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Non-fragile H_∞ filter design with pole placement constraints for Delta operator formulated systems via LMI optimization

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Abstract— **The problem of non-fragile H_∞ filtering for a class of linear systems described by delta operator with circular region pole constraints is investigated. The purpose of the paper is to design a filter such that the error filtering system not only satisfies the prescribed circular pole constraints or *D*-stability constraint, but also meets the prescribed H_∞ norm constraint on the transfer function from the disturbance input to the estimation error. In addition, the filter gain to be designed is assumed to have multiplicative gain variations. A sufficient condition for the existence of such a filter is obtained by using appropriate Lyapunov function and linear matrix inequality (LMI) technique. A numerical example is provided to demonstrate the effectiveness and less conservativeness of the proposed designs.**

I. INTRODUCTION

In the actual engineering systems, the filters and controllers realized by microprocessors/microcontroller do have some uncertainties due to limitation in available microprocessor/microcontroller memory, effects of finite word length of the digital processor and quantization of the A/D and D/A converters and so on[1]. It has shown that optimum and robust controllers designed by modern robust control design techniques should be very sensitive or fragile with respect to error/uncertainty in controller parameters[2]. Therefore, recently, the design of non-fragile (or resilient) controller and filter have been received increasing attention, mainly in additive gain variations[3-5], while the controller and filter with multiplicative gain variations are investigated in [6-7]. [3] investigated a non-fragile nonlinear H_∞ control with additive controller gain variations. [4] studied the problem of non-fragile filter design for continuous-time systems, which the filters to be designed were assumed to be with additive gain variations. The non-fragile H_∞ filtering problem affected by finite word length(FWL) for linear discrete-time systems was investigated in [5]. A robust non-fragile Kalman filtering problem for uncertain linear systems with estimator gain uncertainty was addressed[6], where the multiplicative uncertainty model was used to describe degradations of sensors. [7] presented the non-fragile H_∞ output feedback controller design with multiplicative controller gain variations via Riccati equations method. However, most of the existing results of analysis and synthesis for non-fragile H_∞ filter or controller have been obtained separately for continuous-time and discrete-time system.

Meanwhile, there has also been a rising interest in constructing Delta operator instead of traditional z-transform for sampling continuous system. Two major advantages are known for the use of delta operator parametrization: a theoretically unified formulation of continuous-time and discrete-time systems, see, e.g.,$[8-9]$, and the reference therein, and better numerical properties in FWL implementations when compared with traditional Z transform at high sampling period[10]. On the other hand, as is well known, the estimation dynamics of a linear system is closely related to the location of its poles. By constraining the filter's poles to lie inside a prescribed region in the complex plane, the filter designed would have the expected transient performance[11]. Hence, if combined the delta operator theory and pole-placement method with non-fragile filter theory, the unstable and fragile problem of filtering error system can be solved, and well transient performance can be obtained at the same time. In the past few years, the robust filter problem for delta operator systems has been studied by a number of researchers[12-13], but they are all based on an implicit assumption that the filter will be implemented exactly. Although, the robust non-fragile H_∞ state feedback controller for a class of uncertain system is designed based on delta operator, where the controller and the controlled object parameters are assumed to have additive norm-bounded variations in [14], the non-fragile filter problem for delta operator system remains to be resolved.

Motivated by above points, a non-fragile H_∞ filter with the considerations of the multiplicative gain variations is designed for a class of linear systems described by delta operator with circular region pole constraints. The paper is organized as follows. First of all, we introduce the delta operator model to overcome the unstable problem caused by using the traditional z-transform at high sampling rates. Next, a sufficient condition for the existence of such a non-fragile H_∞ filter is obtained via appropriate Lyapunov function and LMI technique. Further, less conservativeness can be introduced by considering a more general type of filter gain uncertainties. Then, a convex optimization problem is then formulated, and the optimal solutions to the non-fragile H_∞ filter problem with pole location for the domain considered is also provided. Finally, a numerical example is given to illustrate the effectiveness of the developed techniques.

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II. PROBLEM FORMULATION

Consider the following linear continuous system:

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bw(t) \\ y(t) &= Cx(t)+Dw(t) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $x(t)\in\mathbf{R}^n$ is the state, $w(t)\in\mathbf{R}^r$ is the disturbance input which belongs to $l_2[0,\infty)$, $z(t)\in R^q$ is the regulated output and $y(t)\in\mathbf{R}^p$ is the measured output, respectively. The system matrices A, B, C, D and L are known constant matrices of appropriate dimensions.

Then, the delta operator system can be given as following

$$\begin{cases} \delta x(k)=A_\delta x(k)+B_\delta w(k) \\ y(k)=Cx(k)+Dw(k) \\ z(k)=Lx(k) \\ x(k)=0,\ \ k\le 0 \\ A_\delta=(A_z-I)/h, A_z=e^{Ah}, B_\delta=B_z/h, B_z=\int_0^h e^{A\tau}Bd\tau, \\ \delta x(k)=(x(k+1)-x(k))/h \end{cases} \tag{2}$$

Throughout the paper, I denotes an identity matrix of appropriate dimension, and h denotes the sampling period. A_δ and B_δ are the corresponding delta operator system matrices, A_z and B_z are the z-domain discrete system matrices. C, D and L are the same as the z-domain discrete system matrices respectively. In addition, δ is delta operator, which is defined by:

$$\delta x(t)=:\begin{cases} \frac{d}{dt}x(t) & h=0 \\ (x(t+h)-x(t))/h & h\neq 0 \end{cases} \tag{3}$$

On the other hand, it is obvious that

$$\lim_{h\to 0}A_\delta=\lim_{h\to 0}(e^{Ah}-I)/h=A,\ \lim_{h\to 0}B_\delta=B. \tag{4}$$

Consequently, when $h\to 0$, the δ domain discrete system changes to continuous system.

We are interested in designing an delta operator filter

$$\begin{aligned} \delta\bar{x}(k) &= A_{\delta F}\bar{x}(k)+B_{\delta F}y(k) \\ \bar{z}(k) &= C_{\delta F}\bar{x}(k) \end{aligned} \tag{5}$$

where $\bar{x}(t)\in\mathbf{R}^n$ is the filter state, $A_{\delta F}$, $B_{\delta F}$ and $C_{\delta F}$ are the parameters of the filter with multiplicative gain variations described by

$$\begin{gathered} A_{\delta F}=A_{\delta F1}(I+\Gamma_1), B_{\delta F}=B_{\delta F1}(I+\Gamma_2), \\ C_{\delta F}=C_{\delta F1}(I+\Gamma_3) \end{gathered} \tag{6}$$

where $A_{\delta F1}$, $B_{\delta F1}$, $C_{\delta F1}$ are the filter parameters to be designed. Γ_1, Γ_2 and Γ_3 represent the gain variations with the following form

$$\Gamma_1=H_1\Re_1(k)E_1, \Gamma_2=H_2\Re_2(k)E_2, \Gamma_3=H_3\Re_3(k)E_3 \tag{7}$$

where $H_i, E_i, (i=1,2,3)$ are known constant real matrices with appropriate dimensions, $\Re_i(k)$ denotes time-varying parameter uncertainties, and is assumed to be of diagonal form

$$\Re_i(k)=diag\{\Re_{i1}(k),\cdots\Re_{ir}(k)\}, \tag{8}$$

where $\Re_{il}\in\mathbf{R}^{p_l\times q_l}, l=1,\cdots r$, are unknown real time-varying matrices satisfying

$$\Re_{il}^{\mathrm{T}}\Re_{il}\le I,\ where\ k=0,1,2,\cdots.$$

Consider the linear transformation on the filter state

$$\hat{x}(t)=M\bar{x}(t) \tag{9}$$

where M is an invertible matrix to make the design easy, which can be given out during the design of the filter. We have a new representation form of the filter as follows

$$\begin{aligned} \delta\hat{x}(k) &= MA_{\delta F}M^{-1}\hat{x}(k)+MB_{\delta F}y(k) \\ \hat{z}(k) &= C_{\delta F}M^{-1}\hat{x}(k) \end{aligned} \tag{10}$$

Applying the filter (10) to the system (2), we obtain the filtering error system

$$\begin{aligned} \delta\xi(k) &= \bar{A}_\delta\xi(k)+\bar{B}_\delta w(k) \\ e(k) &= \bar{C}_\delta\xi(k) \end{aligned} \tag{11}$$

where $\xi(k)=\begin{bmatrix} x(t) \\ \hat{x}(t)\end{bmatrix}$, $e(k)=z(k)-\hat{z}(k)$ is the estimation error, and $\bar{A}_\delta=\begin{bmatrix} A_\delta & 0 \\ MB_{\delta F}C & MA_{\delta F}M^{-1}\end{bmatrix}$, $\bar{B}_\delta=\begin{bmatrix} B_\delta \\ MB_{\delta F}D\end{bmatrix}$, $\bar{C}_\delta=\begin{bmatrix} L & -C_{\delta F}M^{-1}\end{bmatrix}$.

The transfer function matrix of the filtering error system (11)from $w(k)$ to $e(k)$ is given by

$$G(z)=\bar{C}_\delta(zI-\bar{A}_\delta)^{-1}\bar{B}_\delta. \tag{12}$$

Our objective is to develop a filter of the form (5)(or (10)) such that, for all admissible filter gain variations (6), the filtering error system (11) satisfies the following requirements:

a. While there is no exogenous disturbance, that is $w(k)=0$, the filtering error system (11) is asymptotically stable, and all filtering error system's poles lie in the region $D(a,r)$ in the complex plane with the center at $(a+j0)$ and the radius r, and have

$$\begin{aligned} &\lambda(\bar{A}_\delta)\subset D(a,r) \\ &|a|+r<2/h,\ r<1/h, \end{aligned} \tag{13}$$

where $\lambda(\bar{A}_\delta)$ denote the eigenvalue of $\bar{A}_\delta$.

b. The filtering error system (11) has H_∞ performances, i.e, the transfer function matrix $G(z)$ satisfies

$$\|G(z)\|_\infty<\gamma. \tag{14}$$

Now, we first provide some important lemmas which will be useful in the derivation of our main results.

Lemma 1.[15] All the poles of the matrix $\bar{A}_\delta\in\mathbf{R}^{n\times n}$ are located with a given circular region $D(a,r)$ as shown in figure 1, i.e.,$\lambda(\bar{A}_\delta)\subset D(a,r)$, if and only if there exists a matrix $X>0$ such that

(i)

$$(I+hA_a)^{\mathrm{T}}\frac{X}{h}(I+hA_a)-\frac{X}{h}<0, \tag{15}$$

(ii)

$$\begin{bmatrix} -rX & * \\ X\bar{A}_\delta+\beta X & -rX \end{bmatrix}<0, \tag{16}$$

where $A_a = \frac{\bar{A}_\delta - aI-(1/h)I}{rh}, \beta = r-a-1/h$, and the above two matrix inequalities are equivalent.

III. MAIN RESULTS

A. H_∞ Filtering for Delta Operator Formulated Systems

Before continuing with the solution to the synthesis problem, we present the following theorem which guarantees that the filtering error system (11) is asymptotically stable and has H_∞ performance criterions at the same time.

Theorem 1. Given scalar $\gamma > 0$ and the sampling period h, if there exist some matrices $X = X^{\mathrm{T}} > 0$, $\bar{A}_\delta$, $\bar{B}_\delta$, $\bar{C}_\delta$ such that

$$\begin{bmatrix} hX\bar{A}_\delta + h\bar{A}_\delta^{\mathrm{T}}X & * & * & * \\ h\bar{B}_\delta^{\mathrm{T}}X & -\gamma^2h^2I & * & * \\ hX\bar{A}_\delta & hX\bar{B}_\delta & -X & * \\ h\bar{C}_\delta & 0 & 0 & -I \end{bmatrix} < 0 \tag{17}$$

holds, then the filtering error system (11) is asymptotically stable and satisfies H_∞ performance constraint.

Proof. Due to the limit of the space, the proof is omitted.

B. Non-fragile H_∞ Filter with D-stability Constraints

In this subsection, we develop non-fragile H_∞ filter with D-stability constraints based on LMI technique.

Definition 1. For prescribed $\gamma > 0$ and the sampling period h, assume that there exist $X = X^{\mathrm{T}} > 0$ and filter parameters $\bar{A}_\delta$, $\bar{B}_\delta$ and $\bar{C}_\delta$ satisfying (16) and (17) at the same time, then the filtering error system (11) is D-stable and satisfies H_∞ norm constraint simultaneously.

Remark 1. In the above definition, when the D-stable is not considered, i.e. $D(a,r) = D(-1/T, 1/T)$, the problem is reduced to non-fragile H_∞ filter design without circular pole constraint.

In the following, based on LMI technique, we give the sufficient condition for the existence of the non-fragile H_∞ filter as following theorem.

Theorem 2. For prescribed $\gamma > 0$ and the sampling period h, assume that there exist $S = S^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} > 0$, $\hat{A}_{\delta F}$, $\hat{B}_{\delta F}$, $\hat{C}_{\delta F}$ and $\Lambda_i = diag\{\lambda_{i1}I, \cdots, \lambda_{ir}I\}, (i = 1,2,3)$ such that

$$\Omega_1 = \begin{bmatrix} \Xi_{01} & * \\ \Xi_{02} & \Xi_{03} \end{bmatrix} < 0, \Omega_2 = \begin{bmatrix} \Xi_1 & * \\ \Xi_2 & \Xi_3 \end{bmatrix} < 0, \tag{18}$$

where

$$\Xi_{01} = \begin{bmatrix} -rS & * & * & * \\ -rS & -rR & * & * \\ \emptyset_{21} & \emptyset_{21} & -rS & * \\ \varphi_{31} & \varphi_{32} & -rS & -rR \end{bmatrix},$$

$$\Xi_{02} = \begin{bmatrix} 0 & 0 & 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} \\ \Lambda_1E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} \\ \Lambda_2E_2C & \Lambda_2E_2C & 0 & 0 \end{bmatrix},$$

$$\Xi_{03} = \{-\Lambda_1I, -\Lambda_1I, -\Lambda_2I, -\Lambda_2I\},$$

$$\Xi_1 = \begin{bmatrix} \emptyset_{22} & * & * & * & * & * \\ \Upsilon_{31} & \Upsilon_{32} & * & * & * & * \\ hB_\delta^{\mathrm{T}}S & \Upsilon_{33} & -\gamma^2h^2I & * & * & * \\ hSA_\delta & hSA_\delta & hSB_\delta & -S & * & * \\ \Upsilon_{34} & \Upsilon_{35} & \Upsilon_{36} & -I & -R & * \\ \Upsilon_{37} & hL & 0 & 0 & 0 & -I \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} & 0 & 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} & 0 \\ h\Lambda_2E_2C & h\Lambda_2E_2C & h\Lambda_2E_2D & 0 & 0 & 0 \\ 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} & 0 & 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} & 0 \\ h\Lambda_1E_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -H_3^{\mathrm{T}}\hat{C}_{\delta F}^{\mathrm{T}} \\ h\Lambda_3E_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Xi_3 = \{-\Lambda_2I, -\Lambda_2I, -\Lambda_1I, -\Lambda_1I, -\Lambda_3I, -\Lambda_3I\}$$

with $\emptyset_{21} = SA_\delta + \beta S, \emptyset_{22} = hSA_\delta + hA_\delta^{\mathrm{T}}S, \varphi_{31} = RA_\delta + \hat{B}_{\delta F}C + \hat{A}_{\delta F} + \beta S, \varphi_{32} = RA_\delta + \hat{B}_{\delta F}C + \beta R, \Upsilon_{31} = hA_\delta^{\mathrm{T}}S + hRA_\delta + h\hat{B}_{\delta F}C + h\hat{A}_{\delta F}, \Upsilon_{32} = hA_\delta \mathrm{T}R + hRA_\delta + h\hat{B}_{\delta F}C + hC^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}}, \Upsilon_{33} = hB_\delta^{\mathrm{T}}R + hD\mathrm{T}\hat{B}_{\delta F}^{\mathrm{T}}, \Upsilon_{34} = h(RA_\delta + \hat{B}_{\delta F}C + \hat{A}_{\delta F}), \Upsilon_{35} = hRA_\delta + h\hat{B}_{\delta F}C, \Upsilon_{36} = hRB_\delta + h\hat{B}_{\delta F}D, \Upsilon_{37} = hL - h\hat{C}_{\delta F}$. Then, the filtering error system (11) is D-stable and satisfies H_∞ norm constraint simultaneously.

Moreover, if there exist solutions of these inequalities, the non-fragile filter can be given by

$$A_{\delta F1} = (S-R)^{-1}\hat{A}_{\delta F}, B_{\delta F1} = (S-R)^{-1}\hat{B}_{\delta F}, C_{\delta F1} = \hat{C}_{\delta F}. \tag{19}$$

Proof. Due to the limit of the space, the proof is omitted.

Remark 2. In the above theorem, when the D-stable is not considered, i.e. $D(a,r) = D(-1/h, 1/h)$, the problem is reduced to non-fragile H_∞ filter design without circular pole constraint. Then, the problem of non-fragile H_∞ filter without circular pole constraints design can be resolved by solving LMIs $\Omega_2 < 0$ and

$$\begin{bmatrix} S & S \\ S & R \end{bmatrix} > 0. \tag{20}$$

Remark 3. When the filter gain variations model is the same as the model in [6], i.e.,

$$\Gamma_1 = H_1\Re_1(k)E_1, \Gamma_2 = H_2\Re_2(k)E_2, \Gamma_3 = H_3\Re_3(k)E_3 \tag{21}$$

where $H_i, E_i, (i = 1,2,3)$ are known constant matrices of appropriate dimensions, and $\Re_i (i = 1,2,3)$ are real uncertain matrices with

$$\Re_i^{\mathrm{T}}(k)\Re_i(k) \leq I, i = 1,2,3 \tag{22}$$

where $\Re_i(k)$ without the constraint (8).

Then, the following theorem presents a sufficient condition for the solvability of the non-fragile H_∞ filtering problem with the filter gain variations (21).

Theorem 3. For prescribed $\gamma > 0$ and the sampling period h, assume that there exist $S = S^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} > 0$, $\hat{A}_{\delta F}$, $\hat{B}_{\delta F}$, $\hat{C}_{\delta F}$ and scalars $\lambda_i, (i = 1,2,3)$ such that

$$\begin{bmatrix} \Xi_{01} & * \\ \Xi_{12} & \Xi_{13} \end{bmatrix} < 0, \begin{bmatrix} \Xi_1 & * \\ \Xi_{22} & \Xi_{23} \end{bmatrix} < 0, \tag{23}$$

where

$$\Xi_{12} = \begin{bmatrix} 0 & 0 & 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} \\ \lambda_1 E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} \\ \lambda_2 E_2 C & \lambda_2 E_2 C & 0 & 0 \end{bmatrix},$$

$$\Xi_{13} = \{-\lambda_1 I, -\lambda_1 I, -\lambda_2 I, -\lambda_2 I\}$$

$$\Xi_{22} = \begin{bmatrix} 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} & 0 & 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} & 0 \\ h\lambda_2 E_2 C & h\lambda_2 E_2 C & h\lambda_2 E_2 D & 0 & 0 & 0 \\ 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} & 0 & 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} & 0 \\ h\lambda_1 E_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -H_3^{\mathrm{T}}\hat{C}_{\delta F}^{\mathrm{T}} \\ h\lambda_3 E_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Xi_{23} = \{-\lambda_2 I, -\lambda_2 I, -\lambda_1 I, -\lambda_1 I, -\lambda_3 I, -\lambda_3 I\}$$

with Ξ_{01}, Ξ_1 defined in (18) and β defined in (16), respectively. Then the filtering error system (11) with the filter uncertainties as (21) is D-stable and satisfies H_∞ norm constraint simultaneously. The filter parameters can also be given by (19).

Proof. Due to the limit of the space, the proof is omitted.

Remark 4. When the filter parameter uncertainties are not considered, i.e., $\Gamma_1 = 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, the problem reduces to standard H_∞ filter design with circular pole constraints. Hence, (18) (or (23)) are reduced to the following LMIs:

$$\Xi_{01} < 0, \Xi_1 < 0. \tag{24}$$

C. Comparison with the Existing Design Method

In this subsection, we compare our results, which don't consider the condition of D-stable, with Yang and Che[5](continuous system) and Che and Yang[6](discrete system), respectively. In this subsection, we denote

$$\delta x(t) = \begin{cases} \dot{x}(t), & \textit{Continuous Case}, \\ x(t+1), & \textit{Disctrete Case}, \end{cases}$$

i.e., the signification of δ in this subsection is different from the one in other sections. Further, the filter and filtering error system are similar to those in section 2.

Then, by using the proposed design method as Theorem 2, we can easily obtain the following lemma for continuous system and discrete system, respectively.

Lemma 2. For prescribed $\gamma > 0$ and Ξ_3 defined in (18), the filtering error system is asymptotically stable and satisfies H_∞ norm constraint, if there exist $S = S^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} > 0$, $\hat{A}_{\delta F}$, $\hat{B}_{\delta F}$, $\hat{C}_{\delta F}$ and $\Lambda_i = diag\{\lambda_{i1} I, \cdots, \lambda_{ir} I\}, (i = 1, 2, 3)$ such that

• For continuous system:

$$\begin{bmatrix} \Xi_{31} & * \\ \Xi_{32} & \Xi_3 \end{bmatrix} < 0, \tag{25}$$

where

$$\Xi_{31} = \begin{bmatrix} SA + A^{\mathrm{T}}S & * & * & * \\ \Upsilon_{51} & \Upsilon_{52} & * & * \\ B^{\mathrm{T}}S & \Upsilon_{53} & -\gamma^2 I & * \\ \Upsilon_{54} & L & 0 & -I \end{bmatrix},$$

$$\Xi_{32} = \begin{bmatrix} 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} & 0 & 0 \\ \Lambda_2 E_2 C & \Lambda_2 E_2 C & \Lambda_2 E_2 D & 0 \\ 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} & 0 & 0 \\ \Lambda_1 E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -H_3^{\mathrm{T}}\hat{C}_{\delta F}^{\mathrm{T}} \\ \Lambda_3 E_3 & 0 & 0 & 0 \end{bmatrix},$$

and $\Upsilon_{51} = A^{\mathrm{T}}S + R^{\mathrm{T}}A + \hat{B}_{\delta F}C + \hat{A}_{\delta F}$, $\Upsilon_{52} = A^{\mathrm{T}}R + R^{\mathrm{T}}A + \hat{B}_{\delta F}C + C^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}}$, $\Upsilon_{53} = B^{\mathrm{T}}R + D^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}}$, $\Upsilon_{54} = L - \hat{C}_{\delta F}$.

• For discrete system:

$$\begin{bmatrix} \Xi_{41} & * \\ \Xi_{42} & \Xi_3 \end{bmatrix} < 0, \tag{26}$$

where

$$\Xi_{41} = \begin{bmatrix} -S & * & * & * & * & * \\ -S & -R & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ SA_z & SA_z & SB_z & -S & * & * \\ \Upsilon_{61} & \Upsilon_{62} & \Upsilon_{63} & -S & -R & * \\ \Upsilon_{64} & L & 0 & 0 & 0 & -I \end{bmatrix},$$

$$\Xi_{42} = \begin{bmatrix} 0 & 0 & 0 & 0 & H_2^{\mathrm{T}}\hat{B}_{\delta F}^{\mathrm{T}} & 0 \\ \Lambda_2 E_2 C & \Lambda_2 E_2 C & \Lambda_2 E_2 D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_1^{\mathrm{T}}\hat{A}_{\delta F}^{\mathrm{T}} & 0 \\ \Lambda_1 E_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -H_3^{\mathrm{T}}\hat{C}_{\delta F}^{\mathrm{T}} \\ \Lambda_3 E_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\Upsilon_{61} = RA_z + \hat{B}_{\delta F}C + \hat{A}_{\delta F}$, $\Upsilon_{62} = RA_z + \hat{B}_{\delta F}C$, $\Upsilon_{63} = RB_z + \hat{B}_{\delta F}D$, $\Upsilon_{64} = L - \hat{C}_{\delta F}$. Furthermore, the inequality (20) should be also satisfied for the continuous case and the discrete case. And the designed filter's parameters can also be obtained by (19).

IV. NUMERICAL SIMULATIONS

In this section, numerical simulations are carried out to confirm validity and advantage of the proposed method, which also show the characteristic of discrete-time system and delta operator system in sampling continuous-time system.

A. Simulation for the Proposed Method

Consider a continuous-time system in s-domain:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.7 & 0.4 & 0.6 \\ -0.4 & -0.5 & 0.4 \\ -0.6 & -0.4 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0.05 & 0 \\ 0.05 & 0 \\ 0.06 & 0 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 3 & -2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0.9 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} x(t), \end{aligned}$$

and the part of filter gain variations as following

$$H_1=\begin{bmatrix}1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix},H_3=\begin{bmatrix}1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix},$$

$$E_1=0.02*\begin{bmatrix}1&1&1\\1&2&1\\-1&1&2\\1&2&1\\2&1&3\\1&2&1\end{bmatrix},E_3=0.02*\begin{bmatrix}1&1&2\\1&2&1\\-1&1&1\\1&2&1\\4&1&-1\\1&3&1\end{bmatrix},$$

$$H_2^T=\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix},E_2=0.02*\begin{bmatrix}1\\2\\1\\2\\1\\2\end{bmatrix},\Re_i(k)=\begin{bmatrix}\Re_{i1}(k)&0\\0&\Re_{i2}(k)\end{bmatrix},$$

where $\Re_{i1}(k),\Re_{i2}(k)\in\mathbf{R}^{3\times3}(i=1,2,3,4)$.

By using shift operator and delta operator in sampling the continuous-time system respectively, we get the relevant different discrete-time systems in z-domain and δ-domain.

(1) When $h=0.1(s)$, there exist

$$A_z=\begin{bmatrix}0.9300&0.0365&0.0572\\-0.0388&0.9497&0.0369\\-0.0557&-0.0391&0.9488\end{bmatrix},B_z=\begin{bmatrix}0.0051&0\\0.0049&0\\0.0056&0\end{bmatrix}$$

$$A_\delta=\begin{bmatrix}-0.7005&0.3650&0.5720\\-0.3876&-0.5029&0.3687\\-0.5569&-0.3914&-0.5123\end{bmatrix},B_\delta=\begin{bmatrix}0.0509&0\\0.0489&0\\0.0561&0\end{bmatrix}$$

(2) When $h=0.1(ms)$, there exist

$$A_z=\begin{bmatrix}0.9999&0.0000&0.0001\\-0.0000&0.9999&0.0000\\-0.0001&-0.0000&0.9999\end{bmatrix},B_z=10^{-5}*\begin{bmatrix}0.5000&0\\0.5000&0\\0.5000&0\end{bmatrix}$$

$$A_\delta=\begin{bmatrix}-0.7000&0.4000&0.6000\\-0.4000&-0.5000&0.4000\\-0.6000&-0.4000&-0.5000\end{bmatrix},B_\delta=\begin{bmatrix}0.0500&0\\0.0500&0\\0.0600&0\end{bmatrix}$$

(3) When $h=1(s)$, there exist

$$A_z=\begin{bmatrix}0.3662&0.1311&0.3362\\-0.2599&0.5177&0.1526\\-0.2504&-0.2814&0.4640\end{bmatrix},B_z=\begin{bmatrix}0.0511&0\\0.0366&0\\0.0269&0\end{bmatrix}$$

$$A_\delta=\begin{bmatrix}-0.6338&0.1311&0.3362\\-0.2599&-0.4823&0.1526\\-0.2504&-0.2814&-0.5360\end{bmatrix},B_\delta=\begin{bmatrix}0.0511&0\\0.0366&0\\0.0269&0\end{bmatrix}$$

From above results, we found that when let $h=0.1(ms)$, the Delta operate system is reduced to continuous system, and we can also find the Delta operator model has the advantage of better numerical properties at high sampling rates; and when let $h=1(s)$, the Delta operate system is reduced to the one of discrete system.

Due to the limit of the space, the filter gain matrices and the filtering error system poles are omitted.

For comparison, we compute non-fragile H_∞ performance with different filter gain uncertainties (7) and (21), respectively. The optimal γ under different sampling periods are given in Table 1. From Table 1, the optimal γ by Theorem 2 are smaller than those by Theorem 3 under different sampling periods. Obviously, Theorem 2 is less conservative then Theorem 3.

Table 1 The optimal γ with different uncertainties

h	Thm. 2	Thm. 3
h=0.1(s)	0.2583	0.2620
h=0.1(ms)	6.6361	6.7184
h=1(s)	0.3421	0.3477

Furthermore, in order to demonstrate advantages of the designed filter, we make a comparison between non-fragile H_∞ filter and standard H_∞ one in the presence of filter gain variations (7). For different sampling periods, the optimal γ by different methods are given in Table 2.

Table 2 Comparison of non-fragile H_∞ filter with standard H_∞ filter for the optimal γ

h	Thm.2	Rem.4	Rem.4 with (7)
h=0.1(s)	0.2583	0.2277	28.9412
h=0.1(ms)	6.6361	6.1812	68.2714
h=1(s)	0.3421	0.2190	375.7355

From Table 2, the optimal γ by Remark 4 are obviously smaller than those by Theorem 2 for different sampling periods respectively. However, when the standard filter is with the uncertainties described by (7), the optimal H_∞ performance of the standard filter is serious deteriorative.

On the other hand, in order to further demonstrate the advantage of the non-fragile filter, we assume the disturbance input $w(k)=\begin{bmatrix}w_1^{\mathrm{T}}(k)&w_2^{\mathrm{T}}(k)\end{bmatrix}^{\mathrm{T}}$ as following:

$$w_1(k)=w_2(k)=\begin{cases}0.5 & 10\le k\le 11\\-0.5 & 40\le k\le 41\\0 & otherwise\end{cases}. \tag{27}$$

Figs.1-3 show the response of estimation error e(k) under the disturbance w(k). We can also compute $\|e(k)\|_2=\int_{k=0}^{\infty}e^{\mathrm{T}}(k)e(k)dk$ and $\|w(k)\|_2=\int_{k=0}^{\infty}w^{\mathrm{T}}(k)w(k)dk$ for $h=0.1(ms)$ and $h=0.1(s)$, and $\|e\|_2=\sum_{k=0}^{\infty}e\mathrm{T}(k)e(k)$ and $\|w(k)\|_2=\sum_{k=0}^{\infty}w\mathrm{T}(k)w(k)$ for $h=1(s)$, respectively. We denote $\vartheta=\frac{\|e(k)\|_2}{\|w(k)\|_2}$, and obtain the following table.

Table 3 Comparison of non-fragile H_∞ filter with standard H_∞ filter under the disturbance w(k)

Sampling period	w(k)	NF		SF with (7)	
h	$\|w(k)\|_2$	$\|e(k)\|_2$	ϑ_1	$\|e(k)\|_2$	ϑ_2
h=0.1(s)	1.0071	0.0211	0.0210	0.0325	0.0323
h=0.1(ms)	1.0071	0.1401	0.1391	0.1518	0.1573
h=1(s)	1.4140	0.0443	0.0313	0.1061	0.0750

• NF: Non-fragile filter; • SF: Standard filter

From Table 3, it is easy to find that ϑ_1 are always smaller than ϑ_2 for the different sampling periods, which demonstrate the effectiveness of the proposed method.

B. Comparison with the Existing Works

In this subsection, the results are given to provide a comparison between the non-fragile H_∞ filter designed by the proposed method (Theorem 2) and the non-fragile H_∞ filter designed by the existing method (Lemma 2). Then, the H_∞ performance indexes are shown in Table 4.

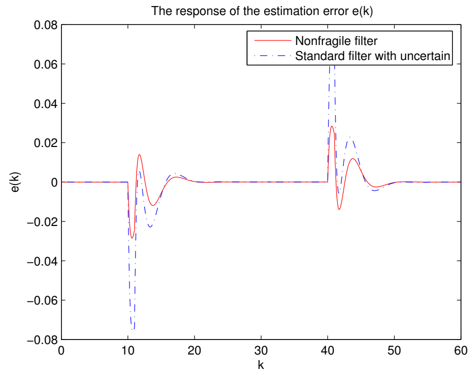

Fig. 1. Comparison of non-fragile H_∞ filter with standard H_∞ filter when $h = 0.1(s)$

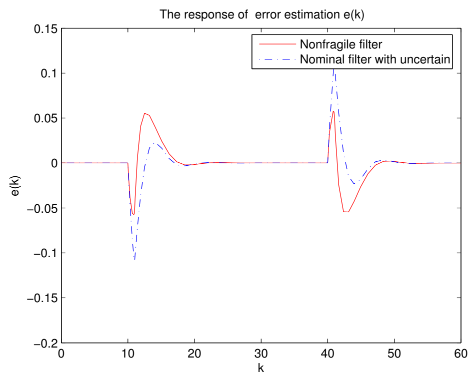

Fig. 2. Comparison of non-fragile H_∞ filter with standard H_∞ filter when $h = 0.1(ms)$

Table 4 Comparison of γ with different methods

Design	methods	γ with different conditions		
		h=0.01(ms)	h=0.1(s)	h=1(s)
Thm. 2	Delta Domain	20.9919	0.2585	0.3142
Lem. 2	z Domain	Infeasible	0.2461	0.2323

In addition, we can obtain the optimal performance index $\gamma = 0.2353$ for continuous system. From Table 4, it is easy to see that Delta operator can solve the unstable problem caused by using traditional z-transform for sampling continuous system at high sampling period though our results are not better than the results of the existing works at low sampling period. On the other hand, our proposed method can unify the related continuous-time and discrete-time systems into the delta operator systems framework. Therefore, the delta operator is widely applied in many fields of engineering such as high-speed digital signal processing, system modeling, and computer control based on fast sampled data.

V. CONCLUSIONS

The problem of a non-fragile H_∞ filter design for a class of linear systems described by delta operator with circular pole constraints is investigated, where the filter to be designed is assumed to be with multiplicative gain variations. It is worth pointing out that the filtering problems of continuous-time and discrete-time systems are investigated in the unified form by using delta operator. A sufficient condition for the existence of the filter to meet H_∞ performance and D-stable is presented via LMI, and the explicit expression of the desired filter is also developed. In addition, the proposed uncertainties are less conservative than the normal norm-bound parameter uncertainties.

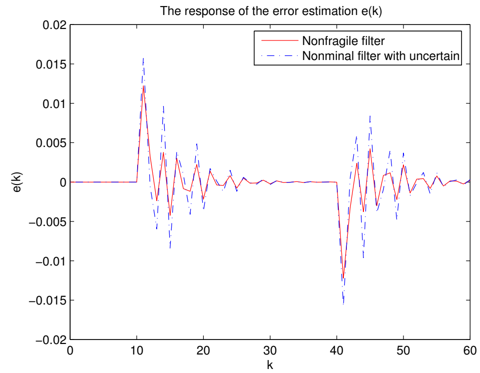

Fig. 3. Comparison of non-fragile H_∞ filter with standard H_∞ filter when $h = 1(s)$

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