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# Robust Quadratic Control of Discrete-Time Singular Markov Jump Systems with Bounded Transition Probabilities

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*Abstract*— **The quadratic control problem for discrete-time singular Markov jump systems with parameter uncertainties is discussed. The weighting matrix in quadratic cost function is indefinite. For full and partial knowledge of transition probabilities cases, state feedback controllers are designed based on linear matrix inequalities (LMIs) methods which guarantee that the closed-loop discrete-time singular Markov jump systems are regular, causal and stochastically stable, and the cost value has a zero lower bound and a finite upper bound. A numerical example to illustrate the effectiveness of the method is given in the paper.**

## I. INTRODUCTION

In practice, many dynamical systems can not be represented by the class of linear time-invariant model since the dynamics of these systems is random with some features, for example, abrupt changes, breakdowns of components, changes in the interconnections of subsystems, etc. Such class of dynamical systems can be adequately described by the class of stochastic hybrid systems. A special class of hybrid systems referred to as Markov jump systems, systems with random structures, has attracted a lot of researchers and many problems have been solved, such as stability and stabilization problems [1-4], LQ control problem [5-8], guaranteed cost problem [9], and $H_\infty$ control problem [10]. Most of results for Markov jump systems are obtained under known transition probabilities [1, 5-9], but in many practical systems, the transition probabilities can not known exactly and therefore they may have uncertainties, so it is also very important to discuss this kind of systems [2-3, 10].

Singular systems, which are also referred to as implicit systems, descriptor systems, differential-algebraic systems, have extensive applications in many practical systems, such as electrical networks [11], power systems [12], economy systems [13], and other areas [14]. So great progress has been made in the theory and applications of the class of systems since 1970s [14-22]. For example, singular LQ problem was discussed in [15]. The robust stability and robust stabilization for discrete singular systems were investigated in [16]. For singular Markov jump systems, the stability problem and the $H_\infty$ control problem for discrete-time singular Markov jump systems were discussed in [17], [18], respectively based on non strict LMIs conditions. [19] discussed the stability and $H_\infty$ control problem for discrete-time singular Markov jump systems by using equivalent system transformation and LMIs method. Boukas [20, 21] discussed the stability and output feedback control for continuous-time singular Markov jump systems. Lu et al [22] discussed guaranteed cost control for continue-time singular Markov jump systems. To the best of our knowledge, the robust quadratic cost control problem for uncertain discrete-time singular Markov jump systems with bounded transition probabilities and indefinite quadratic cost has not been investigated in the literature, this problem is important in both theory and practice.

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In this paper, the robust quadratic cost control problem for uncertain discrete-time singular Markov jump systems is discussed. The weighting matrix in quadratic cost function is indefinite. For full and partial knowledge of transition probabilities cases, state feedback controllers are designed via LMIs methods which guarantee that the closed-loop discrete-time singular Markov jump systems are regular, causal and stochastically stable, and the cost value has a zero lower bound and finite upper one. A numerical example to illustrate the effectiveness of the method is given.

**Notations:** Throughout this paper, $I$ is the identity matrix with appropriate dimension, $\mathcal{Z}$ denotes the set of non-negative integer numbers, and $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation.

## II. DESCRIPTION OF PROBLEM

The discrete-time singular Markov jump system considered in this paper is described by the following dynamics:

$$Ex_{k+1} = A(k, r_k)x_k + B(k, r_k)u_k, \tag{1}$$

where $k \in \mathcal{Z}$, $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^p$ is the control input. $\{r_k, k \in \mathcal{Z}\}$ is a Markov chain taking values in a finite space $\mathcal{S} = \{1, 2, \cdots, N\}$, with the following transition probability from mode $i$ at time $k$ to mode $j$ at time $k+1$, $k \in \mathcal{Z}$:

$$p_{ij} = \Pr\{r_{k+1} = j | r_k = i\} \tag{2}$$

with $p_{ij} \geq 0$ for $i, j \in \mathcal{S}$, and $\sum_{j=1}^{N} p_{ij} = 1$. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, and $\text{rank}(E) = r < n$. For each $i \in \mathcal{S}$, we have

$$A(k, i) = A(i) + \delta A(k, i), \quad B(k, i) = B(i) + \delta B(k, i),$$

where $A(i), B(i)$ are known constant matrices with appropriate dimensions; $\delta A(k, i)$, $\delta B(k, i)$ are unknown matrices, denoting the uncertainties in the system.

The quadratic cost function is described as

$$J = \sum_{k=0}^{\infty} \mathbf{E}\{\begin{bmatrix} x_k^T & u_k^T \end{bmatrix} Q(r_k) \begin{bmatrix} x_k^T & u_k^T \end{bmatrix}^T | r_0\}, \tag{3}$$

where the weighting matrix $Q(i) \in \mathbb{R}^{(n+p)\times(n+p)}$ is known symmetric and constant for each mode $i \in \mathcal{S}$.

In this paper, the uncertainties are norm-bounded and are assumed to be of the following form

$$\begin{bmatrix} \delta A(k,i) & \delta B(k,i) \end{bmatrix} = D(i)\Delta(k,i)\left[F_a(i) \quad F_b(i)\right], \quad (4)$$

where $D(i)$, $F_a(i)$, $F_b(i)$ are known constant matrices with appropriate dimensions, $\Delta(k,i) \in \mathbb{R}^{q\times s}$ are unknown time-varying matrix function satisfying

$$\Delta^T(k,i)\Delta(k,i) \leq I, \quad (5)$$

The transition probabilities are unknown, but the bounds are known, we assume that the following is satisfied

$$0 < \underline{p}_i \leq p_{ij} \leq \bar{p}_i < 1, \forall i,j \in \mathcal{S} \quad (6)$$

where $\underline{p}_i$ and $\bar{p}_i$ are known parameters for each mode.

**Definition 1 [17].** System $Ex_{k+1} = A(r_k)x_k$ (or the pair $(E, A(r_k))$) is said to be

(1) regular if $\det(zE - A(r_k)) \not\equiv 0$ for any $r_k = i$, $i \in \mathcal{S}$.

(2) causal if it is regular and degree $(\det(zE - A(r_k))) = \mathrm{rank}(E)$ for any $r_k = i$, $i \in \mathcal{S}$.

(3) stochastically stable, if for every initial state $x_0$, the condition $\mathbf{E}\{\sum_{k=0}^{\infty} \|x_k\|^2 | x_0, r_0\} < \infty$ is satisfied.

**Remark 1.** In this paper, the weighting matrix $Q(r_k)$ in the quadratic cost function (3) is only symmetric, it does not require positive definite or semi-positive definite, that is, $Q(r_k)$ is indefinite. The problem we are addressing here is different from the singular LQ problem discussed in [15] and the guaranteed cost problem discussed in [22].

**Remark 2.** In practice, the transition probabilities for some systems can not easily be obtained or known exactly. Therefore discussing the stability and control problem for Markov jump systems with partial knowledge of transition probabilities is a very important issue for practical systems. In this paper, we discuss the case of transition probabilities are unknown but bounded with some known bounds.

The purpose of this paper is to design a state feedback controller $u_k = K(r_k)x_k$, develop LMI conditions and find a constant $J_0 \geq 0$ such that the closed-loop system formed by system (1) and $u_k = K(r_k)x_k$ is regular, causal and stochastically stable, and the cost values $0 \leq J \leq J_0$ for all uncertainties satisfying (4) and (5).

**Lemma 1 [17].** System $Ex_{k+1} = A(r_k)x_k$ is regular, causal and stochastically stable, if and only if there exist symmetric matrix $P_i$ such that

$$E^T P_i E \geq 0, \quad A^T(i)\bar{P}_i A(i) - E^T P_i E < 0,$$

where $\bar{P}_i = \sum_{j=1}^{N} p_{ij}P_j$.

**Lemma 2.** Given matrices $X, Y, Z$ with appropriate dimensions, and $Y$ is symmetric. Then there exists scalar $\rho > 0$, such that $\rho I + Y > 0$ and

$$-X^T Z - Z^T X - Z^T Y Z \leq X^T(\rho I + Y)^{-1}X + \rho Z^T Z.$$

If $Y > 0$, then $\rho$ can be taken as $\rho = 0$ [23].

**Lemma 3 [24].** Given a symmetric matrix $\Omega$ and matrices $\Gamma$, $\Xi$ with appropriate dimensions, then $\Omega + \Gamma\Delta\Xi + \Xi^T\Delta^T\Gamma^T < 0$ for all $\Delta$ satisfying $\Delta^T\Delta \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that $\Omega + \epsilon\Gamma\Gamma^T + \epsilon^{-1}\Xi^T\Xi < 0$.

## III. MAIN RESULTS

The closed-loop system formed by system (1) and the state feedback $u_k = K(r_k)x_k$ is

$$Ex_{k+1} = A_K(k, r_k)x_k, \quad (7)$$

where

$$\begin{aligned} A_K(k,r_k) &= A(k,r_k) + B(k,r_k)K(r_k) \\ &= A_K(r_k) + D(r_k)\Delta(k,r_k)F_K(r_k), \\ A_K(r_k) &= A(r_k) + B(r_k)K(r_k), \\ F_K(r_k) &= F_a(r_k) + F_b(r_k)K(r_k), \end{aligned}$$

and the quadratic cost function is changed to

$$J = \sum_{k=0}^{\infty} \mathbf{E}\{x_k^T \left[I_n \quad K^T(r_k)\right] Q(r_k) \left[I_n \quad K^T(r_k)\right]^T x_k | r_0\} \quad (8)$$

**Lemma 4.** For given matrix $K(i)$, if there exist symmetric matrices $Z_i > 0$, $X_i > 0$ and $S_i$ satisfying the following set of coupled LMIs:

$$Q(i) + \begin{bmatrix} A(i) & B(i) \end{bmatrix}^T \Phi^T Z_i \Phi \begin{bmatrix} A(i) & B(i) \end{bmatrix} \geq 0, \quad (9)$$

$$\begin{aligned} \Pi_i = &A_K^T(i)\bar{X}_i A_K(i) - A_K^T(i)\Phi^T\bar{S}_i\Phi A_K(i) - E^T X_i E \\ &+ \left[I_n \quad K^T(i)\right] Q(i) \left[I_n \quad K^T(i)\right]^T < 0, \end{aligned} \quad (10)$$

then system (7) with $\Delta(k,i) = 0$ is regular, causal and stochastically stable, and the cost value satisfies

$$0 \leq J \leq x_0^T E^T X_{r_0} E x_0, \quad (11)$$

where $\bar{X}_i = \sum_{j=1}^{N} p_{ij}X_j$, $\bar{S}_i = \sum_{j=1}^{N} p_{ij}S_j$, $\Phi \in \mathbb{R}^{n\times n}$ is any constant matrix satisfying $\Phi E = 0$, $\mathrm{rank}(\Phi) = n - r$.

**Proof.** First, let us prove that system (7) with $\Delta(k,i) = 0$ is regular, causal and stochastically stable. Since (9) holds, let $\alpha > 0$ and $\alpha I - Z_i > 0$, it follows that

$$\begin{aligned} &\bar{Q}(i) = \\ &Q(i) + \alpha \begin{bmatrix} A(i) & B(i) \end{bmatrix}^T \Phi^T\Phi \begin{bmatrix} A(i) & B(i) \end{bmatrix} \geq 0, \end{aligned} \quad (12)$$

then (10) can be written as

$$\begin{aligned} &A_K^T(i)\sum_{j=1}^{N} p_{ij}(X_j - \Phi^T S_j\Phi - \alpha\Phi^T\Phi)A_K(i) \\ &-E^T X_i E + \left[I_n \quad K^T(i)\right] \bar{Q}(i) \left[I_n \quad K^T(i)\right]^T < 0. \end{aligned} \quad (13)$$

Let $P_i = X_i - \Phi^T S_i \Phi - \alpha\Phi^T\Phi$, from $X_i > 0$, (12) and (13), it follows that $E^T P_i E = E^T X_i E \geq 0$, $A_K^T(i)\bar{P}_i A_K(i) - E^T P_i E < 0$. From Lemma 1, it is obtained that system (7) with $\Delta(k,i) = 0$ is regular, causal and stochastically stable.

Next, to prove that the cost value satisfies (11). Construct a stochastic Lyapunov functional candidate as $V(k, r_k) = x_k^T E^T X_{r_k} E x_k$, where matrices $X_{r_k} > 0$. Let the mode at time $k$ be $i$, that is $r_k = i$. Recall that at time $k+1$, the system may jump to any mode $r_{k+1} = j$. One can then obtain that

$$\begin{aligned}\Delta V(k) &= \mathbf{E}[V(k+1, r_{k+1})|r_k = i] - V(k,i)\\ &= \mathbf{E}[x_{k+1}^T E^T X_{r_{k+1}} E x_{k+1}|r_k = i] - x_k^T E^T X_i E x_k\\ &= x_k^T A_K^T(i)\bar{X}_i A_K(i) x_k - x_k^T E^T X_i E x_k.\end{aligned} \tag{14}$$

From $\Phi E = 0$, the following equation holds for any symmetric matrix $S_i$ with appropriate dimensions and $r_k = i$:

$$\begin{aligned}0 &= -\sum_{j=1}^{N} p_{ij} x_{k+1}^T E^T \Phi^T S_j \Phi E x_{k+1}\\ &= -x_k^T A_K^T(i)\Phi^T \bar{S}_i \Phi A_K(i) x_k.\end{aligned} \tag{15}$$

Then, adding (15) to (14), it is obtained that

$$\begin{aligned}\Delta V(k) = \; & x_k^T (A_K^T(i)\bar{X}_i A_K(i) - E^T X_i E\\ & -A_K^T(i)\Phi^T \bar{S}_i \Phi A_K(i)) x_k.\end{aligned} \tag{16}$$

Consider the quadratic cost function $J$, for $r_k = i$, it follows that

$$\begin{aligned}J &= \sum_{k=0}^{\infty} \mathbf{E}\{x_k^T \left[I_n \quad K^T(i)\right] Q(i) \left[I_n \quad K^T(i)\right]^T x_k|r_0\}\\ &= \sum_{k=0}^{\infty} \mathbf{E}\{x_k^T \left[I_n \quad K^T(i)\right] Q(i) \left[I_n \quad K^T(i)\right]^T x_k\\ &\qquad +\Delta V(k)|r_0\} - \sum_{k=0}^{\infty} \mathbf{E}\{\Delta V(k)|r_0\}\\ &= \sum_{k=0}^{\infty} \mathbf{E}\{x_k^T \Pi_i x_k|r_0\} + \mathbf{E}\{V(0)\}\\ &\leq \mathbf{E}\{V(0)\} = x_0^T E^T X_{r_0} E x_0.\end{aligned} \tag{17}$$

Let

$$J_1 = \sum_{k=0}^{\infty} \mathbf{E}\{\left[x_k^T \quad u_k^T\right] \bar{Q}(r_k) \left[x_k^T \quad u_k^T\right]^T |r_0\}, \tag{18}$$

where $\bar{Q}(r_k)$ is shown as in (12) for $r_k = i$. Since (9) holds, from (12), it follows that $J_1 \geq 0$, and

$$\begin{aligned}J_1 &= \sum_{k=0}^{\infty} \mathbf{E}\{\left[x_k^T \quad u_k^T\right] (Q(r_k) + \alpha \left[\ A(r_k) \quad B(r_k)\ \right]^T\\ &\qquad \cdot\Phi^T \Phi \left[\ A(r_k) \quad B(r_k)\ \right]) \left[x_k^T \quad u_k^T\right]^T |r_0\}\\ &= \sum_{k=0}^{\infty} \mathbf{E}\{\left[x_k^T \quad u_k^T\right] Q(r_k) \left[x_k^T \quad u_k^T\right]^T\\ &\qquad +\alpha(A(r_k)x_k + B(r_k)u_k)^T\\ &\qquad \cdot\Phi^T \Phi (A(r_k)x_k + B(r_k)u_k)|r_0\}\\ &= \sum_{k=0}^{\infty} \mathbf{E}\{\left[x_k^T \quad u_k^T\right] Q(r_k) \left[x_k^T \quad u_k^T\right]^T\\ &\qquad +\alpha x_{k+1}^T E^T \Phi^T \Phi E x_{k+1}|r_0\} = J \geq 0.\end{aligned} \tag{19}$$

According to (17)-(19), it is obtained that (11) holds. The proof is completed.

**Remark 3.** Although the weighting matrix $Q(r_k)$ in (3) is indefinite for each $r_k = i$, the cost value also can satisfy $J \geq 0$. This is determined by the singularity of the matrix $E$. From (19), it is shown that the cost value of $J_1$ has no relation with the introduction of the scalar $\alpha$. If $Q(r_k) \geq 0$, then $J \geq 0$ holds directly, and it only needs to consider the upper bound of $J$.

**Remark 4.** The solvability of (9) is independence of (10), and $J \geq 0$ is guaranteed by (9). If (9) does not hold, then (12) cannot hold, since $Q(i)$ is indefinite, the regularity, causality and stochastic stability cannot be obtained if only (10) holds.

From the proof of Lemma 4, we know that if (9) holds, then $J = J_1 \geq 0$. So the following lemma can be obtained by replacing $Q(i)$ in (10) with $\bar{Q}(i)$ directly.

**Lemma 5.** For given matrix $K(i)$, if there exist symmetric matrices $Z_i > 0$, $X_i > 0$, $S_i$ and a scalar $\alpha > 0$ satisfying (9) and

$$\alpha I - Z_i \geq 0, \tag{20}$$

$$\begin{aligned}&A_K^T(i)\bar{X}_i A_K(i) - A_K^T(i)\Phi^T \bar{S}_i \Phi A_K(i) - E^T X_i E\\ &\quad + \left[I_n \quad K^T(i)\right] \bar{Q}(i) \left[I_n \quad K^T(i)\right]^T < 0,\end{aligned} \tag{21}$$

then system (7) with $\Delta(k,i) = 0$ is regular, causal and stochastically stable, and the cost value satisfies (11), where

$$\bar{Q}(i) = Q(i) + \alpha \left[\ A(i) \quad B(i)\ \right]^T \Phi^T \Phi \left[\ A(i) \quad B(i)\ \right]. \tag{22}$$

In the following, we consider system (7) and quadratic cost function (8), we will design a state feedback controller and find a scalar $J_0 \geq 0$ such that system (7) is regular, causal and stochastically stable and $0 \leq J \leq J_0$ for all uncertainties satisfying (4) and (5).

**Theorem 1.** For given scalars $\lambda_i > 0$, $\rho > 0$, $\epsilon_1$ and $\epsilon_2$, if there exist matrices $V_i$, $Z_i > 0$, $Y_i > 0$, $\bar{K}(i)$, nonsingular matrix $R_i$, symmetric matrix $S_i$, and scalars $\alpha > 0$, $\epsilon > 0$ such that (20) and

$$\begin{bmatrix} \Psi_{i11} & V_i & V_i\Phi D(i)\\ * & Z_i & 0\\ * & * & \epsilon I \end{bmatrix} \geq 0, \tag{23}$$

$$\begin{bmatrix} \Theta_{i11} & \Lambda_{1i}^T W_i & \epsilon_1 I & \rho\Lambda_{1i}^T\Phi^T & \Theta_{i15} & \Theta_{i16} & \lambda_i\Lambda_{2i}^T\\ * & -\bar{Y} & 0 & 0 & 0 & W_i^T D(i) & 0\\ * & * & \Theta_{i33} & 0 & 0 & 0 & 0\\ * & * & * & -\rho I & 0 & \rho\Phi D(i) & 0\\ * & * & * & * & -I & 0 & 0\\ * & * & * & * & * & \Theta_{i66} & 0\\ * & * & * & * & * & * & -\lambda_i I \end{bmatrix} < 0, \tag{24}$$

where

$$\begin{aligned}\Psi_{i11} &= Q(i) + [A(i) \quad B(i)]^T \Phi^T V_i^T + V_i \Phi [A(i) \quad B(i)]\\ &\quad -\epsilon [F_a(i) \quad F_b(i)]^T [F_a(i) \quad F_b(i)]\\ \Lambda_{1i} &= A(i)R_i + B(i)\bar{K}(i),\ \Lambda_{2i} = F_a(i)R_i + F_b(i)\bar{K}(i),\\ \Theta_{i11} &= \epsilon_1\Lambda_{1i}^T\Phi^T + \epsilon_1\Phi\Lambda_{1i} + \epsilon_2 R_i^T E^T + \epsilon_2 E R_i + \epsilon^2 Y_i,\\ \Theta_{i15} &= \left[\ R_i^T \quad \bar{K}^T(i)\ \right] \bar{Q}^{\frac{1}{2}}(i),\\ \Theta_{i16} &= \epsilon_1\Phi D(i) + \alpha\Lambda_{1i}^T\Phi^T\Phi D(i),\\ \Theta_{i33} &= -\rho I - \bar{S}_i,\ \bar{S}_i = \sum_{j=1}^{N} p_{ij} S_j,\\ \Theta_{i66} &= -\lambda_i I + \alpha D^T(i)\Phi^T\Phi D(i),\\ W_i &= \left[\ \sqrt{p_{i1}}I \quad \cdots \quad \sqrt{p_{iN}}I\ \right], \bar{Y} = \mathrm{diag}\{Y_1, \cdots, Y_N\},\end{aligned} \tag{25}$$

hold, then there exists a state feedback controller such that system (7) is regular, causal and stochastically stable, and the cost value satisfies

$$0 \leq J \leq x_0^T E^T Y_{r_0}^{-1} E x_0, \tag{26}$$

for all uncertainties satisfying (4) and (5), and the state feedback controller is given by $u_k = \bar{K}(i)R_i^{-1}x_k$.

**Proof.** First, from (23), using Schur complement, it is obtained that the following holds

$$\Xi_i = \Psi_{i11} - \epsilon^{-1}V_i\Phi D(i)D^T(i)\Phi V_i^T - V_iZ_i^{-1}V_i^T \geq 0. \tag{27}$$

Since $Z_i > 0$, based on Lemma 2 and Lemma 3, it is obtained that

$$\begin{aligned}\tilde{Q}(i) &= Q(i) + \begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix}^T \Phi^T \\ &\quad \cdot Z_i\Phi \begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix} \\ &\geq Q(i) + \begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix}^T \Phi^T V_i^T \\ &\quad + V_i\Phi \begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix} - V_iZ_i^{-1}V_i^T \\ &= Q(i) + \begin{bmatrix} A(i) & B(i) \end{bmatrix}^T \Phi^T V_i^T \\ &\quad + V_i\Phi \begin{bmatrix} A(i) & B(i) \end{bmatrix} - V_i^TZ_i^{-1}V_i \\ &\quad + \begin{bmatrix} F_a(i) & F_b(i) \end{bmatrix}^T \Delta^T(k,i)D^T(i)\Phi^T V_i^T \\ &\quad + V_i\Phi D(i)\Delta(k,i) \begin{bmatrix} F_a(i) & F_b(i) \end{bmatrix} \\ &\geq \Xi_i \geq 0.\end{aligned} \tag{28}$$

Let

$$\begin{aligned}\hat{Q}(i) &= Q(i) + \alpha \begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix}^T \Phi^T \\ &\quad \cdot \Phi \begin{bmatrix} A(k,i) & B(k,i) \end{bmatrix} \\ &= \bar{Q}(i) + \delta\bar{Q}(i)\end{aligned} \tag{29}$$

where

$$\begin{aligned}\delta\bar{Q}(i) &= \alpha \begin{bmatrix} A(i) & B(i) \end{bmatrix}^T \Phi^T\Phi \begin{bmatrix} \delta A(k,i) & \delta B(k,i) \end{bmatrix} \\ &+ \alpha \begin{bmatrix} \delta A(k,i) & \delta B(k,i) \end{bmatrix}^T \Phi^T\Phi \begin{bmatrix} A(i) & B(i) \end{bmatrix} \\ &+ \alpha \begin{bmatrix} \delta A(k,i) & \delta B(k,i) \end{bmatrix}^T \Phi^T\Phi \begin{bmatrix} \delta A(k,i) & \delta B(k,i) \end{bmatrix}.\end{aligned}$$

From (20) and (28), it follows that $\hat{Q}(i) \geq 0$, and then $\bar{Q}(i) \geq 0$.

Based on Lemma 2, for any matrix $L_{1i}$ with appropriate dimensions, there exists $\rho > 0$ such that

$$\rho I + \bar{S}_i > 0, \tag{30}$$

$$\begin{aligned}&-A_K^T(k,i)\Phi^T\bar{S}_i\Phi A_K(k,i) \\ &\leq A_K^T(k,i)\Phi^T L_{1i}^T + L_{1i}\Phi A_K(k,i) \\ &\quad + L_{1i}(\rho I + \bar{S}_i)^{-1}L_{1i}^T + \rho A_K^T(k,i)\Phi^T\Phi A_K(k,i).\end{aligned} \tag{31}$$

Since $X_i > 0$, from Lemma 2, it is obtained that

$$-E^TX_iE \leq E^TL_{2i}^T + L_{2i}E + L_{2i}X_i^{-1}L_{2i}^T. \tag{32}$$

From (29)-(32), it is obtained that

$$\begin{aligned}\hat{\Pi}_i &= A_K^T(k,i)\bar{X}_iA_K(k,i) - A_K^T(k,i)\Phi^T\bar{S}_i\Phi A_K(k,i) \\ &- E^TX_iE + \begin{bmatrix} I_n & K^T(i) \end{bmatrix} \hat{Q}(i) \begin{bmatrix} I_n & K^T(i) \end{bmatrix}^T \\ &\leq A_K^T(k,i)\bar{X}_iA_K(k,i) + A_K^T(k,i)\Phi^T L_{1i}^T \\ &+ L_{1i}\Phi A_K(k,i) + L_{1i}(\rho I + \bar{S}_i)^{-1}L_{1i}^T + E^TL_{2i}^T \\ &+ \rho A_K^T(k,i)\Phi^T\Phi A_K(k,i) + L_{2i}E + L_{2i}X_i^{-1}L_{2i}^T \\ &+ \begin{bmatrix} I_n & K^T(i) \end{bmatrix} (\bar{Q}(i) + \delta\bar{Q}(i)) \begin{bmatrix} I_n & K^T(i) \end{bmatrix}^T = \breve{\Pi}_i.\end{aligned} \tag{33}$$

Applying Schur complement, $\breve{\Pi}_i < 0$ is equivalent to

$$\begin{bmatrix} \bar{\Theta}_{i11} & A_K^T(k,i)W_i & L_{1i} & \rho A_K^T(k,i)\Phi^T & [I_n \;\; K^T(i)]\bar{Q}^{\frac{1}{2}}(i) & \bar{\Theta}_{i16} \\ * & -\bar{Y} & 0 & 0 & 0 & 0 \\ * & * & -\rho I - \bar{S}_i & 0 & 0 & 0 \\ * & * & * & -\rho I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\alpha^{-1}I \end{bmatrix} < 0, \tag{34}$$

holds, where

$$\begin{aligned}\bar{\Theta}_{i11} &= A_K^T(k,i)\Phi^T L_{1i}^T + L_{1i}\Phi A_K(k,i) \\ &\quad + E^TL_{2i}^T + L_{2i}E + L_{2i}Y_iL_{2i}^T \\ &\quad + \alpha\bar{\Theta}_{i16}\Phi A_K(i) + \alpha A_K^T(i)\Phi^T\bar{\Theta}_{i16}^T, \\ \bar{\Theta}_{i16} &= (\delta A(k,i) + \delta B(k,i)K(i))^T\Phi^T, \;\; Y_i = X_i^{-1},\end{aligned}$$

from (33), if (34) holds, then $\hat{\Pi}_i < 0$. Let

$$\tilde{\Pi}_i = \begin{bmatrix} \tilde{\Theta}_{i11} & A_K^T(i)W_i & L_{1i} & \rho A_K^T(i)\Phi^T & [I_n \;\; K^T(i)]\bar{Q}^{\frac{1}{2}}(i) & 0 \\ * & -\bar{Y} & 0 & 0 & 0 & 0 \\ * & * & -\rho I - \bar{S}_i & 0 & 0 & 0 \\ * & * & * & -\rho I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\alpha^{-1}I \end{bmatrix},$$

where

$$\begin{aligned}\tilde{\Theta}_{i11} = \; & A_K^T(i)\Phi^T L_{1i}^T + L_{1i}\Phi A_K(i) \\ & + E^TL_{2i}^T + L_{2i}E + L_{2i}Y_iL_{2i}^T.\end{aligned}$$

From (4), (34) can be rewritten as

$$\tilde{\Pi}_i + \Omega_{1i}\Delta(k,i)\Omega_{2i} + (\Omega_{1i}\Delta(k,i)\Omega_{2i})^T < 0, \tag{35}$$

where

$$\begin{aligned}\Omega_{1i}^T &= \big[\; (L_{1i}\Phi D(i) + \alpha A_K^T(i)\Phi^T\Phi D(i))^T \\ &\quad\;\; (W_i^TD(i))^T \quad 0 \quad \rho(\Phi D(i))^T \quad 0 \quad (\Phi D(i))^T \;\big], \\ \Omega_{2i} &= \begin{bmatrix} F_K(i) & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

From Lemma 3, a necessary and sufficient condition guaranteeing (35) is that there exists a scalar $\lambda_i > 0$ such that

$$\tilde{\Pi}_i + \lambda_i^{-1}\Omega_{1i}\Omega_{1i}^T + \lambda_i\Omega_{2i}^T\Omega_{2i} < 0, \tag{36}$$

Using now Schur complement, (36) becomes

$$\begin{bmatrix} \tilde{\Pi}_i & \Omega_{1i} & \lambda_i\Omega_{2i}^T \\ * & -\lambda_iI & 0 \\ * & * & -\lambda_iI \end{bmatrix} < 0. \tag{37}$$

Let

$$L_{1i} = \epsilon_1 L_i, L_{2i} = \epsilon_2 L_i, R_i = L_i^{-T}, \bar{K}(i) = K(i)R_i,$$

From (37), it follows that

$$\begin{aligned}\tilde{\Theta}_{i11} = \quad & (\epsilon_1 A_K^T(i)\Phi^T + \epsilon_2 E^T)L_i^T \\ & + L_i(\epsilon_1 \Phi A_K(i) + \epsilon_2 E) + \epsilon_2^2 L_i Y_i L_i^T < 0\end{aligned}$$

then $L_i$ is nonsingular. Let $T_i = \operatorname{diag}\{L_i^{-1}, I, I, I, I, I, I\}$, pre- and postmultiply (37) by $T_i$ and $T_i^T$, and by using Schur complement, (37) is equivalent to (24). Then $\hat{\Pi}_i < 0$, together with (28), (20), and $Y_i = X_i^{-1}$, $\bar{K}(i) = K(i)R_i$, and by Lemma 5, the conclusion is obtained. The proof is completed.

**Theorem 2.** For given scalars $\lambda_i > 0$, $\rho > 0$, $\epsilon_1$ and $\epsilon_2$, if there exist matrices $V_i$, $Z_i > 0$, $Y_i > 0$, $\bar{K}(i)$, nonsingular matrix $R_i$, symmetric matrix $S_i$, and scalars $\alpha > 0$, $\epsilon > 0$ such that (20), (23) and

$$\begin{bmatrix} \Theta_{i11} & \Lambda_{1i}^T \bar{W}_i & \epsilon_1 I & \rho \Lambda_{1i}^T \Phi^T & \Theta_{i15} & \Theta_{i16} & \lambda_i \Lambda_{2i}^T \\ * & -\bar{Y} & 0 & 0 & 0 & \bar{W}_i^T D(i) & 0 \\ * & * & \hat{\Theta}_{i33} & 0 & 0 & 0 & 0 \\ * & * & * & -\rho I & 0 & \rho \Phi D(i) & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & \Theta_{i66} & 0 \\ * & * & * & * & * & * & -\lambda_i I \end{bmatrix} < 0, \tag{38}$$

$$\rho I + S_j > 0 \tag{39}$$

where

$$\bar{W}_i = \begin{bmatrix} \sqrt{\bar{p}_i} I & \cdots & \sqrt{\bar{p}_i} I \end{bmatrix}, \hat{\Theta}_{i33} = -\bar{\rho}_i I - \hat{S}_i,$$
$$\bar{\rho}_i = N \underline{p}_i \rho, \ \hat{S}_i = \sum_{j=1}^{N} \underline{p}_i S_j,$$

hold, then there exists a state feedback controller such that system (7) is regular, causal and stochastically stable, and the cost value satisfies (26) for all uncertainties satisfying (4) and (5), and the transition probabilities satisfying (6), the state feedback controller is given by $u_k = \bar{K}(i) R_i^{-1} x_k$.

**Proof.** From (39) and (6), it follows that

$$\begin{aligned}\rho I + \bar{S}_i \quad & = \sum_{j=1}^{N} p_{ij}(\rho I + S_j) \geq \sum_{j=1}^{N} \underline{p}_i(\rho I + S_j) \\ & = \bar{\rho}_i I + \hat{S}_i > 0.\end{aligned} \tag{40}$$

Since $X_i > 0$, together with (6), (31)-(33) and (40), it is obtained that

$$\begin{aligned}\hat{\Pi}_i \quad & \leq A_K^T(k,i) \hat{X}_i A_K(k,i) + A_K^T(k,i) \Phi^T L_{1i}^T \\ & + L_{1i} \Phi A_K(k,i) + L_{1i}(\bar{\rho}_i I + \hat{S}_i)^{-1} L_{1i}^T \\ & + \rho A_K^T(k,i) \Phi^T \Phi A_K(k,i) \\ & + E^T L_{2i}^T + L_{2i} E + L_{2i} X_i^{-1} L_{2i}^T \\ & + \begin{bmatrix} I_n & K^T(i) \end{bmatrix} (\bar{Q}(i) + \delta \bar{Q}(i)) \begin{bmatrix} I_n & K^T(i) \end{bmatrix}^T,\end{aligned}$$

where $\hat{X}_i = \sum_{j=1}^{N} \bar{p}_i X_j$. Applying Schur complement, similar to the proof of Theorem 1, the conclusion is obtained. The proof is completed.

**Remark 5.** In Theorem 1, (20), (23) are independence of (24). (24) depends on (20), (23) since $\bar{Q}(i)$ is in (24). So in Theorem 1, we can solve (20), (23) first, if (20), (23) are solvable, then solve (24). The method to solve Theorem 2 is similar to Theorem 1.

**Remark 6.** The method in this paper only gives the sufficient condition such that the quadratic cost function has a zero lower bound and finite upper one. The minimal value of $J$ cannot be obtained by the method given in this paper.

## IV. EXAMPLE

Consider the following data for the control problem we are addressing in this paper:

$$E = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A(1) = \begin{bmatrix} 4 & -3 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \ B(1) = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix},$$
$$D(1) = \begin{bmatrix} 0.002 & 0.001 & 0.002 \end{bmatrix}^T,$$
$$F_a(1) = \begin{bmatrix} 0.005 & 0.003 & 0.001 \end{bmatrix}, F_b(1) = 0.002,$$

$$A(2) = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \ B(2) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$
$$D(2) = \begin{bmatrix} 0.001 & 0.002 & 0.003 \end{bmatrix}^T,$$
$$F_a(2) = \begin{bmatrix} 0.005 & 0.005 & 0.002 \end{bmatrix}, F_b(2) = 0.001,$$

$$A(3) = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \ B(3) = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix},$$
$$D(3) = \begin{bmatrix} 0.003 & 0.002 & 0.001 \end{bmatrix}^T,$$
$$F_a(3) = \begin{bmatrix} 0.002 & 0.004 & 0.002 \end{bmatrix}, F_b(3) = 0.002,$$

$$Q(1) = \begin{bmatrix} 1 & -2 & -4 & 4 \\ -2 & 4 & -2 & 2 \\ -4 & -2 & -1 & 4 \\ 4 & 2 & 4 & -1 \end{bmatrix},$$

$$Q(2) = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 3 & 0 & -1 \\ 0 & 0 & 5 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix},$$

$$Q(3) = \begin{bmatrix} 2 & -2 & -6 & -8 \\ -2 & 4 & -3 & -4 \\ -6 & -3 & -4 & -12 \\ -8 & -4 & -12 & -12 \end{bmatrix}.$$

The transition probabilities are assomed to satisfy

$$0.1 \leq p_{1j} \leq 0.5, \ 0.2 \leq p_{2j} \leq 0.6, \ 0.2 \leq p_{3j} \leq 0.5.$$

Notice that $Q(1)$, $Q(2)$ and $Q(3)$ are indefinite. Let $\Phi = \operatorname{diag}\{0, 5, 0\}$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\epsilon_1 = \epsilon_2 = 1$, $\rho = 1$. Solving the LMIs (20), (23), (38), (39), they are feasible and the results are given by

$$Y_1 = \begin{bmatrix} 5.4773 & 0.0119 & -1.5136 \\ 0.0119 & 0.0197 & 0.0012 \\ -1.5136 & 0.0012 & 0.4301 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 5.5408 & -0.0030 & -0.9621 \\ -0.0030 & 0.0352 & -0.0039 \\ -0.9621 & -0.0039 & 0.4320 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 5.6191 & 0.0106 & -0.9925 \\ 0.0106 & 0.0374 & 0.0022 \\ -0.9925 & 0.0022 & 0.1911 \end{bmatrix},$$

the gain matrices of a state feedback controller can be obtained as

$$\begin{aligned} K(1) &= \begin{bmatrix} -0.1316 & -0.0118 & -0.1807 \end{bmatrix}, \\ K(2) &= \begin{bmatrix} -0.3150 & -0.6976 & -1.2108 \end{bmatrix}, \\ K(3) &= \begin{bmatrix} -0.2872 & 0.2970 & 0.0928 \end{bmatrix}, \end{aligned}$$

and the cost value satisfies that $0 \leq J \leq x_0 E^T Y_{r_0}^{-1} E^T x_0$, $x_0$ is the initial value.

Figure 1 and Figure 2 give the simulation results for state trajectories of the open-loop system and the closed-loop system with the mode shown in Figure 3 and initial state $Ex_0 = [1 \;\; 0 \;\; -1]^T$, respectively. The cost function satisfies $0 \leq J \leq 48.7584$.

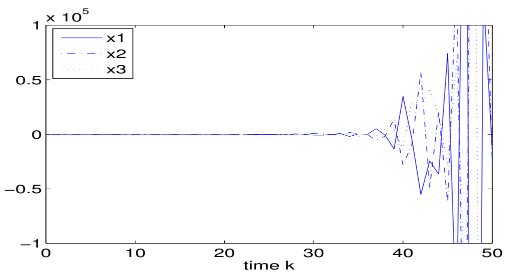

Figure 1. The state trajectories of the open-loop system

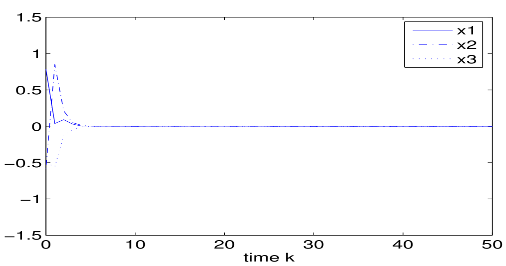

Figure 2. The state trajectories of the closed-loop system

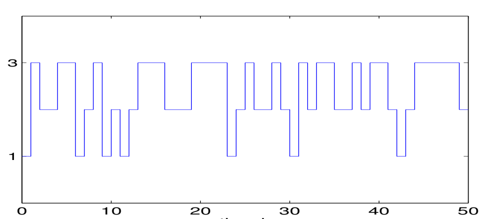

Figure 3. The mode $r_k$

## V. CONCLUSIONS

In this paper, the quadratic cost control problem for discrete-time singular Markov jump singular systems with parameter uncertainties is discussed. The weighting matrix in quadratic cost function is indefinite. For full and partial knowledge of transition probabilities cases, state feedback controllers are designed using LMIs setting, which guarantee the closed-loop discrete-time singular Markov jump systems to be regular, causal and stochastically stable, and the cost value has a zero lower bound and a finite upper one.

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