q_r)/4 \\ (Q_c - Q_r)/4 & (Q_c + Q_r)/4 \end{bmatrix} x(i) \right. \\ & \left. + v^T(i) R v(i) \right\} \\ & + x^T(f) \begin{bmatrix} (Q_{cf} + Q_{rf})/4 & (Q_{cf} - Q_{rf})/4 \\ (Q_{cf} - Q_{rf})/4 & (Q_{cf} + Q_{rf})/4 \end{bmatrix} x(f) \end{aligned}$$

where Q_c, Q_{cf}, Q_r, and Q_{rf} are semi-positive matrices, respectively.

Problem 2 can be transformed into the following mixed integer quadratic programming (MIQP) problem

$$\begin{aligned} \min_{\bar{v} \in \mathcal{V}} \quad & \bar{v}^T M_1 \bar{v} + \bar{v}^T M_2 x_0 \\ \text{subject to} \quad & L_1 \bar{v} \le L_2 x_0 + L_3 \end{aligned}$$

where the input set $\mathcal{V}$ is a set of $(\mathcal{R}^m \times \mathcal{R}^{m_1} \times \{0,1\}^{m_2})^f$, and M_1, M_2, L_1, L_2, L_3 are some matrices/vectors.

Remark 2: The MIFT problem and the MIQP problem can be solved by using a suitable solver, e.g. ILOG CPLEX. Unfortunately, solving the MIFT problem and the MIQP problem for large f becomes prohibitive. So it is one of significant works to decrease the computation time to solve these problems.

VI. Numerical Example

A. Plant

As a numerical example, consider the following piecewise linear (PWL) system

$$x(k+1) = \alpha(k) \begin{bmatrix} \cos\beta(k) & -\sin\beta(k) \\ \sin\beta(k) & \cos\beta(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

where $\alpha(k)$ is a uncertain parameter given by $\alpha(k) \in [0.50, 0.60]$, and $\beta(k)$ is given by

$$\beta(k) = \begin{cases} -\pi/3, & \text{if } [\ 1 \ \ 0\ ] x(k) < 0, \\ +\pi/3, & \text{if } [\ 1 \ \ 0\ ] x(k) \ge 0, \end{cases}$$

and state and input constraints are given by

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x(k) \le \begin{bmatrix} +10 \\ +10 \end{bmatrix}, \quad -1 \le u(k) \le +1.$$

Using the result of Section IV, this PWL system is transformed into the MLD model (13) with $n = 2$, $m = 1$, $m_1 = 25$, $m_2 = 10$. For the obtained MLD system, we will consider the trajectory generation problem and the optima control problem.

B. Example 1: Trajectory Generation Problem

First, consider the trajectory generation problem. Suppose that the terminal time, the interval of the initial state and the input sequence are given by $f = 5$,

$$x_0 = \begin{bmatrix} \underline{x}_0 \\ \overline{x}_0 \end{bmatrix} = \begin{bmatrix} -1.05 \\ +0.95 \\ \hline -0.95 \\ +1.05 \end{bmatrix}, \qquad (16)$$

and $u(0) = u(1) = \cdots = u(f-1) = 0$, respectively. Note here that the initial state is given as an interval, not a point. Of course, the initial state may be given as a point. The obtained state trajectory is given by Fig. 1, where we used ILOG CPLEX 11.0 on the computer with the Intel Core 2 Duo 3.0GHz processor and the 4GB memory. Also, the computation time to solve this trajectory generation problem was 0.02 [sec]. From Fig. 1, we see that the state trajectory converges to a neighborhood of the origin. On the other hand, in the proposed method, an over-approximation of an interval is computed by using Lemma 1. Here, comparing between the obtained approximate trajectory and the accurate trajectory, the error of the infimum and the supremum of the state at each time is less than 10^{-4}. So we see that in this example, the obtained over-approximation is very tight.

C. Example 2: Optimal Control Problem

Next, consider the optimal control problem. In this example, we use a cost function (15), and consider the following two cases, i.e.,

Case 1: $Q_c = Q_{cf} = 100 I_2$, $Q_r = Q_{rf} = 0_{2\times 2}$,

Case 2: $Q_c = Q_{cf} = 0_{2\times 2}$, $Q_r = Q_{rf} = 100 I_2$.

In both cases, suppose that the terminal time, the initial state and the input weighting matrix are given by $f = 10$, (16) and $R = \text{block-diag}(1, 0_{35\times 35})$, respectively.

Fig. 2 shows the state trajectory in Case 1, where the box at each time is suitably omitted. The computation time to solve the optimal control problem was 2.36 [sec]. From Fig. 2, we see that the state trajectory converges to a neighborhood of the origin faster than that in the case of $u(0) = u(1) = \cdots = u(f-1) = 0$.

Fig. 3 shows the state trajectory in Case 2, where the box at each time is suitably omitted. The computation time to solve the optimal control problem was 20.01 [sec]. Comparing between Fig. 2 and Fig. 3, it seems that the difference is small. Note that the cost function of Case 2 is given for minimizing the radius of an interval of the state at each time.

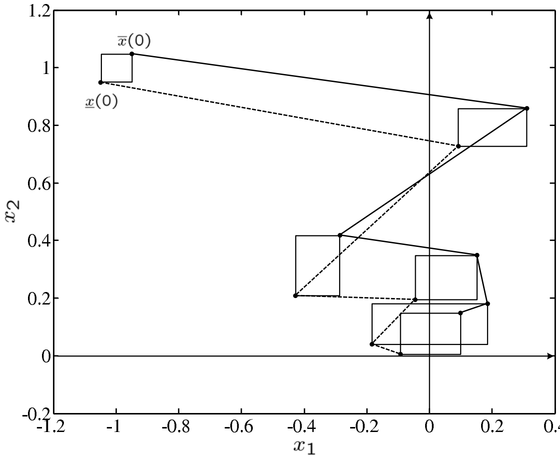

Fig. 1. State trajectory for $u(k) = 0$

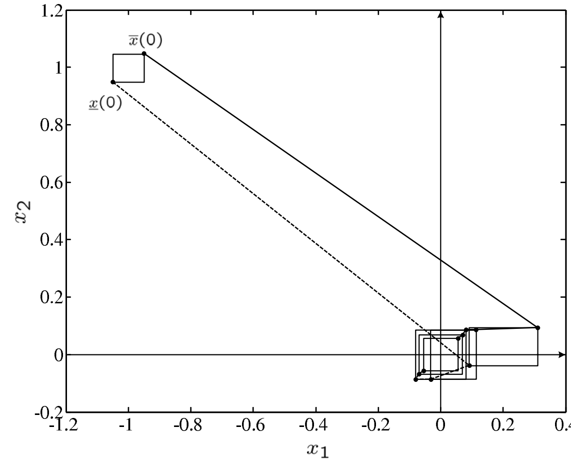

Fig. 2. State trajectory in Case 1

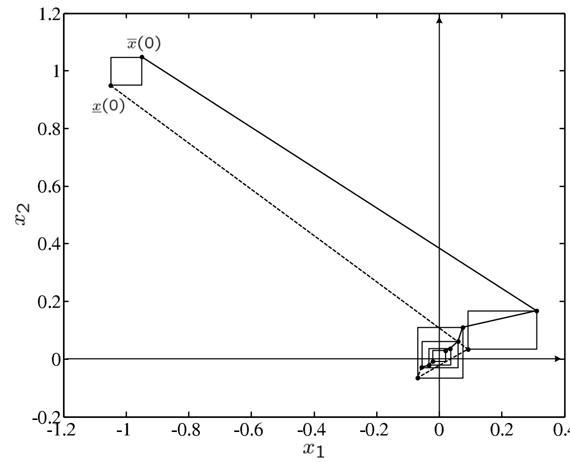

Fig. 3. State trajectory in Case 2

Here, a sum of an area of the state interval at each time is derived as

$$\kappa = \sum_{k=0}^{f} \prod_{i=1}^{n} 2x_r^{(i)}(k), \quad x_r(k) = \frac{1}{2}(-\underline{x}(k) + \overline{x}(k)).$$

Then for each case, κ is calculated as follows:

Case of $u(k) = 0$: $\kappa = 0.1789$,
Case 1: $\kappa = 0.1226$,
Case 2: $\kappa = 0.0799$.

Therefore, we see that in Case 2, the expansion of an area at each time is restrained.

VII. Conclusion

In this paper, for discrete-time piecewise affine systems with parameter uncertainty, applying of interval methods has been proposed. By approximately using interval arithmetic, discrete-time uncertain hybrid systems can be expressed as the MLD model. From the result of this paper, we will give a new viewpoint for the MLD model framework.

Some of future works have been already explained (see Remarks 1 and 2). As other future works, it is one of interesting works to clarify the relation between the proposed method and predicate abstraction techniques [4], which are one of discrete abstraction techniques of hybrid systems [3]. Furthermore, the relation to the concept of box invariance [1], [2] is also interesting.

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