2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Stabilizing Control for an Inverted Pendulum with Restricted Travel

Kazunobu Yoshida
Faculty of Science and Engineering
Shimane University
Matsue, Shimane 690-8504, Japan
kyoshida@ecs.shimane-u.ac.jp

Itaru Matsumoto
Department of Mechanical Engineering
Yonago National College of Technology
Yonago, Tottori 683-8502, Japan
i-matsum@yonago-k.ac.jp

***Abstract*— For the problem of stabilizing an inverted pendulum with restricted travel, a saturating control law is developed that satisfies the amplitude constraint on the cart and has a large region of attraction. The explicit expression of the region of attraction is also obtained. The analysis and design are performed based on the linearized model of the system, as in the study of Lin et al. The control law has two design parameters: T and k, which are both positive. As T approaches 0, the region of attraction approaches the maximal one. These parameters are chosen to optimize the transient response of the closed-loop system considering the input constraint, observation noise, etc. The effectiveness of the control law is demonstrated by simulations and experiments.**

I. INTRODUCTION

Inverted pendulums (cart-and-pole systems) have been in widespread use in control laboratories as controlled objects for testing control laws developed. Also, since they are interesting controlled objects with the properties that they are underactuated, nonlinear, and unstable, methods for controlling them are still being studied.

This paper investigates the problem of stabilizing an inverted pendulum considering the amplitude constraint of the cart. For a similar problem, Lin et al. [1] have developed a linear state feedback control law minimizing the amplitude of the cart from a given initial attitude. The closed-loop system has four poles. Let ω_n be the natural angular frequency of the pendulum, and ϵ a small positive number. Lin et al.'s control law places one of the poles at $-\omega_n$, another at $-\epsilon$, and the remaining two at points whose real parts are very small (very large in the negative direction). If ϵ is made to approach 0, then the maximum amplitude of the cart approaches the theoretical lower limit. However, there is a slow mode in the motion of the cart due to the pole at $-\epsilon$. Yoshida et al. [2] have proposed a linear state feedback control law that makes the amplitude of the cart small, placing two of the four closed-loop poles at complex conjugate points, $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ ($\zeta = 0.6 \sim 1$), and the other two at a single real point whose real part is very small. While this control achieves good response characteristics, it entails a larger amplitude of the cart than Lin et al.'s one. Also, Wei et al. [3] proposed a nonlinear control, using the pendulum's phase trajectory and switching functions, that performs a bang-bang control of the acceleration of the cart. Although this control can achieve a good control performance under the amplitude constraint of the cart, such a bang-bang control is susceptible to measurement noise, and also a driving system is required that can precisely control the acceleration of the cart.

For a given control law, it is desirable to find the set of all initial states from which the system can be stabilized, i.e., the region of attraction, where the control can be used. However, in general, it is not easy to find such a set; in fact, in [1], [2], [3] such sets were not obtained.

For the problem of stabilizing an inverted pendulum with restricted cart travel, this paper proposes a saturating control that satisfies the constraint and makes the region of attraction as large as possible; also, the expression of the region of attraction is obtained. The control law has a structure similar to the one proposed by Teel [4], [5] realizing semi-global stabilization for a class of single-input partially linear composite systems. Although both controls can keep some of the state variables of the control system small by using a state feedback and a saturating control, the proposed method designs the feedback compensation so that the state constraint is satisfied. The analysis and design are performed based on the linearized model of the system, as in the study of Lin et al. [1].

The proposed control law has two design parameters, $T > 0$ and $k > 0$. T is the time constant of the servo system for the cart, and if T is made to approach 0, then the region of attraction approaches its upper limit (the maximal region of attraction in the sense that a larger region of attraction cannot be obtained by any other controls satisfying the constraint). Also, if the control law is used in the unsaturated range, then the closed-loop poles are $\{-1/T, -1/T, -\omega_n, -k\}$. Unlike Lin et al.'s control law, k is not required to be small to make the region of attraction large, and thus a control system with good response characteristics can be designed.

The effectiveness of the control law is examined by simulations and experiments.

II. MATHEMATICAL MODEL OF THE CONTROLLED OBJECT AND PROBLEM STATEMENT

Fig.1 shows the inverted pendulum considered in this study. It is assumed that the pendulum can rotate without friction around the pivot attached to the cart. Let $\theta(t)$, $r(t)$, and $u(t)$ be, respectively, the angular displacement of the pendulum, the displacement of the cart, and the force applied to the cart, at time t. Moreover, let M denote the mass of the cart, m the mass of the pendulum, L_G the distance from the pivot to the center of gravity of the pendulum, J_G the

978-1-4244-4524-0/09/$25.00 ©2009 AACC

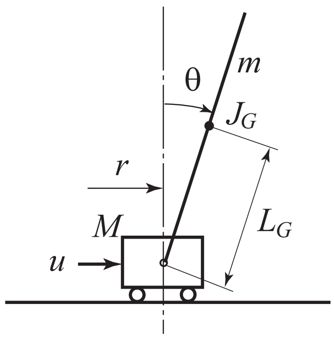

Fig. 1. Inverted pendulum.

moment of inertia of the pendulum around the center of gravity, and g the acceleration of gravity.

The linearized equations of motion of the inverted pendulum can be written as [6]

$$M\ddot{r} = u - F\dot{r} \tag{1}$$

$$\ddot{\theta} = \frac{g}{L}\theta - \frac{\ddot{r}}{L} \tag{2}$$

where the equation for the cart, (1), was simplified by supposing $m \ll M$, and where F is the viscous friction coefficient of the cart driving system and L is the effective pendulum length defined by

$$L = \frac{mL_G^2 + J_G}{mL_G}. \tag{3}$$

Let $a > 0$ be the maximum allowable amplitude of r; that is, let r be constrained as

$$|r(t)| \leq a. \tag{4}$$

The problem is to find, for the inverted pendulum described by (1) and (2), a control law that asymptotically stabilizes the system around the upright equilibrium point under condition (4) with the region of attraction being made as large as possible.

III. Design method

A. Reduction to a problem with constrained input

Introduce the variable $v(t)$ as a new input for the cart driving system. Let $v(s)$ and $r(s)$ be the Laplace transforms of $v(t)$ and $r(t)$, respectively, and $G(s)$ the transfer function from $v(s)$ to $r(s)$. $G(s)$ is designed so as to be

$$G(s) = \frac{r(s)}{v(s)} = \frac{1}{(1+Ts)^2} \tag{5}$$

where $T > 0$ is a design parameter. Then the following relation holds for the 1-norm of $G(s)$, denoted $\|G(s)\|_1$:

$$\|G(s)\|_1 := \int_0^\infty |g(t)|dt = 1 \tag{6}$$

where $g(t)$ is the impulse response of $G(s)$.

The relation (5) can be written in the time domain as

$$\ddot{r} = -\frac{1}{T^2}r - \frac{2}{T}\dot{r} + \frac{1}{T^2}v. \tag{7}$$

The input u making (7) hold is obtained from (1) and (7) as

$$u = M\left\{-\frac{r}{T^2} - \left(\frac{2}{T} - \frac{F}{M}\right)\dot{r} + \frac{v}{T^2}\right\}. \tag{8}$$

Let $\mathcal{R}$ be the set of all solutions of (7), $[r(t)\ \dot{r}(t)]'$, $\forall t \geq 0$, reachable from the origin by some input v satisfying $|v(t)| \leq a$.

Thanks to (6), condition (4) is satisfied if the following two conditions hold (see Appendix I).

$$[r(0)\ \dot{r}(0)]' \in \mathcal{R} \tag{9}$$

$$|v(t)| \leq a, \quad \forall t \geq 0 \tag{10}$$

With the state variables

$$\begin{aligned} x_1 &= r \\ x_2 &= \dot{r} \\ x_3 &= r + L\theta \\ x_4 &= \dot{r} + L\dot{\theta} \end{aligned} \tag{11}$$

which were also used in [6], the equations of motion (2) and (7) can be written in state equation form as

$$\dot{x} = Ax + Bv \tag{12}$$

where

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}'$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{T^2} & -\frac{2}{T} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_n^2 & 0 & \omega_n^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{T^2} \\ 0 \\ 0 \end{bmatrix}$$

with ω_n being defined by

$$\omega_n = \sqrt{\frac{g}{L}}.$$

The problem has been simplified to the one with constrained input: find a control law that asymptotically stabilizes the system (12) under conditions (9) and (10) with the region of attraction being made as large as possible. Since a solution of this problem satisfies condition (4), it is also a solution of the original problem.

B. Stabilization by partial state feedback

Decomposing the system (12) into a stable and an unstable subsystem by a change of coordinates, we shall solve the problem in Section III-A by reducing it to a much simpler problem where a stabilizing control for the unstable subsystem, a first-order system, is required to be found under the amplitude constraint on v.

Introduce the change of coordinates

$$w = Sx \tag{13}$$

where S is given by

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1+T(T\omega_n - 2) & -T^2 & s_{33} & s_{34} \\ T(T\omega_n+2) & T^2 & s_{43} & s_{44} \end{bmatrix}$$

with

$$s_{33} = -\frac{(T\omega_n - 1)^2 + \omega_n}{\omega_n}, \; s_{34} = \frac{(T\omega_n - 1)^2 + \omega_n}{\omega_n^2}$$

$$s_{43} = -\frac{(T\omega_n + 1)^2}{\omega_n}, \; s_{44} = -\frac{(T\omega_n + 1)^2}{\omega_n^2}.$$

The fact that S is nonsingular can be seen from the relation

$$|S| = \frac{2}{\omega_n^3}(T\omega_n + 1)^2\{(T\omega_n - 1)^2 + \omega_n\} \neq 0. \qquad (14)$$

The system (12) is transformed by S as

$$\dot{w} = \tilde{A}w + \tilde{B}v \qquad (15)$$

where

$$\tilde{A} = SAS^{-1}, \quad \tilde{B} = SB$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -\frac{1}{T^2} & -\frac{2}{T} & 0 & 0 \\ 0 & 1 & -\omega_n & 0 \\ \hline 0 & 0 & 0 & \omega_n \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c} 0 \\ \frac{1}{T^2} \\ -1 \\ \hline 1 \end{array}\right].$$

This can be seen by direct computation.

Then the state w is partitioned conformably with the partition of $\tilde{A}$ and $\tilde{B}$ as

$$w = \begin{bmatrix} w_s \\ w_u \end{bmatrix}$$

where

$$w_s = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \; w_u = w_4.$$

The idea of the design is as follows: construct a saturating control $v(t) = f(w_u(t))$ that locally asymptotically stabilizes the w_u subsystem, a first-order system, around the equilibrium point $w_u = 0$ and apply it to the whole system. Then, for initial states in the region of attraction, the control $v(t)$ approaches 0 because so does $w_u(t)$; and moreover $w_s(t)$ also approaches 0 since the w_s subsystem is asymptotically stable. Therefore, the problem in Section III-A is further simplified as follows: find a control v that locally asymptotically stabilizes the w_u subsystem under the condition $|v(t)| \leq a$.

From (15), the w_u subsystem is written as

$$\dot{w}_u = \omega_n w_u + v. \qquad (16)$$

A control v that locally asymptotically stabilizes this system and satisfies the constraint on v is given by the saturating control

$$v = -\text{sat}((\omega_n + k)w_u, a), \qquad k > 0 \qquad (17)$$

where the function $\text{sat}(\cdot,\cdot)$ is defined by

$$\text{sat}(\xi, a) = \text{sgn}(\xi)\min\{|\xi|, a\}. \qquad (18)$$

Let s_4 denote the fourth row of S. Then, since $w_u = s_4x$ from (13), (17) can be written in terms of x as

$$v = -\text{sat}((\omega_n + k)s_4x, a), \qquad k > 0. \qquad (19)$$

Moreover, the region of attraction is given by

$$\mathcal{X}_0 = \left\{x : |s_4x| < \frac{a}{\omega_n}\right\}. \qquad (20)$$

For the proofs of these results, see Appendix II.

It can be thought from (20) that since a/ω_n is a constant, the magnitude of the set $\mathcal{X}_0$ is inversely proportional to $\|s_4\|$. Since $\|s_4\|$ is a monotone increasing function of T, when $T \to 0$, $\|s_4\|$ becomes minimal, or equivalently the magnitude of the set $\mathcal{X}_0$ becomes maximal. In fact, when $T \to 0$, the set (20) approaches the maximum region of attraction of the original problem in the sense that a larger region of attraction cannot be obtained by any other controls satisfying the constraint of r (see Appendix III). Note that when $T \to 0$, the magnitude of the manipulated input u becomes infinitely large (see (8)); in practice, u is also constrained and thus T cannot be chosen too small.

Specifically, the resulting control input u is obtained by substituting v in (19) into (8). Also, from (9) and (20) we see that the initial state has to be in the set

$$\mathcal{X}_1 = \left\{x : |s_4x| < \frac{a}{\omega_n}, \; [x_1 \; x_2]' \in \mathcal{R}\right\}. \qquad (21)$$

Remark 1: If the control (19) is used in the unsaturated range, the closed-loop poles are

$$\left\{-\frac{1}{T}, -\frac{1}{T}, -\omega_n, -k\right\}.$$

This can be seen from the fact that substitution of the unsaturated control $v = -(\omega_n + k)w_u$ into (15) gives the closed-loop system

$$\dot{w} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -\frac{1}{T^2} & -\frac{2}{T} & 0 & * \\ 0 & 1 & -\omega_n & * \\ \hline 0 & 0 & 0 & -k \end{array}\right] w$$

where the symbol $*$ denotes a number that may not be 0.

Remark 2: The region where the control (19) does not saturate is given by

$$\left\{x : |s_4x| \leq \frac{a}{\omega_n + k}\right\}. \qquad (22)$$

It becomes smaller as k becomes larger.

Remark 3: Suppose that v is not constrained and let $T \to 0$. Then the peak value of $|r(t)|$ for the initial state $[r(0) \; \dot{r}(0) \; \theta(0) \; \dot{\theta}(0)]' = [0 \; 0 \; \theta_0 \; 0]'$ becomes a monotone increasing function of k; moreover, when $k \to 0$, the peak value of $|r(t)|$ approaches its lower limit, i.e., $L\theta_0$ (see Appendix IV). Then the control (19) used in the unsaturated range becomes the same kind of control as the one proposed by Lin et al.[1].

IV. Numerical results

Let the value of the equivalent pendulum length L be that of the apparatus used in this study. That is,

$$L = \frac{g}{\omega_n^2} = 0.265\,[\text{m}]. \tag{23}$$

From this, ω_n can be computed as

$$\omega_n = \sqrt{\frac{g}{L}} = 6.09\,[\text{rad/s}] \tag{24}$$

where $g = 9.81\,[\text{m/s}^2]$ was used. Suppose that the cart system has been compensated so as to have the transfer function (5). Let the maximum allowable amplitude of the cart a be given by

$$a = 0.2\,[\text{m}].$$

The design parameter k was chosen as

$$k = w_n = 6.09$$

and the following three Ts were considered:

$$T = \frac{1}{n\omega_n}\,[\text{s}],\ \ n = 1,\,2,\,5$$

or

$$T = 0.164\,[\text{s}],\ \ 0.0821\,[\text{s}],\ \ 0.0328\,[\text{s}].$$

The initial state variables were given as

$$r(0) = 0,\ \ \dot{r}(0) = 0,\ \ \theta(0) = 0.151\,[\text{rad}],\ \ \dot{\theta}(0) = 0 \tag{25}$$

which were obtained by multiplying by 0.8 a state on the boundary of the region of attraction for $T = 0.164\,[\text{s}]$. Figs.2 and 3 show the numerical results for the nonlinear and linear plant model, respectively. The difference between the two is very small, because the initial state is small. Also, the amplitude constraint of the cart $|r(t)| \le 0.2\,[\text{m}]$ is satisfied. It is seen that as T becomes smaller, the response becomes faster and the peak value of the cart amplitude becomes smaller.

Fig.4 shows the relationship between T and $\|s_4\|$; recall that $\|s_4\|$ is inversely proportional to the magnitude of the region of attraction $\mathcal{X}_0$. It is confirmed from Fig.4 that $\|s_4\|$ is a monotone increasing function of T.

V. Experimental results

Fig.5 shows a view of the experimental system. The value of the parameter L is given by (23). The cart system was compensated so as to have the transfer function (5). The maximum allowable amplitude of the cart a was set to be

$$a = 0.2\,[\text{m}]. \tag{26}$$

The design parameters T and k were chosen as

$$T = \frac{1}{\omega_n} = 0.164\,[\text{s}],\ \ k = \omega_n = 6.09. \tag{27}$$

Fig.6 shows the experimental results. First the pendulum was swung up from the pendant position ($\theta = -\pi$) using the method in [7], and then stabilized at the upright position with

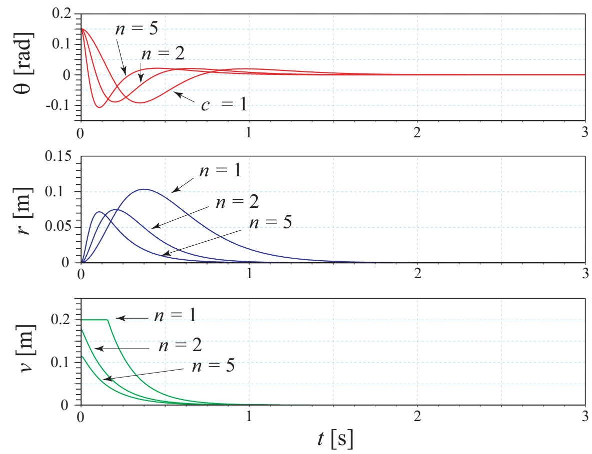

Fig. 2. Numerical results for the nonlinear model.

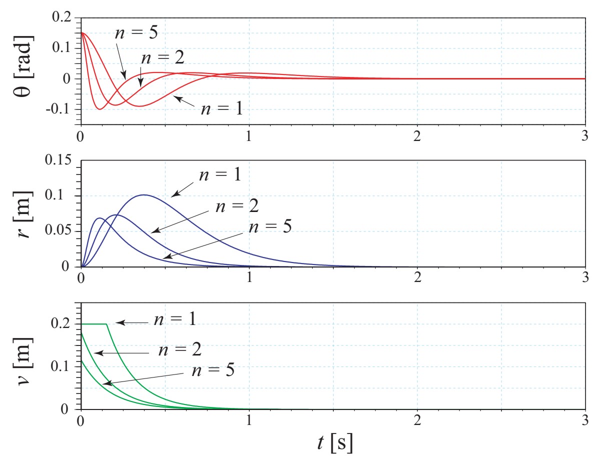

Fig. 3. Numerical results for the linear model.

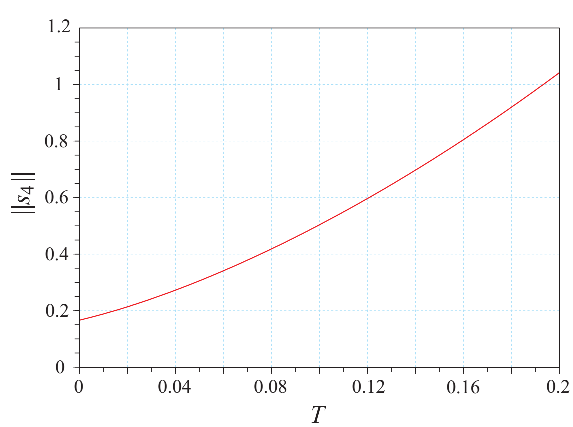

Fig. 4. Relationship between T and $\|s_4\|$.

the proposed saturating control. As the switching criterion of control law, the following logic was adopted: if

$$|\theta| < 0.5\,[\text{rad}]\ \ \text{and}\ \ |s_4 x| < 0.8\frac{a}{\omega_n} \tag{28}$$

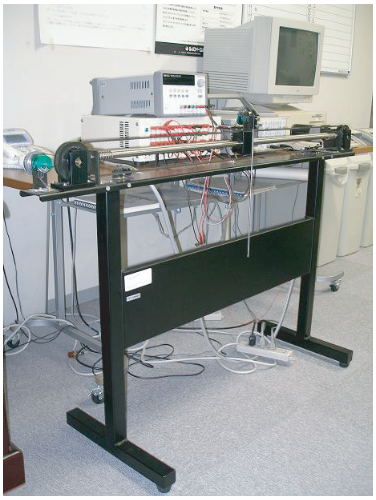

Fig. 5. View of the experimental system.

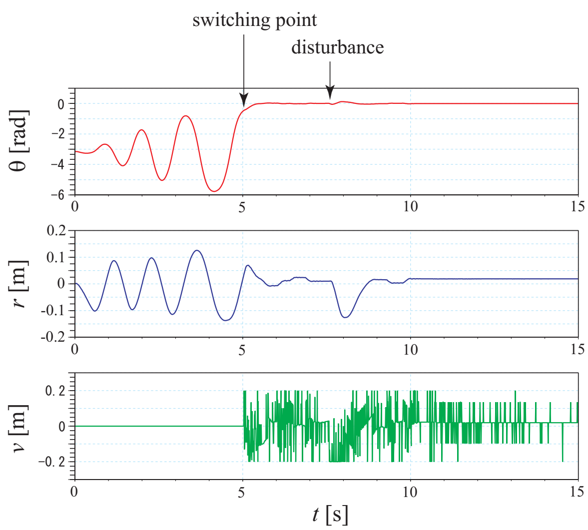

Fig. 6. Experimental results.

hold, then switch the control mode from swing-up to stabilization. The first condition is used to determine if the pendulum is close to the upright position. The second condition means that the state x is in the subset of $\mathcal{X}_0$, i.e., 0.8 times $\mathcal{X}_0$ (see (20)). Also, after the switching, a disturbance was applied to the pendulum by patting it with a finger. Fig.7 shows an enlarged graph of the part of the disturbed response. It can be seen from these experiments that the pendulum is stabilized with the amplitude constraint of the cart being satisfied.

VI. Conclusion

The stabilization problem for an inverted pendulum with restricted cart travel was reduced to the stabilization problem for an unstable first-order system with bounded input, by compensating the cart transfer function to be a second-order lag and then decomposing the system into the stable and unstable subsystem by a change of coordinates. A saturating

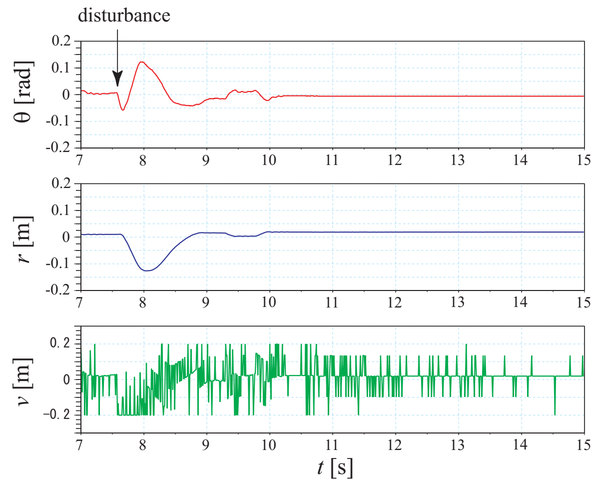

Fig. 7. Experimental results enlarged.

control solving the original problem and the expression of the region of attraction were obtained from the investigation of the reduced problem. These results were derived based on the linearized model of the system, so they are valid in a small region of the state space around the equilibrium point.

The stable subsystem has the poles $\{-1/T, -1/T, -\omega_n\}$, where T is the design parameter; and the unstable subsystem is stabilized by the saturating control with parameter k. A control system with good response characteristics can be designed by adjusting the parameters T and k. Actually, these parameters are designed considering response characteristics, actuator saturation, and measurement noise.

Appendix I
Proof of the fact that if (9) and (10) hold, then (4) is satisfied.

Conditions (9) and (10) can be replaced by

$$\begin{bmatrix} r(-\infty) & \dot{r}(-\infty) \end{bmatrix}' = 0, \quad |v(t)| \le a, \quad \forall t > -\infty.$$

From these, $r(t)$ is computed as

$$r(t) = \int_{-\infty}^{t} g(t-\tau)v(\tau)d\tau.$$

Use of the change of variable $\eta = t - \tau$ yields

$$r(t) = \int_{0}^{\infty} g(\eta)v(t-\eta)d\eta.$$

From this and (6), the following inequality is obtained.

$$|r(t)| \le \int_{0}^{\infty} |g(\eta)| \cdot |v(t-\eta)|d\eta \le a$$

Appendix II
Proof of the fact that v in (17) asymptotically stabilizes the w_u subsystem for $x(0) \in \mathcal{X}_0$.

Consider the positive definite function of w_u

$$V = \frac{1}{2}w_u^2.$$

The time derivative of this function is given by

$$\begin{aligned}\dot{V} &= w_u \dot{w}_u = w_u(\omega_n w_u + v) \\ &= w_u(\omega_n w_u - \text{sat}((\omega_n + k)w_u, a)).\end{aligned}$$

From this and the fact that $k > 0$, it is seen that if $|\omega_n w_u(0)| < a$, then $\dot{V}(t) \leq 0$, and moreover the equality holds only when $w_u = 0$. Therefore, from LaSalle's invariance theorem [8], the w_u subsystem is asymptotically stable for $w(0)$ satisfying $|\omega_n w_u(0)| < a$, i.e., $|s_4 x(0)| < a/\omega_n$. Conversely, if $|\omega_n w_u(0)| \geq a$, then $\dot{V}(t) \geq 0$, so $w_u(t)$ does not converge to zero.

APPENDIX III
PROOF OF THE FACT THAT THE REGION OF ATTRACTION (20) BECOMES MAXIMAL WHEN $T \to 0$.

Take from (11)

$$x_3 = r + L\theta. \tag{29}$$

Differentiating this twice with respect to t and using (2) and (11) give

$$\dot{x}_4 = \omega_n^2 x_3 - \omega_n^2 r. \tag{30}$$

Note that when $T \to 0$, the variable r can be considered as the input of the system (30), because then, from (5), $r = v$ (an ideal case where an infinitely large input would be required). Also, we see from (20) and the definition of S that when $T \to 0$, the set (20) becomes

$$\left\{ x : \left| x_3 + \frac{1}{\omega_n} x_4 \right| < a \right\}. \tag{31}$$

Define the variable

$$\lambda := x_3 + \frac{1}{\omega_n} x_4 \tag{32}$$

and consider the positive definite function of λ

$$V_1 = \frac{1}{2}\lambda^2. \tag{33}$$

The time derivative of V_1 is computed as

$$\begin{aligned}\dot{V}_1 &= \lambda\dot{\lambda} = \lambda\left(x_4 + \frac{1}{\omega_n}\dot{x}_4\right) \\ &= \lambda\left\{x_4 + \frac{1}{\omega_n}(\omega_n^2 x_3 - \omega_n^2 r)\right\} \\ &= \lambda(\omega_n x_3 + x_4 - \omega_n r) \\ &= \omega_n(\lambda^2 - \lambda r).\end{aligned} \tag{34}$$

Here (30) and (32) were used. From (34), we get

$$\min_{|r|\leq a} \dot{V}_1 = \omega_n(\lambda^2 - |\lambda| a). \tag{35}$$

Therefore, if $|\lambda| \geq a$, then

$$\min_{|r|\leq a} \dot{V}_1 \geq 0 \tag{36}$$

which shows that $|\lambda(t)|$ is a monotone nondecreasing function of t, and thus the system is unstable for any $r(t)$ satisfying $|r(t)| \leq a$. Hence, for the initial state which does not belong to the set (31), the system cannot be stabilized by changing $r(t)$ within the limited range. This means that the set (31) is the maximum region of attraction.

APPENDIX IV
ON THE BEHAVIOR OF $r(t)$ WHEN $T \to 0$.

Let the initial state be

$$[r(0)\ \dot{r}(0)\ \theta(0)\ \dot{\theta}(0)]' = [0\ 0\ \theta_0\ 0]'. \tag{37}$$

Letting $T \to 0$ in (13) yields

$$w_u = w_4 = -\frac{1}{\omega_n}x_3 - \frac{1}{\omega_n^2}x_4. \tag{38}$$

From (11) and (37), we get

$$x_3(0) = L\theta_0,\ x_4(0) = 0. \tag{39}$$

Substituting (39) into (38) gives

$$w_u(0) = -\frac{L\theta_0}{\omega_n}. \tag{40}$$

From (16) and (17), the equation of motion for w_u with v being unsaturated is given by

$$\dot{w}_u = -k w_u. \tag{41}$$

From (40) and (41), $w_u(t)$ is found to be

$$w_u(t) = -e^{-kt}\frac{L\theta_0}{\omega_n}. \tag{42}$$

Also, since $r = v$ when $T \to 0$, $r(t)$ is represented by

$$r(t) = v(t) = -(\omega_n + k)w_u(t) = \frac{(\omega_n + k)L\theta_0}{\omega_n}e^{-kt}. \tag{43}$$

From this, we have

$$\max_{t\geq 0}|r(t)| = \frac{(\omega_n + k)L\theta_0}{\omega_n}$$

which is a monotone increasing function of $k > 0$. Therefore, when $k \to 0$, the maximum amplitude of $r(t)$ approaches its lower limit, $L\theta_0$.

REFERENCES

[1] Z. Lin, A. Saberi, M. Gutmann and Y.A. Shamash, "Linear controller for an inverted pendulum having restricted travel: a high-and-low gain approach", *Automatica*, vol.32, no.6, pp.933–937, Jun. 1996.

[2] K. Yoshida and I. Matsumoto, "Swing-up control of a translational inverted pendulum having restricted travel", *Trans. of the Institute of Systems, Control and Information Engineers*", Vol.16, No.8, pp.397–405, Aug. 2003(in Japanese).

[3] Q. Wei, W.P. Dayawansa and W.S. Levine, "Nonlinear controller for an inverted pendulum having restricted travel", *Automatica*, Vol.31, No.6, pp.841–850, Jun. 1995.

[4] A. R. Teel, "Using saturation to stabilize a class of single-input partially linear composite systems," *Pro. of the IFAC Nonlinear Control Systems Design*, pp.379–384, Jun. 1992.

[5] A. R. Teel, "Semi-global stabilization of the 'ball and beam' using 'output' feedback," *Proc. Amer. Contr. Conf.*, pp. 2577–2581, Jun. 1993.

[6] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, pp.4–6, 1972.

[7] K. Yoshida, "Swing-up control for an inverted pendulum by energy-based methods", in *Proc. Amer. Contr. Conf.*, San Diego, CA, Jun. 1999, pp.4045–4047.

[8] H.K. Khalil, *Nonlinear Systems*, 3rd Ed. Upper Saddle River, NJ: Prentice Hall, pp.126–128, 2002.