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Vibration Suppression Control for a Structural System using an AMD with Restricted Stroke

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***Abstract*— The problem of suppressing the vibrations of a one-degree-of-freedom structural system using an AMD (active mass damper) with restricted stroke is considered; it can be formulated as the problem of stabilizing a linear system with a state constraint. First this problem is reduced to the one with an input constraint by compensating the driving system for the weight (the auxiliary mass) of the AMD to have a second-order lag transfer function. Then a saturating control is designed by a partial state feedback technique. The control law has the following properties: 1) a good control performance is obtained controlling the weight of the AMD within the stroke constraint; 2) the control law is simple and easily implemented; and 3) in the case where the controller is used in the linear range a good frequency response property is obtained that the peak gains of the frequency responses of the weight and the structure for disturbances are lowered. The effectiveness of the control law is demonstrated by experiments and simulations.**

I. INTRODUCTION

An active mass damper (AMD) is a mechanical device that suppresses the vibrations of a structural system using reaction forces generated by moving the auxiliary mass with an actuator connected between the structure and the auxiliary mass. Many applications of AMDs to real buildings and civil structures have already been reported. The number of applications to high-rise buildings has been over sixty since the Kyobashi Center building, the former Kyobashi Seiwa building, an 11-story building in Tokyo, Japan, was constructed by the Kajima Corporation in 1989; and these applications have been conducted mainly in Japan [1], [2].

The auxiliary mass of an AMD for a building structure is usually less than 0.4% of the total mass of the structure [3]; that is, a relatively small mass is to be moved with a limited amplitude to suppress the vibrations of the structure. Therefore, AMDs intrinsically dot not have an ability to absorb the energy due to large earthquake excitations, but they aim to suppress the vibrations of structures under strong winds or moderate earthquakes. Generally, the allowable amplitude of an auxiliary mass is severely restricted due to the size of the AMD or the installation space. In fact, the amplitude constraint of an auxiliary mass is one of the main reasons for the limited performance of AMDs. Thus, it is desired to develop a control law that effectively suppresses the vibrations of a structural system under the amplitude constraint of the auxiliary mass.

Control laws for AMDs that have been in practical use are mainly state feedback controls with constant gain, obtained by LQ optimal control theory or H_∞ control theory [3]. These gains are designed to satisfy the amplitude constraint of the auxiliary mass for the maximum possible disturbance (wind or earthquake excitation), so they do not make the most of the performance of the AMD for ordinary disturbances.

For the problems considering the limited amplitude of an auxiliary mass, the following control laws have been developed: gain-scheduling control laws changing the feedback gain according to the magnitude of the disturbance [4], [5], [6], [7], [8]; gain-scheduling control laws with a nonlinear spring [9], [10]; and saturating control laws that limit the input to the actuator [11], [12]. These have the drawback that the control algorithm is complex. Also, except the control law in [4] that uses the technique developed in [13], they do not theoretically assure that the auxiliary mass satisfies the amplitude constraint or that the closed-loop system is asymptotically stable.

This paper proposes a control law for an AMD that effectively suppress the vibrations of a one-degree-of-freedom structural system under the amplitude constraint of the auxiliary mass. The proposed control law has the following features:

1) The control algorithm is simple; the control law is constructed based on a linear saturating control and can be computed as easily as an LQ optimal regulator. Thus, it can also be applied to systems where an LQ optimal regulator is already implemented.
2) The asymptotic stability of the control system is theoretically guaranteed. Moreover, it provides a good control performance.
3) When the control law is used in the linear range, the frequency response for disturbances has good characteristics. That is, for sinusoidal disturbances, it effectively suppresses the vibrations of the structural system, keeping the amplitude of the auxiliary mass small.

The effectiveness of the control law is examined by simulations and experiments.

II. MATHEMATICAL MODEL OF THE CONTROLLED OBJECT AND PROBLEM STATEMENT

Fig.1 shows the analytical model of the apparatus, a one-degree-of-freedom structural system with an AMD, used in this study. The apparatus consists of a pair of flexible beams,

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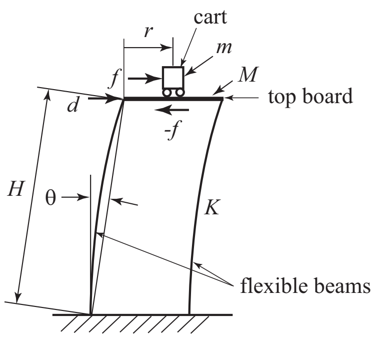

Fig. 1. One-degree-of-freedom structural system with an AMD.

a rigid top board, and a cart as the additive mass; the cart can be moved on the top board without slip with a DC motor and a rack and pinion gear set. In the modeling the following are assumed: only the first mode of oscillation appears in each beam; the amplitude of the oscillation is sufficiently small; and the mass of each beam can be neglected. With these assumptions the paired beams can be regarded as a linearly elastic spring.

The symbol $\theta(t)$ represents the angular displacement the line between the ends of each beam makes with the vertical line, $r(t)$ the position of the cart measured from the nominal point on the top board, $f(t)$ the manipulated force applied between the cart and the top board, generated by the DC motor, and $d(t)$ the disturbance applied to the top board, at time t. Let M denote the mass of the top board, m the mass of the cart (the auxiliary mass of the AMD), H the length of each beam, and K the rotational spring constant of the paired beams.

The linearized equations of motion are obtained from the balance of forces or moments as

$$m(\ddot{r} + H\ddot{\theta}) = f \tag{1}$$

$$MH^2\ddot{\theta} + Hf + K\theta = Hd \tag{2}$$

or equivalently

$$\ddot{r} = \frac{K}{MH}\theta + \frac{m+M}{mM}f \tag{3}$$

$$\ddot{\theta} = -\frac{K}{MH^2}\theta - \frac{1}{MH}f + \frac{1}{MH}d. \tag{4}$$

The variable r is supposed to be constrained as

$$|r(t)| \leq a, \quad \forall t \geq 0 \tag{5}$$

where $a > 0$ is the maximum allowable amplitude of the cart.

The problem is to find a control law for the AMD that quickly suppresses the vibrations of the structural system generated by an impulsive disturbance, under condition (5).

III. Design method

A. Reduction to a problem with constrained input

Let $v(t)$ be a new input for the cart system, $r(s)$ and $v(s)$ the Laplace transforms of $r(t)$ and $v(t)$, respectively, and $G(s)$ the transfer function from $v(s)$ to $r(s)$. Let $V_{\text{in}}(t)$ be the input voltage of the driver amplifier for the cart's DC motor. The input V_{in} is designed so that $G(s)$ takes the form of a second-order lag

$$G(s) = \frac{r(s)}{v(s)} = \frac{1}{(1+Ts)^2} \tag{6}$$

with $T > 0$ being a design parameter; a second-order dynamics is natural because the cart-motor system is well modeled by a second-order system. Then the following holds for the 1-norm of $G(s)$, denoted $\|G(s)\|_1$:

$$\|G(s)\|_1 := \int_0^\infty |g(t)|dt = 1 \tag{7}$$

where $g(t)$ is the impulse response of $G(s)$.

The time-domain expression of (6) is

$$\ddot{r} = -\frac{1}{T^2}r - \frac{2}{T}\dot{r} + \frac{1}{T^2}v \tag{8}$$

and the force f making (8) hold is obtained from (3) and (8) as

$$f = \frac{mM}{m+M}\left(-\frac{1}{T^2}r - \frac{2}{T}\dot{r} - \frac{K}{MH}\theta + \frac{1}{T^2}v\right). \tag{9}$$

Let $\mathcal{R}$ be the set of all solutions of (8), $[r(t)\ \dot{r}(t)]'$, $\forall t \geq 0$, reachable from the origin by some input v satisfying $|v(t)| \leq a$.

Thanks to (7), condition (5) is satisfied if the following two conditions hold (see Appendix I).

$$[r(0)\ \dot{r}(0)]' \in \mathcal{R} \tag{10}$$

$$|v(t)| \leq a, \quad \forall t \geq 0 \tag{11}$$

Also, substitution of (9) into (4) yields

$$\ddot{\theta} = a_{41}r + a_{42}\dot{r} - \omega_n^2\theta + b_4 v + \frac{1}{MH}d \tag{12}$$

where

$$a_{41} = \frac{m}{(m+M)HT^2}, \quad a_{42} = \frac{2m}{(m+M)HT}$$

$$\omega_n^2 = \frac{K}{(m+M)H^2}, \quad b_4 = -\frac{m}{(m+M)HT^2}.$$

Now define the state as

$$x = \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} \end{bmatrix}'. \tag{13}$$

Then the equations of motion of the structural system after compensated by (9), i.e., (8) and (12), are represented by the following state equation:

$$\dot{x} = Ax + Bv + B_d d \tag{14}$$

where

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_d = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ \frac{1}{MH} \end{array}\right]$$

with

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{T^2} & -\frac{2}{T} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ a_{41} & a_{42} \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{1}{T^2} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ b_4 \end{bmatrix}.$$

It is seen from the structures of A and B that (A, B) is controllable.

A solution of the problem will be obtained by solving the more tractable problem: find a control law that asymptotically stabilizes the system (14) under conditions (10) and (11).

B. Stabilization by partial state feedback

The problem in Section III-A will be solved by reducing the problem to a much easier one, by decomposing the system (14) into the stable and unstable subsystem by a change of coordinates, where a control law is to be found that asymptotically stabilizes the unstable subsystem, a second-order system, under the constraint of the input v.

Let S be a coordinate transform matrix such that by the change of coordinates

$$w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}' = Sx \tag{15}$$

the system (14) can be transformed as

$$\dot{w} = \tilde{A}w + \tilde{B}v + \tilde{B}_d d \tag{16}$$

where

$$\tilde{A} = SAS^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$\tilde{B} = SB = \begin{bmatrix} B_1 \\ \tilde{B}_2 \end{bmatrix} = \left[\begin{array}{c} 0 \\ \frac{1}{T^2} \\ \hline 0 \\ 1 \end{array}\right], \quad \tilde{B}_d = SB_d.$$

Such a matrix S is given by (see Appendix II)

$$S = \begin{bmatrix} I_2 & 0 \\ Y^{-1}X & Y^{-1} \end{bmatrix} \tag{17}$$

where I_2 is the 2×2 identity matrix and

$$Y = \begin{bmatrix} A_{22}(XB_1 + B_2) & XB_1 + B_2 \end{bmatrix}$$

with X being the solution of the Sylvester equation

$$-A_{22}X + XA_{11} + A_{21} = 0. \tag{18}$$

Since A_{11} and A_{22} have no common eigenvalues, (18) can be solved uniquely [14].

The disturbance signal d is not necessary for the development of the stabilizing control law, so let $d = 0$ for the moment.

The state w is partitioned as

$$w = \begin{bmatrix} w_s \\ w_u \end{bmatrix} \tag{19}$$

where

$$w_s = \begin{bmatrix} w_1 & w_2 \end{bmatrix}', \quad w_u = \begin{bmatrix} w_3 & w_4 \end{bmatrix}'.$$

With the change of coordinates (15) the system has been decomposed into the two subsystems:

$$\dot{w}_s = A_{11}w_s + B_1 v \tag{20}$$

$$\dot{w}_u = A_{22}w_u + \tilde{B}_2 v. \tag{21}$$

The w_s subsystem is asymptotically stable, while the w_u subsystem is unstable (its poles are $\pm j\omega_n$). Thus, a control law $v = f_1(w_u)$ that asymptotically stabilizes the w_u subsystem under $|v(t)| \le a$ asymptotically stabilizes the whole system; moreover, if condition (10) also holds, then the control law satisfies the constraint on r, i.e., (5), and thus it is also a solution of the original problem.

A saturating control that globally asymptotically stabilizes the w_u subsystem is given by (see Appendix III)

$$v = \mathrm{sat}(\tilde{F}_0 w, a) \tag{22}$$

where

$$\tilde{F}_0 = \begin{bmatrix} 0 & 0 & 0 & -2\zeta\omega_n \end{bmatrix} \tag{23}$$

with $\zeta > 0$ being a design parameter. Here sat$(\cdot)$ is the saturating function with amplitude a defined by

$$\mathrm{sat}(\mu, a) = \mathrm{sgn}(\mu)\min\{|\mu|, a\}. \tag{24}$$

Substituting (15) into (22) and letting

$$F_0 = \tilde{F}_0 S \tag{25}$$

yield

$$v = \mathrm{sat}(F_0 x, a). \tag{26}$$

The f obtained by substituting (26) into (9) is the proposed control law.

IV. Frequency characteristics of the control law when used in the linear range

In this section it is assumed that the input v is not constrained, or the control is used in the linear range. Let $\tilde{F}$ be a feedback gain in a class of gains including $\tilde{F}_0$ as

$$\tilde{F} = \begin{bmatrix} 0 & 0 & -k_1 & -k_2 \end{bmatrix} \tag{27}$$

and consider the control

$$v = \tilde{F}\tilde{x}. \tag{28}$$

Note that the control in (28) has so sufficient degrees of freedom for the control of the w_u subsystem that with it the eigenvalues of the w_u subsystem can be allocated arbitrarily. Suppose that k_1 and k_2 are taken so that the closed-loop eigenvalues of the w_u subsystem have negative real parts (the w_u subsystem is asymptotically stabilized).

For the closed-loop system with the control (28), the transfer functions from d to r, denoted $G_{rd}(s)$, and from d to

θ, denoted $G_{\theta d}$, can be computed as follows (see Appendix IV):

$$G_{rd}(s) = \frac{(-2k_1T - k_2 + k_2T^2\omega_n^2)s + k_1T^2\omega_n^2 - k_1 + 2k_2T\omega_n^2}{\epsilon\omega_n^2 M(1+Ts)^2(s^2 + k_2 s + \omega_n^2 + k_1)} \tag{29}$$

$$G_{\theta d}(s) = \frac{(1+Ts)^2\omega_n^2 + (2k_1T + k_2)s + k_1}{\omega_n^2 MH(1+Ts)^2(s^2 + k_2 s + \omega_n^2 + k_1)} \tag{30}$$

where ϵ was defined by

$$\epsilon := \frac{m}{m+M}.$$

From (29), as T tends to 0, $G_{rd}(s)$ approaches

$$G_{rd}(s) = -\frac{k_2 s + k_1}{s^2 + k_2 s + \omega_n^2 + k_1} \cdot \frac{1}{\epsilon\omega_n^2 M}. \tag{31}$$

From this, the magnitude of $G_{rd}(j\omega)$ is computed as

$$|G_{rd}(j\omega)| = \sqrt{\frac{k_1^2 + k_2^2\omega^2}{(\omega_n^2 + k_1 - \omega^2)^2 + k_2^2\omega^2}} \cdot \frac{1}{\epsilon\omega_n^2 M}. \tag{32}$$

This takes the following constant value at $\omega = \omega_n$, irrespective of k_1 and k_2:

$$|G_{rd}(j\omega_n)| = \frac{1}{\epsilon\omega_n^2 M}.$$

Moreover, the function $|G_{rd}(j\omega)|$ becomes maximum at $\omega = \omega_n$ when $k_1 = 0$. In fact, substituting $k_1 = 0$ into (32) gives

$$|G_{rd}(j\omega)| = \frac{|k_2\omega|}{\sqrt{(\omega_n^2 - \omega^2)^2 + k_2^2\omega^2}} \cdot \frac{1}{\epsilon\omega_n^2 M} \le \frac{1}{\epsilon\omega_n^2 M}.$$

Therefore, when T tends to 0, the control system minimizing $\|G_{rd}(j\omega)\|_\infty$ (the maximum magnitude of the frequency response from d to r) can be obtained by setting $\tilde{F} = \tilde{F}_0$, i.e., $k_1 = 0$ and $k_2 = 2\zeta\omega_n$.

$|G_{rd}(j\omega)|$ and $|G_{\theta d}(j\omega)|$ with $\tilde{F} = \tilde{F}_0$ have the following upper bounds (see Appendix V):

$$|G_{rd}(j\omega)| \le \frac{1}{\epsilon\omega_n^2 M}(1 + 4\omega_n T\zeta M_p(\zeta)) \tag{33}$$

$$|G_{\theta d}(j\omega)| \le \frac{1}{\omega_n^2 MH}(M_p(\zeta) + 1). \tag{34}$$

where

$$M_p(\zeta) = \begin{cases} \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} & \text{if } 0 < \zeta < \frac{1}{\sqrt{2}} \\ 1 & \text{if } \zeta \ge \frac{1}{\sqrt{2}} \end{cases}.$$

Note from (33) that the upper bound of $|G_{rd}(j\omega)|$ is a linear function of T, which approaches its minimum $1/\epsilon\omega_n^2 M$ when T tends to 0, and from (34) that $|G_{\theta d}(j\omega)|$ can have an upper bound that does not depend on T.

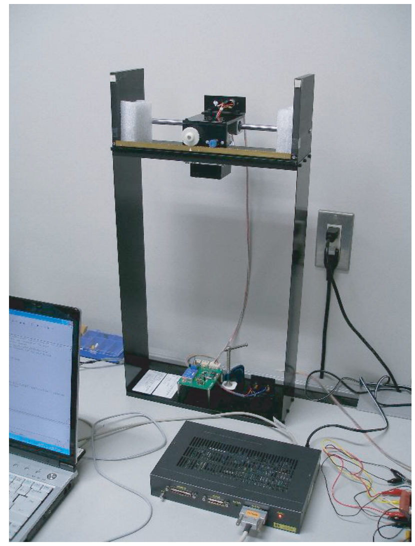

Fig. 2. View of the experimental system.

V. Experimental results

Fig.2 shows a view of the experimental system. The parameters of the apparatus are as follows: $M = 1.2$ [kg], $m = 0.455$ [kg], $H = 0.5$ [m], $K = 74$ [Nm/rad]. The outputs are the position of the cart, r, measured by a potentiometer, and the angular acceleration of the top board, $\ddot{\theta}$, measured by an accelerometer. So the output equation is

$$y = Cx + Dv \tag{35}$$

where

$$y = \begin{bmatrix} r \\ \ddot{\theta} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{41} & a_{42} & -\omega_n^2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ b_4 \end{bmatrix}.$$

The equation of $\ddot{\theta}$ is obtained from (12) with $d = 0$. The nonmeasured variables of the state, $\dot{r}$, θ, and $\dot{\theta}$, were estimated by a full-order observer, desiged based on (14) and (35) with the poles being $\{-20, -20, -20, -20\}$.

The DC motor for the cart was compensated in advance by rate feedback and a first-order-lag filter to have a robust input-output property; the resulting transfer function of the drive unit (from the input voltage of the motor driver, V_{in}, to the output, r) was an integrator plus a first-order lag. Moreover, when compensating the cart system to have the transfer function $G(s)$, only r and $\dot{r}$ were used; f was not computed and the force relating to $\ddot{\theta}$ was ignored and treated as a disturbance.

In the experiment the maximum allowable cart's amplitude a and the initial conditions were given as

$$a = 0.03 \text{ [m]}$$

$$r(0) = 0, \ \dot{r}(0) = 0, \ \theta(0) = 0, \ \dot{\theta}(0) = 0.$$

Also, the design parameters were chosen as

$$T = 0.2 \text{ [s]}, \quad \zeta = 0.7.$$

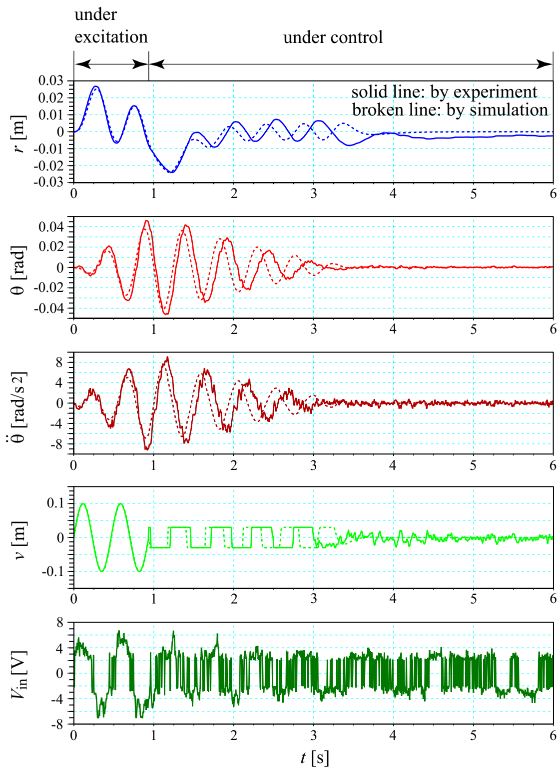

Fig. 3. Experimental (solid lines) and numerical (broken lines) results for $T = 0.2\text{s},\ \zeta = 0.7$.

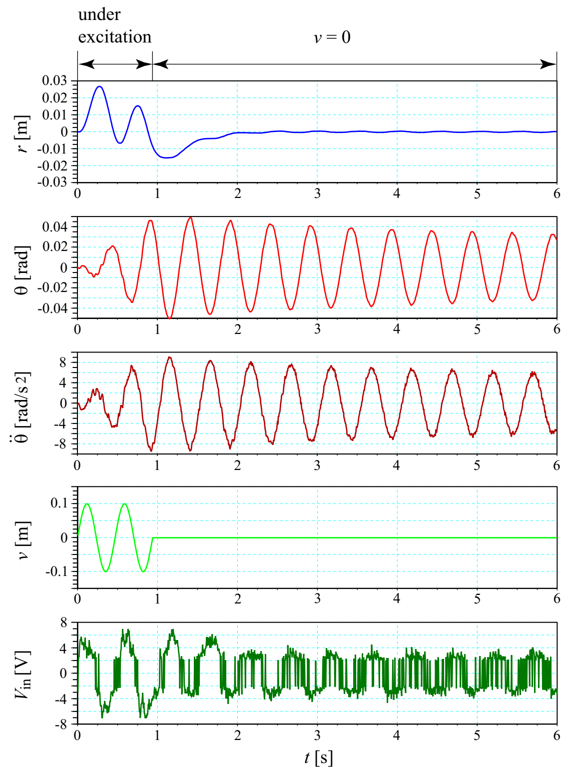

Fig. 4. Experimental results for $v = 0$.

Fig.3 show the results of the experiment and those of the corresponding simulation. The structural system was first excited by an input $v(t) = 0.1 \sin \omega_n t\,[\text{m}]$ for the first $4\pi/\omega_n \approx 0.94\,[\text{s}]$, and then the control was applied. A vibration suppression control is done, under the constraint of r, providing a comparable rate of damping with that of the simulation in spite of sensor noise and modeling errors. The amplitude of r is maintained at a certain level until the vibration is settled, enabling an effective damping. For comparison, the response of the apparatus with $v = 0$ is shown in Fig.4; it is confirmed that the inherent damping of the apparatus is very small. These figures also show the control v and the input voltage V_{in} to the driver of the DC motor; a constant voltage of $2.0\,[\text{V}]$ or $-2.0\,[\text{V}]$ is added to the input to compensate the dead zone of the drive unit.

When T is made smaller, the speed of response of the w_s subsystem becomes faster and the control performance is more improved; then, on the other hand, the magnitude of the manipulated input f becomes larger and the robustness against sensor noise and modeling errors becomes lower. The parameter ζ can be used to improve the speed of response of the w_u subsystem.

VI. Concluding remarks

If the w_u subsystem is controlled time-optimally (for second-order systems time-optimal control is easily obtained [15]) and moreover T is made to approach 0, then the structural system is controlled time-optimally under the stroke constraint of the AMD. However, such a control has no robustness against sensor noise and modeling errors, and requires an infinitely large manipulated input when T approaches 0. Therefore, a practical control law was constructed by using a saturating control to stabilize the w_u subsystem and choosing T that is not too small.

The equations of motion of the structural system under an earthquake has the same structure as those considered here (see appendix VI). Thus the proposed control law can also be applied to such a case.

Appendix I
Proof of the fact that if (10) and (11) hold, then (5) is satisfied.

Conditions (10) and (11) can be replaced by

$$\begin{bmatrix} r(-\infty) & \dot{r}(-\infty) \end{bmatrix}' = 0, \quad |v(t)| \le a, \quad \forall t > -\infty.$$

From these, $r(t)$ is computed as

$$r(t) = \int_{-\infty}^{t} g(t-\tau) v(\tau) d\tau.$$

Use of the change of variable $\eta = t - \tau$ yields

$$r(t) = \int_{0}^{\infty} g(\eta) v(t-\eta) d\eta.$$

From this and (7), the following inequality is obtained.

$$|r(t)| \le \int_{0}^{\infty} |g(\eta)| \cdot |v(t-\eta)| d\eta \le a$$

APPENDIX II
PROOF OF THE FACT THAT THE COORDINATE TRANSFORM MATRIX S IS GIVEN BY (17)

Define the following matrices using X, the solution of (18).

$$S_1 = \begin{bmatrix} I_2 & 0 \\ X & I_2 \end{bmatrix}, \quad S_1^{-1} = \begin{bmatrix} I_2 & 0 \\ -X & I_2 \end{bmatrix}.$$

Then we have

$$S_1 A S_1^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \tag{36}$$

$$S_1 B = \begin{bmatrix} B_1 \\ XB_1 + B_2 \end{bmatrix}. \tag{37}$$

Since (A, B) is controllable, $(A_{22}, XB_1 + B_2)$ is also controllable. Therefore, the following matrix is nonsingular.

$$Y = \begin{bmatrix} A_{22}(XB_1 + B_2) & XB_1 + B_2 \end{bmatrix}$$

This and the relation $A_{22}^2 = -\omega_n^2 I_2$ yield

$$A_{22}Y = YA_{22}, \quad XB_1 + B_2 = Y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or

$$Y^{-1}A_{22}Y = A_{22}, \quad Y^{-1}(XB_1 + B_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{38}$$

Letting

$$S_2 = \begin{bmatrix} I_2 & 0 \\ 0 & Y^{-1} \end{bmatrix}$$

transforming the matrices in (36) and (37) by S_2, and using (38) yield

$$S_2 \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} S_2^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} = \tilde{A}$$

$$S_2 \begin{bmatrix} B_1 \\ XB_1 + B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \tilde{B}.$$

It is seen from the above that S is obtained by combining S_1 and S_2 as follows:

$$S = S_2 S_1 = \begin{bmatrix} I_2 & 0 \\ Y^{-1}X & Y^{-1} \end{bmatrix}.$$

APPENDIX III
PROOF OF THE FACT THAT THE CONTROL (22) GLOBALLY ASYMPTOTICALLY STABILIZES THE w_u SUBSYSTEM

Let $d = 0$ and define the following positive definite function of the state $w_u = [w_3 \; w_4]'$:

$$V_1 := \frac{1}{2}(\omega_n^2 w_3^2 + w_4^2).$$

The time derivative of V_1 is computed as

$$\begin{aligned} \dot{V}_1 &= w_4(\omega_n^2 w_3 + \dot{w}_4) = w_4 v \\ &= -w_4 \mathrm{sat}(2\zeta\omega_n w_4, a) \le 0 \end{aligned}$$

where (21) and (22) were used. Moreover, $\dot{V}_1 \not\equiv 0$ except where $w_u = 0$. In fact, if $\dot{V}_1 \equiv 0$, then we have $w_4 \equiv 0$, which means that $\dot{w}_4 \equiv 0$ and $v \equiv 0$. Then from (21) we get $\dot{w}_4 = -\omega_n^2 w_3$, so we also have $w_3 \equiv 0$. Therefore, it follows from LaSalle's invariance theorem [16] that any initial state $w_u(0)$ is made to converge to 0 by the control v in (22). That is, the control (22) globally asymptotically stabilize the w_u subsystem.

APPENDIX IV
COMPUTING $G_{rd}(s)$ AND $G_{\theta d}(s)$

Although $G_{rd}(s)$ and $G_{\theta d}(s)$ can in principle be derived from the state equation of w, this way needs to compute the explicit expression of S. So we shall use the state equation in another coordinate system where the computation of S is not necessary.

From (3) and (4), we have

$$\ddot{\theta} = -\omega_n^2 \theta - \frac{\epsilon}{H}\ddot{r} + \frac{1}{MH}d. \tag{39}$$

Define the variable

$$p := \epsilon r + H\theta. \tag{40}$$

The equation (39) can be written in terms of p as

$$\ddot{p} = -\omega_n^2 p + \epsilon\omega_n^2 r + \frac{1}{M}d. \tag{41}$$

Define the new state

$$\bar{x} = \begin{bmatrix} r & \dot{r} & p & \dot{p} \end{bmatrix}'. \tag{42}$$

Then the equations of motion with the servo system for the cart, i.e., (8) and (41), are represented by the following state equation:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}v + \bar{B}_d d \tag{43}$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{T^2} & -\frac{2}{T} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon\omega_n^2 & 0 & -\omega_n^2 & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0 \\ \frac{1}{T^2} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix}.$$

The characteristic equation of $\tilde{A} + \tilde{B}\tilde{F}$ is

$$\frac{1}{T^2}(1 + Ts)^2(s^2 + k_2 s + \omega_n^2 + k_1) = 0. \tag{44}$$

Since the structural system is a single-input controllable system, if we find $\bar{F}$ such that the characteristic equation of $\bar{A} + \bar{B}\bar{F}$ is equal to (44), then we have $v = \tilde{F}w = \bar{F}\bar{x}$. Therefore, $\bar{F}$ can explicitly be computed by pole placement methods as the feedback gain realizing the same closed-loop poles as those of (44) as follows:

$$\bar{F} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 & \bar{f}_3 & \bar{f}_4 \end{bmatrix} \tag{45}$$

with

$$\bar{f}_1 = -(k_1 T + 2k_2)T, \quad \bar{f}_2 = -k_2 T^2$$

$$\bar{f}_3 = -\frac{1}{\epsilon\omega_n^2}\{k_1 - (k_1 T + 2k_2)T\omega_n^2\}$$

$$\bar{f}_4 = -\frac{1}{\epsilon\omega_n^2}(2k_1T + k_2 - k_2T^2\omega_n^2).$$

Noting that

$$r = C_r\bar{x}, \quad \theta = C_\theta\bar{x} \tag{46}$$

where

$$C_r = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]$$

$$C_\theta = \left[\begin{array}{cccc} -\frac{\epsilon}{H} & 0 & \frac{1}{H} & 0 \end{array}\right]$$

we have

$$G_{rd}(s) = C_r\left\{sI - (\bar{A} + \bar{B}\bar{F})\right\}^{-1}\bar{B}_d \tag{47}$$

$$G_{\theta d}(s) = C_\theta\left\{sI - (\bar{A} + \bar{B}\bar{F})\right\}^{-1}\bar{B}_d. \tag{48}$$

Computing these equations yields (29) and (30). The above computations were performed with the aid of the software Mathematica.

Appendix V
Upper bounds of $|G_{rd}(j\omega)|$ and $|G_{\theta d}(j\omega)|$

An upper bound of $|G_{rd}(j\omega)|$ can be computed as follows. From (29), we have

$$\begin{aligned} G_{rd}(s) &= \frac{2\zeta\omega_n(\omega_n^2T^2 - 1)s + 4\zeta\omega_n^3T}{\epsilon\omega_n^2M(1+Ts)^2(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &= -\frac{1}{\epsilon\omega_n^2M(1+Ts)^2}\cdot\frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &+\frac{2\zeta\omega_n T}{\epsilon\omega_n^2 M}\cdot\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{Ts+2}{(1+Ts)^2}. \end{aligned} \tag{49}$$

Note that

$$\left|\frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right|_{s=j\omega} \le 1$$

$$\left|\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right|_{s=j\omega} \le M_p(\zeta)$$

$$\left|\frac{1}{(1+Ts)^2}\right|_{s=j\omega} \le 1, \quad \left|\frac{Ts+2}{(1+Ts)^2}\right|_{s=j\omega} \le 2.$$

Equation (49) and the above inequalities give (33).

An upper bound of $|G_{\theta d}(j\omega)|$ can similarly be computed using the following relation:

$$\begin{aligned} G_{\theta d}(s) &= \frac{(1+Ts)^2\omega_n^2 + 2\zeta\omega_n s}{\omega_n^2MH(1+Ts)^2(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &= \frac{1}{\omega_n^2MH}\cdot\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &+\frac{1}{\omega_n^2MH(1+Ts)^2}\cdot\frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \end{aligned} \tag{50}$$

This and related inequalities yield (34).

Appendix VI
Equations of motion of the structural system under an earthquake

Let $u_0(t)$ be the horizontal displacement of the ground. Then the equations of motion of the structural system are written as

$$m(\ddot{r} + H\ddot{\theta}) = \tilde{f}$$

$$MH^2\ddot{\theta} + H\tilde{f} + K\theta = -(m+M)H\ddot{u}_0$$

where

$$\tilde{f} = f - m\ddot{u}_0.$$

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