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Stability and H_∞ Control for Discrete-Time Singular Systems subject to Actuator Saturation

Shuping Ma and El-Kébir Boukas

Abstract— **In this paper, sufficient conditions which guarantee that the discrete-time singular systems with actuator saturation are admissible with γ- disturbance attenuation are established. With these conditions, the estimation of stability region, state feedback design and H_∞ state feedback controllers are obtained by solving corresponding LMIs optimization problems. A numerical example to illustrate the effectiveness of the proposed methods is given.**

I. INTRODUCTION

Singular systems, which are also referred to as implicit systems, descriptor systems, have extensive applications in many practical systems, such as circuit boundary control systems, chemical processes, economy systems, and other areas [1]. Hence a great number of fundamental notions and results in control and system theory based on standard state-space systems have been extended to singular systems, such as stability, stabilization and H_∞ control problems [2-7]. On the other hand, actuator saturation can lead to poor performance of the closed-loop system and sometimes destabilizes the system. The analysis and design for systems with actuator saturation have received a lot of attentions [8-11]. For singular systems with actuator saturation, readers may refer to [12-15]. In particular, [12] discussed the semiglobal stabilization and output regulation problems of singular systems with actuator saturation. [13] gave sufficient conditions for the stability of closed-loop systems with actuator saturation. [14] not only gave the sufficient conditions for the stability of closed-loop systems with actuator saturation, but also the estimation the domain of attraction and the design of state feedback gain matrix via linear matrix inequality (LMI) technique. [15] discussed the L_2 and L_∞ problem of closed-loop systems with actuator saturation. All the results mentioned above are for continuous-time singular systems with actuator saturation. For discrete-time singular systems with actuator saturation, since the Lyapunov matrix is indefinite, it is more difficult to deal with the stability of the closed-loop systems and to design the controller compared to continuous-time singular systems with actuator saturation by using similar LMI method. To the best of our knowledge, the stability condition and H_∞ control for discrete-time singular systems with actuator saturation is an important but unexplored research topic.

In this paper, the stability and H_∞ control for discrete-time singular systems with actuator saturation are discussed. First, a sufficient condition which guarantees that the discrete-time singular systems with actuator saturation are admissible is established. Using this condition, the estimation of stability region and the state feedback design are obtained by solving a convex optimization problem. Then, the condition and the convex optimization problem such that the discrete-time singular systems with actuator saturation are admissible with γ- disturbance attenuation are obtained. Finally, a numerical example to illustrate the effectiveness of the proposed method is given.

Shuping Ma is with School of Mathematics and System Science, Shandong University, Jinan, 250100, China mashup@sdu.edu.cn

El-Kébir Boukas is with Mechanical Engineering Department, École Polytechnique de Montréal, P.O. Box 6079, Station “centre-ville”, Montréal, Québec, Canada, H3C 3A7 el-kebir.boukas@polymtl.ca

II. PROBLEM DESCRIPTION AND PRELIMINARIES

Consider a discrete-time singular system subject to actuator saturation with the following dynamics

$$\begin{cases} Ex(k+1) = Ax(k) + B\text{sat}(u(k)) + B_w w(k), \\ z(k) = Cx(k) + D\text{sat}(u(k)) + D_w w(k), \end{cases} \quad (1)$$

where $k \in \mathcal{Z}$, $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^p$ is the control input, and sat: $\mathbb{R}^p \to \mathbb{R}^p$ is the standard saturation function defined as follows:

$$\text{sat}(u(k)) = [\text{sat}(u_1(k)) \quad \text{sat}(u_2(k)) \quad \cdots \quad \text{sat}(u_p(k))]^T,$$

where $\text{sat}(u_i(k)) = \text{sign}(u_i(k)) \min\{1, |u_i(k)|\}$. Here the notation of sat$(\cdot)$ is abused to denote the scalar values and the vector valued saturation functions [9,10]. $w(k) \in \mathbb{R}^q$ is the disturbance input which belongs to $l_2 = \{\{a_k, k \in \mathcal{Z}\} | \sum_{k=0}^{\infty} \|a_k\|^2 < \infty\}$, $z(k) \in \mathbb{R}^m$ is the controlled output. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, and rank$(E) = r < n$. A, B, B_w, C, D, D_w are known constant matrices with appropriate dimensions.

Remark 1. In this paper, only the case of $|u_i(k)| \leq 1, i = 1, 2, \cdots, p$ is discussed. If $|u_i(k)| \leq \tilde{u}_i$, let $\hat{u}_i(k) = \frac{1}{\tilde{u}_i} u_i(k)$, then $|\hat{u}_i(k)| \leq 1$.

Definition 1 [1]. System $Ex_{k+1} = Ax_k$ (or the pair (E, A)) is said to be regular if $\det(zE - A) \not\equiv 0$. The pair (E, A) is said to be causal if it is regular and degree $(\det(zE-A))) = \text{rank}(E)$. The pair (E, A) is said to be stable, if it is regular, and all the roots of $\det(zE-A) = 0$ lie inside the unit disk with center at the origin.

Definition 2 [2,3]. The pair (E, A) is said to be admissible if it is regular, causal and stable.

Definition 3. System (1) with $u_k = 0$ is said to be admissible with γ - disturbance attenuation, if it is admissible, and for a given scalar $\gamma > 0$, for any disturbance $w_k \in l_2$, the H_∞ performance $\sum_{k=0}^{\infty} \|z(k)\|^2 < \gamma^2 \sum_{k=0}^{\infty} \|w(k)\|^2$ is satisfied with the zero initial conditions.

In this paper, we study the design of the state feedback controller of the following form:

$$u(k) = Fx(k), \quad F = \bar{F}E, \tag{2}$$

where $F \in \mathbb{R}^{p\times n}$ and $\bar{F} \in \mathbb{R}^{p\times n}$.

Remark 2. In general, the state feedback is taken as $u(k) = Fx(k)$, and there is no structural restriction on the feedback gain matrix F. However, for singular system (1), the general state feedback case may lead to that the solution is not unique, hence the feedback gain matrix is assumed to be of the structure $F = \bar{F}E$ to overcome this. To show that is required to guarantee the uniqueness of the solution of (1), notice that since rank$(E) = r < n$, there exist two nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$ such that

$$\begin{cases} MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \ MAN = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \\ FN = \begin{bmatrix} F_1 & F_2 \end{bmatrix}, \bar{F}M^{-1} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix}, \\ MBN = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ x(k) = N \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \end{cases} \tag{3}$$

where $x_1(k) \in \mathbb{R}^r, \ x_2(k) \in \mathbb{R}^{n-r}$. If $u(k) = Fx(k)$ and there is no structural assumption on F, then system (1) with $w(k) = 0$ is r.s.e. (restricted system equivalent) [1] to the following system

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + A_2x_2(k) \\ &\quad + B_1\text{sat}(F_1x_1(k) + F_2x_2(k)), \\ 0 &= A_3x_1(k) + A_4x_2(k) + B_2\text{sat}(F_1x_1(k) + F_2x_2(k)). \end{aligned} \tag{4}$$

If $x_1(0)$ is given, since $x_2(0)$ is in the function sat, from the second equation of (4), the unique solution of $x_2(0)$ cannot be obtained directly even A_4 is nonsingular. So system (4) has unique solution if $B_2 = 0$, that is, B satisfies that $B = E\bar{B}$. If $u(k) = \bar{F}Ex(k)$, then system (1) is r.s.e. to

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + A_2x_2(k) + B_1\text{sat}(\bar{F}_1x_1(k)), \\ 0 &= A_3x_1(k) + A_4x_2(k) + B_2\text{sat}(\bar{F}_1x_1(k)), \end{aligned} \tag{5}$$

the unique solution of state $x_2(k)$ can be obtained when A_4 is nonsingular.

Remark 3. For system (1) with the state feedback in (2), if the initial condition is given as $x(0)$, then from (3), $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = N^{-1}x(0)$, and from the second equation of (5), another $x_2(0)$ is obtained. In general, these two $x_2(0)$ are different. In such case, the initial condition is not compatible, in order to guarantee that the initial condition for system (1) is compatible, the initial condition is given as $Ex(0) = x_0$, in this case, only $x_1(0)$ is given, and $x_2(0)$ is obtained by

$$0 = A_3x_1(0) + A_4x_2(0) + B_2\text{sat}(\bar{F}_1x_1(0)). \tag{6}$$

In this paper, two objectives will be achieved. The first one is to obtain a stability condition for the closed-loop system

$$Ex(k+1) = Ax(k) + B\text{sat}(\bar{F}Ex), \tag{7}$$

and design a state feedback controller of the form (2) such that the closed-loop system (7) is admissible. The second one is to design a state feedback controller of the form (2) for system (1) such that the closed-loop system is admissible with γ- disturbance attenuation.

For the rest of our paper, let us recall some important notions and results which have been given in [9,10].

For a matrix $F \in \mathbb{R}^{p\times n}$, denote the ith row of F as f_i and define $\mathcal{L}(F)$ as

$$\mathcal{L}(F) = \{x(k) \in \mathbb{R}^n : |f_ix(k)| \leq 1, i = 1, 2, \cdots, p\}.$$

Let $P \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $E^TPE \geq 0$, and denote by $\Omega(P)$ the following set:

$$\Omega(E^TPE) = \{x(k) \in \mathbb{R}^n : x^T(k)E^TPEx(k) \leq 1\}.$$

Let $\mathcal{D}$ be the set of $p \times p$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose each element of $\mathcal{D}$ is labelled as D_i, $i = 1, 2, \cdots, 2^p$, and denote $D_i^- = I - D_i$. Clearly, if $D_i \in \mathcal{D}$, then $D_i^- \in \mathcal{D}$.

Lemma 1 [9]. Let $F, H \in \mathbb{R}^{p\times n}$. Then for any $x(k) \in \mathcal{L}(H)$

$$\text{sat}(Fx(k)) \in \text{co}\{D_iFx(k) + D_i^-Hx(k), i = 1, 2, \cdots, 2^p\}$$

or, equivalently,

$$\text{sat}(Fx(k)) = \sum_{i=1}^{2^p} \alpha_i(k)(D_iF + D_i^-H)x(k)$$

where co stands for the convex hull, $\alpha_i(k)$ for $i = 1, 2, \cdots, 2^p$ are some scalars which satisfy $0 \leq \alpha_i(k) \leq 1$ and $\sum_{i=1}^{2^p} \alpha_i(k) = 1$.

The following lemma also will be used in the state feedback controller design.

Lemma 2. Given matrices X, Y, Z with appropriate dimensions, and Y is symmetric. Then there exists scalar $\rho > 0$, such that $\rho I + Y > 0$ and

$$-X^TZ - Z^TX - Z^TYZ \leq X^T(\rho I + Y)^{-1}X + \rho Z^TZ.$$

If $Y > 0$, then ρ can be taken as $\rho = 0$ [16].

III. STABILITY CONDITION AND CONTROLLER DESIGN

In this section, first of all, we consider the stability condition for system (7).

Theorem 1. Let $F = \bar{F}E$ be the state feedback controller gain matrix. If there exist a symmetric matrix P and matrices $\bar{F}$, H such that

$$E^TPE \geq 0, \tag{8}$$

$$A_{iF}^TPA_{iF} - E^TPE < 0, \ i = 1, 2, \cdots, 2^p \tag{9}$$

and $\Omega(E^TPE) \subset \mathcal{L}(H)$, then the closed-loop system (7) is admissible within $\Omega(E^TPE)$, where

$$A_{iF} = A + B(D_i\bar{F}E + D_i^-H). \tag{10}$$

Proof. First, let us prove that the closed-loop system (7) is regular and causal. Let

$$M^{-T}PM^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, HN = \begin{bmatrix} H_1 & H_2 \end{bmatrix}. \tag{11}$$

From (8), together with (3), it follows that

$$N^TE^TPEN=\begin{bmatrix} P_1 & 0\\ 0 & 0\end{bmatrix}\geq 0,$$

it is obtained that $P_1 \geq 0$. From the set inclusion condition $\Omega(E^TPE)\subset\mathcal{L}(H)$, it is obtained that $H_2=0$. Otherwise, let $x(k)=N\begin{bmatrix} x_1(k)\\ x_2(k)\end{bmatrix}$, $x_1(k)=0$ and $|h_{2i}x_2(k)|>1$, then $x^T(k)E^TPEx(k)=0$, $|h_ix(k)|=|h_{2i}x_2(k)|>1$, it contradicts that $\Omega(E^TPE)\subset\mathcal{L}(H)$, where h_{2i}, h_i are the ith row of matrices H_2, H, respectively. Pre- and postmultiply inequality (9) by N^T and N, respectively, and together with (3), it follows that

$$\begin{bmatrix} \bar{A}_1 & A_2\\ \bar{A}_3 & A_4\end{bmatrix}^T\begin{bmatrix} P_1 & P_2\\ P_2^T & P_3\end{bmatrix}\begin{bmatrix} \bar{A}_1 & A_2\\ \bar{A}_3 & A_4\end{bmatrix}-\begin{bmatrix} P_1 & 0\\ 0 & 0\end{bmatrix}<0, \tag{12}$$

where

$$\bar{A}_1=A_1+B_1D_i\bar{F}_1+B_1D_i^-H_1,$$
$$\bar{A}_3=A_3+B_2D_i\bar{F}_1+B_2D_i^-H_1.$$

From Dai [1], the closed-loop system (7) is regular and causal if and only if A_4 is nonsingular. If A_4 is singular, then there exists a vector $\xi\in\mathbb{R}^{n-r}$ and $\xi\neq 0$ such that $A_4\xi=0$. Let $\zeta=[0\ \ \xi^T]^T\in\mathbb{R}^n$, then pre- and postmultiply inequality (12) by ζ^T and ζ, respectively, it is obtained that $\xi^TA_2^TP_1A_2\xi<0$, which contradicts $P_1\geq 0$.

Next, we prove that system (7) is stable. Since A_4 is nonsingular, let $T_i=\begin{bmatrix} I_r & 0\\ -A_4^{-1}\bar{A}_3 & I_{n-r}\end{bmatrix}$, then pre- and postmultiply inequality (12) by T_i^T and T_i, respectively, it is obtained that $\begin{bmatrix} Q_1 & \star\\ \star & \star\end{bmatrix}<0$, where $Q_1=(\bar{A}_1-A_2A_4^{-1}\bar{A}_3)^TP_1(\bar{A}_1-A_2A_4^{-1}\bar{A}_3)-P_1$, and $\star$ represents the matrix block we do not need. From $Q_1<0$, it follows that $P_1>0$. On the other hand, from Lemma 1, for every $x(k)\in\Omega(E^TPE)$, we have

$$\text{sat}(\bar{F}Ex(k))\in\text{co}\{(D_i\bar{F}E+D_i^-H)x(k),\ i=1,2,\cdots,2^p\}.$$

It follows that

$$Ax(k)+B\text{sat}(\bar{F}Ex(k))\in\text{co}\{A_{iF}x(k),\ i=1,2,\cdots,2^p\}.$$

Define the Lyapunov function candidate as

$$V(x(k))=x_1^T(k)P_1x_1(k)=x^T(k)E^TPEx(k),$$

then

$$\begin{aligned}\Delta V(k)&=V(x(k+1))-V(x(k))\\ &=x^T(k+1)E^TPEx(k+1)-x^T(k)E^TPEx(k)\\ &=(Ax(k)+B\text{sat}(\bar{F}Ex(k)))^TP\\ &\quad\cdot(Ax(k)+B\text{sat}(\bar{F}Ex(k)))-x^T(k)E^TPEx(k)\\ &\leq\max_{i\in[1,2^p]}x^T(k)A_{iF}^TPA_{iF}x(k)<0,\\ &\quad\forall x(k)\in\Omega(E^TPE)\setminus 0,\end{aligned}$$

which indicates that the closed-loop system (7) is stable within $\Omega(E^TPE)$. The proof is completed. □

Theorem 2. Let $F=\bar{F}E$ be the state feedback controller gain matrix. If there exist a positive definite matrix X, a symmetric matrix S and matrices $\bar{F}$, H such that

$$A_{iF}^T(X-L^TSL)A_{iF}-E^TXE<0,\ i=1,2,\cdots,2^p, \tag{13}$$

and $\Omega(E^TXE)\subset\mathcal{L}(H)$, then the closed-loop system (7) is admissible within $\Omega(E^TXE)$, where $L\in\mathbb{R}^{n\times n}$ is any constant matrix satisfying $LE=0$, rank$(L)=n-r$.

Proof. If (10) holds, let $P=X-L^TSL$, then $E^TPE=E^TXE\geq 0$, $\Omega(E^TXE)=\Omega(E^TPE)$ and (13) is equivalent to (9). The proof is completed. □

In the following, we give the estimation of the set $\Omega(E^TXE)$ with respect to a shape reference set X_R. It can be solved by the following optimization problem

OP1 :

$$\sup_{X>0,S,\bar{F},H}\ \alpha$$

subject to $(i)\ \alpha X_R\subset\Omega(E^TXE)$
(ii) (13)
$(iii)\ \Omega(E^TXE)\subset\mathcal{L}(H)$

The above optimization problem is non-convex, in order to formulate this problem into a convex one, we have the following discussion. If X_R has the form as

$$X_R=\text{co}\{r_1,r_2,\cdots,r_l\}, r_i\in\mathbb{R}^n$$

then (i) is equivalent to

$$\alpha^2r_i^TE^TXEr_i\leq 1,\ i=1,2,\cdots,l,$$

by Schur complement, which is equivalent to

$$\begin{bmatrix} \alpha^{-2} & r_i^TE^TX\\ XEr_i & X\end{bmatrix}\geq 0,\ i=1,2,\cdots,l. \tag{14}$$

If X_R has the following form

$$X_R=\{x(k)\in\mathbb{R}^n: x^T(k)E^T\bar{R}Ex(k)\leq 1\},$$

where $\bar{R}>0$. Let

$$M^{-T}XM^{-1}=\begin{bmatrix} X_1 & X_2\\ X_2^T & X_3\end{bmatrix},\ X_1>0,$$
$$M^{-T}\bar{R}M^{-1}=\begin{bmatrix} \bar{R}_1 & \bar{R}_2\\ \bar{R}_2^T & \bar{R}_3\end{bmatrix}, \bar{R}_1>0,$$

then

$$x^T(k)E^TXEx(k)=x_1^T(k)X_1x_1(k),$$
$$x^T(k)E^T\bar{R}Ex(k)=x_1^T(k)\bar{R}_1x_1(k).$$

(i) is equivalent to $\alpha^2X_1\leq\bar{R}_1$, that is, $\alpha^2E^TXE\leq E^T\bar{R}E$, which is equivalent to

$$\begin{bmatrix} \alpha^{-2}E^T\bar{R}E & E^TX\\ XE & X\end{bmatrix}\geq 0 \tag{15}$$

by Schur complement.

Consider (ii), by Lemma 2, for any scalars ϵ_1, there exists $\rho>0$ such that

$$\rho I+S>0, \tag{16}$$

$$\begin{aligned}-A_{iF}^TL^TSLA_{iF}\ &\leq\epsilon_1A_{iF}^TL^T+\epsilon_1LA_{iF}\\ &\quad+\epsilon_1^2(\rho I+S)^{-1}+\rho A_{iF}^TL^TLA_{iF}.\end{aligned} \tag{17}$$

By Schur complement, and from (16), (17), if

$$\begin{bmatrix} \Theta_{11} & A_{iF}^T & \epsilon_1I_n & \rho A_{iF}^TL^T\\ * & -X^{-1} & 0 & 0\\ * & * & -\rho I-S & 0\\ * & * & * & -\rho I\end{bmatrix}<0, \tag{18}$$

$i=1,2,\cdots,2^p$

where

$$\Theta_{11} = -E^T XE + \epsilon_1 A_{iF}^T L^T + \epsilon_1 LA_{iF} \tag{19}$$

holds, then condition (ii) holds. Here, two methods can be used to deal with X^{-1} in (18). One consists of using the following inequality

$$-X^{-1} \leq -2\epsilon_2 I + \epsilon_2^2 X \tag{20}$$

obtained by Lemma 2, therefore, if

$$\begin{bmatrix} \Theta_{11} & A_{iF}^T & \epsilon_1 I_n & \rho A_{iF}^T L^T \\ * & -2\epsilon_2 I + \epsilon_2^2 X & 0 & 0 \\ * & * & -\rho I - S & 0 \\ * & * & * & -\rho I \end{bmatrix} < 0, \quad i = 1, 2, \cdots, 2^p \tag{21}$$

holds, then (18) holds. Another is letting $X^{-1} = Z$ directly, $XZ = I$, then (18) is transformed into

$$\begin{bmatrix} \Theta_{11} & A_{iF}^T & \epsilon_1 I_n & \rho A_{iF}^T L^T \\ * & -Z & 0 & 0 \\ * & * & -\rho I - S & 0 \\ * & * & * & -\rho I \end{bmatrix} < 0, \quad i = 1, 2, \cdots, 2^p. \tag{22}$$

From

$$x^T(k)E^T XEx(k) = x_1^T(k)X_1 x_1(k), Hx(k) = H_1 x_1(k),$$

the condition (iii) is equivalent to

$$h_{1i} X_1^{-1} h_{1i}^T \leq 1, \ i = 1, 2, \cdots, p,$$

by Schur complement, which is equivalent to

$$\begin{bmatrix} 1 & h_{1i} \\ h_{1i}^T & X_1 \end{bmatrix} \geq 0, \ i = 1, 2, \cdots, p$$

or

$$\begin{bmatrix} 1 & [h_{1i} \ \ 0] \\ [h_{1i} \ \ 0]^T & \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \geq 0, \quad i = 1, 2, \cdots, p. \tag{23}$$

Pre- and postmultiply inequality (23) by diag$\{1, N^{-T}\}$ and diag$\{1, N^{-1}\}$, respectively, together with (3), it follows that

$$\begin{bmatrix} 1 & h_i \\ h_i^T & E^T XE \end{bmatrix} \geq 0, \ i = 1, 2, \cdots, p. \tag{24}$$

Notice that $H_2 = 0$, $[h_{1i} \ \ 0] = [h_{1i} \ \ \star] \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, (24) is also equivalent to

$$\begin{bmatrix} 1 & \bar{h}_i E \\ E^T \bar{h}_i^T & E^T XE \end{bmatrix} \geq 0, \ i = 1, 2, \cdots, p, \tag{24$'$}$$

where $\bar{h}_i$ is the ith row of $\bar{H}$ and $H = \bar{H}E$.

Remark 4. If ϵ_1, $\epsilon_2 > 0$ and $\rho > 0$ are given first in (21) and (22), then (21) is a LMI, and (22) can be solved by cone complement method [17]. The optimal values of ϵ_1, ϵ_2 and ρ can be obtained by using a numerical optimization algorithm, such as fminsearch in Optimization Toolbox.

Then the optimization problem **OP1** can be transformed to the following LMI problem:

OP2 :
$$\inf_{X>0, S, \bar{F}, H} \beta$$
subject to inequalities (14)(or (15)), (21) and (24)(or (24$'$)),

or

OP2 $'$:
$$\inf_{X>0, Z>0, S, \bar{F}, H} \beta$$
subject to inequalities (14)(or (15)), (22) with $XZ = I$ and (24)(or (24$'$)),

where $\beta = \alpha^{-2}$, ϵ_1, $\epsilon_2 > 0$ and $\rho > 0$ are given scalars. The optimal state feedback controller gain $F = \bar{F}E$ can be obtained by solving **OP2** or **OP2$'$**, directly.

IV. H_∞ CONTROLLER DESIGN

In this section, we consider the admissibility with γ-disturbance attenuation for system (1) with $u(k) = \bar{F}Ex(k)$. First, in order to solve this problem by using LMI approach, we assume that $q = n$.

Remark 5. Generally, $q \neq n$. If $q < n$, then let $\hat{w}(k) = \begin{bmatrix} w(k) \\ \bar{w}(k) \end{bmatrix} \in \mathbb{R}^{n \times n}$, and $\bar{B}_w = [B_w \ \ 0]$, $\bar{D}_w = [D_w \ \ 0]$; If $q > n$, then let $\hat{x}(k) = \begin{bmatrix} x(k) \\ \bar{x}(k) \end{bmatrix} \in \mathbb{R}^{q \times q}$, and $\hat{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{C} = [C \ \ 0]$, $\hat{B}_w = \begin{bmatrix} B_w \\ 0 \end{bmatrix}$. Therefore, the input-output relation of system (1) does not change.

The closed-loop system of system (1) with $u(k) = \bar{F}Ex(k)$ is

$$\begin{cases} Ex(k+1) = Ax(k) + B\text{sat}(\bar{F}Ex(k)) + B_w w(k), \\ z(k) = Cx(k) + D\text{sat}(\bar{F}Ex(k)) + D_w w(k). \end{cases} \tag{25}$$

Theorem 3. For given scalars $\gamma > 0$, ϵ_1, $\epsilon_2 > 0$, ϵ_3 and $\rho > 0$, if there exists matrices $X > 0$, S, $\bar{F}$ and H such that

$$\Psi = \begin{bmatrix} \Theta_{11} & \epsilon_3 A_{iF}^T L^T + \epsilon_1 LB_w & C_{iF}^T & A_{iF}^T & \epsilon_1 I_n & \rho A_{iF}^T L^T \\ * & -\gamma^2 I + \epsilon_3 B_w + \epsilon_3 B_w^T & D_w^T & B_w^T & \epsilon_3 I_n & \rho B_w^T L^T \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -X^{-1} & 0 & 0 \\ * & * & * & * & -\rho I - S & 0 \\ * & * & * & * & * & -\rho I \end{bmatrix} < 0, \tag{26}$$

$$i = 1, 2, \cdots, 2^p$$

and $\Omega(E^TXE) \subset \mathcal{L}(H)$, then the closed-loop system (25) is admissible with γ- disturbance attenuation within $\Omega(E^TXE)$, where

$$C_{iF} = C + D(D_i\bar{F}E + D_i^- H). \tag{27}$$

Proof. First, from (26), it follows that (18) holds, then from Theorem 2, system (25) with $w(k) = 0$ is admissible within $\Omega(E^TXE)$. Next, let us prove that system (25) satisfy H_∞ performance. Let $P = X - L^TSL$, construct the Lyapunov function as $V(x(k)) = x^T(k)E^TPEx(k)$, from Lemma 1, it is obtained that

$$\begin{aligned}\Delta V(k) = \ & [\sum_{i=1}^{2^p}\alpha_i(k)(A_{iF}x(k) + B_w w(k))]^T \\ & \cdot P[\sum_{i=1}^{2^p}\alpha_i(k)(A_{iF}x(k) + B_w w(k))] \\ & -x^T(k)E^TPEx(k).\end{aligned}$$

Then, it follows that

$$\begin{aligned} J &= \sum_{k=0}^{\infty}(z^T(k)z(k) - \gamma^2 w^T(k)w(k)) \\ &\le \sum_{k=0}^{\infty}(z^T(k)z(k) - \gamma^2 w^T(k)w(k) + \Delta V(k)) \\ &= \sum_{k=0}^{\infty}\begin{bmatrix} x^T(k) & w^T(k)\end{bmatrix}\Phi\begin{bmatrix} x^T(k) & w^T(k)\end{bmatrix}^T, \end{aligned} \tag{28}$$

where

$$\begin{aligned}\Phi = \ & (\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} C_{iF}^T \\ D_w^T\end{bmatrix})(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} C_{iF}^T \\ D_w^T\end{bmatrix}^T) \\ & + \begin{bmatrix} -E^TPE & 0 \\ 0 & -\gamma^2 I\end{bmatrix} \\ & +(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} A_{iF}^T \\ B_w^T\end{bmatrix})P(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} A_{iF}^T \\ B_w^T\end{bmatrix}^T).\end{aligned}$$

If $\Phi < 0$, then system (25) satisfies the H_∞ performance. Notice that from (26), it can be obtained that

$$\sum_{i=1}^{2^p}\alpha_i(k)\Psi < 0. \tag{29}$$

According to the Schur complement, (29) is equivalent to

$$\begin{aligned}\bar{\Psi} = & \begin{bmatrix} \sum_{i=1}^{2^p}\alpha_i(k)\Theta_{11} & \sum_{i=1}^{2^p}\alpha_i(k)\epsilon_3 A_{iF}^T L^T + \epsilon_1 LB_w \\ * & -\gamma^2 I + \epsilon_3 B_w^T + \epsilon_3 B_w\end{bmatrix} \\ & +(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} C_{iF}^T \\ D_w^T\end{bmatrix})(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} C_{iF}^T \\ D_w^T\end{bmatrix}^T) \\ & +(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} A_{iF}^T \\ B_w^T\end{bmatrix})X(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} A_{iF}^T \\ B_w^T\end{bmatrix}^T) \\ & +\rho(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} A_{iF}^T \\ B_w^T\end{bmatrix})L^TXL(\sum_{i=1}^{2^p}\alpha_i(k)\begin{bmatrix} A_{iF}^T \\ B_w^T\end{bmatrix}^T) \\ & + \begin{bmatrix}\epsilon_1 I \\ \epsilon_3 I\end{bmatrix}(\rho I + S)^{-1}\begin{bmatrix}\epsilon_1 I \\ \epsilon_3 I\end{bmatrix}^T < 0\end{aligned}$$

and $\rho I + S > 0$. From Lemma 2 and $P = X - L^TSL$, it is obtained that $\Phi < \bar{\Psi}$. Therefore, $\Phi < 0$. The proof is completed. □

Notice that we have X and X^{-1} in (26) at the same time, we can deal with X^{-1} in (26) similarly to the method used in (18). The first one, from (20), let X^{-1} be replaced by $-2\epsilon_2 I + \epsilon_2^2 X$, and another is that let $X^{-1} = Z$, $XZ = I$.

Theorem 4. For given scalars $\gamma > 0$, ϵ_1, $\epsilon_2 > 0$, ϵ_3 and $\rho > 0$, if the following optimization problem

OP3 :
$\inf_{X>0,S,\bar{F},H} \beta$
subject to inequalities (14)(or (15)),
(26) with X^{-1} being replaced by $-2\epsilon_2 I + \epsilon_2^2 X$
and (24)(or (24′)),

or

OP3 ′ :
$\inf_{X>0,Z>0,S,\bar{F},H} \beta$
subject to inequalities (14)(or (15)),
(26) with $X^{-1} = Z$ and (24)(or (24′)),

have solution, then there exists a H_∞ state feedback controller of the form as (2), and the feedback gain is taken as $F = \bar{F}E$.

Remark 6. The conditions given in Theorems 1-4 and **OP1 -OP3** are also valid for the case of $B = E\bar{B}$. In this case, it needs only to replaced $B, \bar{F}E$ with $E\bar{B}, F$ in all the conditions, respectively, and the state feedback controller gain matrix is F.

V. EXAMPLE

Consider the stability and H_∞ control for system (1) with the following coefficient matrices

$$E = \begin{bmatrix} 5 & 10 & 5 \\ 0 & 2.5 & 2.5 \\ 0 & 0 & 0\end{bmatrix}, A = \begin{bmatrix} 10 & 12 & 6 \\ 9 & 5.5 & 5.5 \\ 2.5 & 1 & 2.5\end{bmatrix},$$
$$B = \begin{bmatrix} 3 & 2 & 0.5\end{bmatrix}^T, B_w = \begin{bmatrix} 0.1 & 0.2 & 0.3\end{bmatrix}^T,$$
$$C = \begin{bmatrix} 0.2 & 0 & 0.5\end{bmatrix}, \ D = 0.1, \ D_w = 0.1.$$

Let $\gamma = 0.8$, $\bar{R} = I_3$, $L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1\end{bmatrix}$.

1. The stability problem. Let $\epsilon_1 = -11$, $\epsilon_2 = 1$, $\rho = 0.01$. Solve **OP2** subject to (15), (21) and (24), it is obtained that $\beta_{\inf} = 0.6019$, and

$$X = \begin{bmatrix} 0.5246 & -0.0567 & -0.0277 \\ -0.0567 & 0.5579 & -0.0267 \\ -0.0277 & -0.0267 & 0.1886\end{bmatrix},$$

the controller gain matrix is given by

$$F = \begin{bmatrix} -1.6397 & -1.4403 & 0.1995\end{bmatrix}.$$

The bigger ellipsoid shown in Figure 1 is the ellipsoid $\Omega(E^TXE, 1)$ for stability under the transformation

$$y(k) = \bar{N}x(k), \bar{N} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1\end{bmatrix}. \tag{30}$$

Figure 2 gives the controller $u(k)$ behavior with respect to time. Figures 3 and 4 give simulations for the state trajectories of the open-loop system and the closed-loop system, respectively, the initial condition values are $Ex(0) = \begin{bmatrix} -1 & 0.75 & 0 \end{bmatrix}^T$. It can be seen the designed controller stabilizes on system.

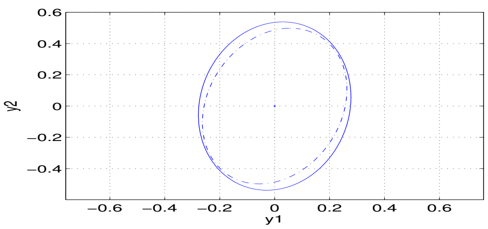

Figure 1. The ellipsoids $\Omega(E^TXE, 1)$ for stability and H_∞ control, respectively

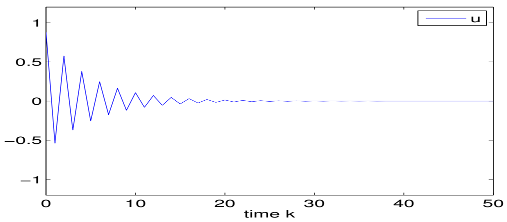

Figure 2. The controller $u(k)$

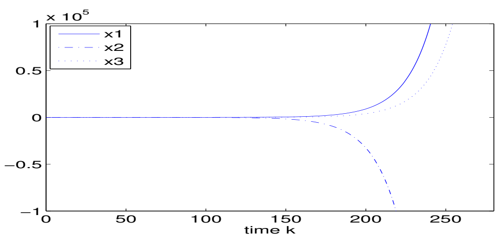

Figure 3. The state trajectories of the open-loop system

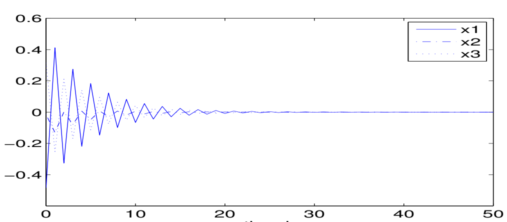

Figure 4. The state trajectories of the closed-loop system

2. The H_∞ control problem. Let $\epsilon_1 = -11$, $\epsilon_2 = 1$,$\epsilon_3 = -0.01$, $\rho = 0.01$. Since $q = 1$, $n = 3$, $q < n$. From Remark 7, let $B_w = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0.3 & 0 & 0 \end{bmatrix}$, $D_w = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}$, then solving **OP3** subject to (15), (26) with X^{-1} being replaced by $-2\epsilon_2 I + \epsilon_2^2 X$ and (24), it is obtained that $\beta_{\text{inf}} = 0.7643$, and

$$X = \begin{bmatrix} 0.6057 & -0.1373 & 0.2108 \\ -0.1373 & 0.6446 & 0.1883 \\ 0.2108 & 0.1883 & 0.1922 \end{bmatrix},$$

and the controller gain matrix is given by

$$F = \begin{bmatrix} -1.5489 & -1.1684 & 0.3804 \end{bmatrix}.$$

The smaller ellipsoid (in dash) shown in Figure 1 is the ellipsoid $\Omega(E^TXE, 1)$ for the H_∞ control under the transformation (30).

VI. CONCLUSIONS

In this paper, the stability and the H_∞ control problems for discrete-time singular systems with actuator saturation are discussed. The sufficient conditions which guarantee that the discrete-time singular systems with actuator saturation are admissible, admissible with γ- disturbance attenuation are established. With these conditions, the estimation of stability region and the state feedback controller, and the H_∞ state feedback controller are obtained by solving corresponding LMIs optimization problems.

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