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Robust Adaptive Optimal Control Modification with Large Adaptive Gain

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***Abstract*— A new adaptive control modification is presented that can achieve robust adaptation with a large adaptive gain without incurring high-frequency oscillations as with the standard model-reference adaptive control. The modification is based on an optimal control formulation that minimizes the $\mathcal{L}_2$ norm of the tracking error. The optimality condition is used to derive the modification using the gradient method. The adaptive optimal control modification results in a stable adaptation and allows a large adaptive gain to be used for better tracking performance with improved stability robustness. Simulations demonstrate the effectiveness of the proposed modification.**

I. Introduction

In recent years, adaptive control has been receiving a significant amount of attention. Adaptive control provides the ability to accommodate system uncertainties and to improve fault tolerance of a control system. Various model-reference adaptive control (MRAC) methods have been investigated [1], [2], [4], [3], [5], [6], [7], [8], [9], [10]. In the conventional MRAC framework, the upper bound on the steady state tracking error is generally inversely proportional to the magnitude of the adaptive gain. Thus, in the presence of large uncertainties, fast adaptation using a large adaptive gain can be used to reduce the tracking error rapidly. However, a large adaptive gain can lead to high-frequency oscillations which can adversely affect robustness of an MRAC law.

Various modifications were developed to increase robustness of MRAC by adding damping to the adaptive law. Two well-known modifications in adaptive control are the σ-modification [11] and ε_1- modification [12]. These modifications have been used extensively in adaptive control. This paper introduces a new modification based on an optimal control formulation that minimizes the $\mathcal{L}_2$-norm of the tracking error. The optimality condition results in a damping term proportional to the persistent excitation. The analysis shows that this modification can allow fast adaptation with a large adaptive gain without causing high-frequency oscillations and can provide improved stability robustness while preserving the tracking performance.

II. Optimal Control Modification

A direct MRAC problem is posed as follows:

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Given a nonlinear plant as

$$\dot{x} = Ax + B\left[u + f(x)\right] \tag{1}$$

where $x(t) : [0,\infty) \to \mathbb{R}^n$ is a state vector, $u(t) : [0,\infty) \to \mathbb{R}^p$ is a control vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known such that the pair (A,B) is controllable, and $f(x) : \mathbb{R}^n \to \mathbb{R}^p$ is a bounded unstructured uncertainty.

Assumption 1: The uncertainty $f(x)$ can be approximated using a feedforward neural network in the form

$$f(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \varepsilon(x) = \Theta^\top \Phi(x) + \varepsilon(x) \tag{2}$$

where $\Theta \in \mathbb{R}^{m\times p}$ is an unknown constant weight matrix that represents a parametric uncertainty, $\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector of known bounded basis functions with Lipschitz nonlinearity, and $\varepsilon(x) : \mathbb{R}^n \to \mathbb{R}^p$ is an approximation error.

Since $\Phi(x)$ is Lipschitz, then

$$\|\Phi(x) - \Phi(x_0)\| \le C\|x - x_0\| \tag{3}$$

for some constant $C > 0$, which implies a bounded partial derivative

$$\left\|\frac{\partial \Phi(x)}{\partial x}\right\| \le L \tag{4}$$

for some constant $L > 0$.

The set of basis functions $\Phi(x)$ is chosen such that the approximation error $\varepsilon(x)$ becomes small on a compact domain $x \in \mathbb{R}^n$. The universal approximation theorem for sigmoidal neural networks by Cybenko can be used for selecting a good set of basis functions $\Phi(x)$ [13]. Alternatively, the Micchelli's theorem provides theoretical basis for a neural net design of $\Theta^\top\Phi(x)$ using radial basis functions to keep the approximation error $\varepsilon(x)$ small [14].

Assumption 2: The set of basis functions $\Phi(x)$ satisfies the persistent excitation condition for some $\alpha_0, \alpha_1, T_0 \ge 0$

$$\alpha_1 I \ge \frac{1}{T_0}\int_t^{t+T_0} \Phi(x(\tau))\Phi^\top(x(\tau))\,d\tau \ge \alpha_0 I, \ \forall t > 0 \tag{5}$$

where I is an identity matrix.

The objective is to design a controller that enables the plant to follow a reference model

$$\dot{x}_m = A_m x_m + B_m r \tag{6}$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwitz and known, $B_m \in \mathbb{R}^{n\times p}$ is also known, and $r(t) : [0,\infty) \to \mathbb{R}^p \in \mathcal{L}_\infty$ is a command vector with $\dot{r} \in \mathcal{L}_\infty$.

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Defining the tracking error as $e = x_m - x$, then the controller $u(t)$ is specified by

$$u = K_x x + K_r r - u_{ad} \tag{7}$$

where $K_x \in \mathbb{R}^{p\times n}$ and $K_r \in \mathbb{R}^{p\times p}$ are known nominal gain matrices, and $u_{ad} \in \mathbb{R}^p$ is a direct adaptive signal.

Then, the tracking error equation becomes

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e + (A_m - A - BK_x)x + (B_m - BK_r)r + B\left[u_{ad} - \Theta^\top \Phi(x) - \varepsilon(x)\right] \tag{8}$$

The gain matrices K_x and K_r are chosen to satisfy the model matching conditions $A + BK_x = A_m$ and $BK_r = B_m$. The adaptive signal u_{ad} is an estimator of the parametric uncertainty in the plant such that

$$u_{ad} = \hat{\Theta}^\top \Phi(x) \tag{9}$$

where $\hat{\Theta} \in \mathbb{R}^{m\times p}$ is an estimate of Θ.

Let $\tilde{\Theta} = \hat{\Theta} - \Theta$ be an estimation error. Then the tracking error equation can be expressed as

$$\dot{e} = A_m e + B\left(\tilde{\Theta}^\top \Phi - \varepsilon\right) \tag{10}$$

Proposition 1: The following adaptive law provides an update law that minimizes $\|e\|_{\mathscr{L}_2}$

$$\dot{\hat{\Theta}} = -\Gamma\Phi\left(e^\top P - \nu\Phi^\top\hat{\Theta}B^\top P A_m^{-1}\right)B \tag{11}$$

where $\Gamma = \Gamma^\top > 0 \in \mathbb{R}^{m\times m}$ is an adaptive gain matrix, $\nu > 0 \in \mathbb{R}$ is a weighting constant, and $P = P^\top > 0 \in \mathbb{R}^{n\times n}$ solves

$$PA_m + A_m^\top P = -Q \tag{12}$$

where $Q = Q^\top > 0 \in \mathbb{R}^{n\times n}$.

Proof: The adaptive law seeks to minimize the cost function

$$J = \frac{1}{2}\int_0^{t_f} (e-\Delta)^\top Q(e-\Delta)\,dt \tag{13}$$

subject to Eq. (10) where Δ is the tracking error at $t = t_f$.

This optimal control problem can be formulated by the Pontryagin's Minimum Principle. Defining a Hamiltonian

$$H\left(e,\tilde{\Theta}\right) = \frac{1}{2}(e-\Delta)^\top Q(e-\Delta) + p^\top\left(A_m e + B\tilde{\Theta}^\top\Phi - B\varepsilon\right) \tag{14}$$

where $p(t) : [0,\infty) \to \mathbb{R}^n$ is an adjoint variable, then the necessary condition gives

$$\dot{p} = -\nabla H_e^\top = -Q(e-\Delta) - A_m^\top p \tag{15}$$

with the transversality condition $p\left(t_f\right) = 0$ since $e(0)$ is known. The optimality condition is obtained by

$$\nabla H_{\tilde{\Theta}^\top} = \Phi\nabla H_{\tilde{\Theta}^\top\Phi} = \Phi p^\top B \tag{16}$$

The adaptive law is formulated by the gradient method as

$$\dot{\tilde{\Theta}} = -\Gamma\nabla H_{\tilde{\Theta}^\top} = -\Gamma\Phi p^\top B \tag{17}$$

The solution of p can be obtained using a "sweeping" method [15] by letting $p = Pe + S\hat{\Theta}^\top\Phi$. Then

$$\dot{P}e + P\left(A_m e + B\hat{\Theta}^\top\Phi - B\Theta^\top\Phi - B\varepsilon\right) + \dot{S}\hat{\Theta}^\top\Phi + S\frac{d\left(\hat{\Theta}^\top\Phi\right)}{dt} = -Q(e-\Delta) - A_m^\top\left(Pe + S\hat{\Theta}^\top\Phi\right) \tag{18}$$

which yields the following equations

$$\dot{P} + PA_m + A_m^\top P + Q = 0 \tag{19}$$

$$\dot{S} + PB + A_m^\top S = 0 \tag{20}$$

$$Q\Delta - S\frac{d\left(\hat{\Theta}^\top\Phi\right)}{dt} + PB\left(\Theta^\top\Phi + \varepsilon\right) = 0 \tag{21}$$

subject to the transversality conditions $P\left(t_f\right) = 0$ and $S\left(t_f\right) = 0$.

The existence and uniqueness of the solution of the Lyapunov differential equation (19) is well-established. It follows that Eq. (20) also has a stable, unique solution in time-to-go $\tau = t_f - t$.

Since $\dot{r} \in \mathscr{L}_\infty$, Φ is bounded and Lipschitz, and $p\left(t_f\right) = 0$ from the transversality condition, then as $t_f \to \infty$, $\lim_{t_f\to\infty}\left|d\left(\hat{\Theta}^\top\Phi\right)/dt\right|$ exists, where

$$\lim_{t_f\to\infty}\left|\frac{d\left(\hat{\Theta}^\top\Phi\right)}{dt}\right| = \lim_{t_f\to\infty}\left|-B^\top p\left(t_f\right)\Phi^\top\Gamma\Phi + \hat{\Theta}^\top\frac{\partial\Phi}{\partial x}\left[\dot{x}_m - B\left(\tilde{\Theta}^\top\Phi - \varepsilon\right)\right]\right| \le \lim_{t_f\to\infty}\left|\hat{\Theta}^\top L\mathscr{I}\left[-A_m^{-1}B_m\dot{r} - B\left(\tilde{\Theta}^\top\Phi - \varepsilon\right)\right]\right| = \sigma_t \tag{22}$$

for some constant vector $\sigma_t > 0 \in \mathbb{R}^n$, and $\mathscr{I} \in \mathbb{R}^{m\times n}$ is a matrix whose elements are all equal to one.

Consider an infinite time-horizon problem as $t_f \to \infty$. Then $P(t) \to P(0)$ and $S(t) \to S(0)$ are determined by their steady state solutions from Eqs. (19) and (20) as

$$PA_m + A_m^\top P = -Q \tag{23}$$

$$S = -A_m^{-\top}PB \tag{24}$$

The adjoint p is then obtained as

$$p = Pe - \nu A_m^{-\top}PB\hat{\Theta}^\top\Phi \tag{25}$$

where ν is introduced as a weighting constant to allow for adjustments of the second term in the adaptive law. Since Θ is constant, then the adaptive law (11) is obtained from Eqs. (17) and (25).

Defining $\delta_\varepsilon = \sup_t|\varepsilon|$ and $\varphi = \sup_t\left|\Theta^\top\Phi\right|$, then for $\nu = 1$ the unknown tracking error Δ at $t = t_f \to \infty$ is bounded by

$$\|\Delta\| \le \frac{\lambda_{max}(P)\|B\|}{\lambda_{min}(Q)}\left[\|\varphi\| + \|\delta_\varepsilon\| + \frac{\|\sigma_t\|}{\sigma_{min}(A_m)}\right] \tag{26}$$

where λ and σ denote the eigenvalue and singular value, respectively.

Theorem 1: The adaptive law (11) results in stable and uniformly bounded tracking error outside a compact set

$$\mathscr{C} = \left\{ e, \tilde{\Theta}^\top \Phi \in \mathbb{R}^n : \lambda_{min}(Q)\left[(\|e\| - \Delta_1)^2 - \Delta_1^2\right] \right.$$
$$\left. + \nu \lambda_{min}\left(B^\top A_m^{-\top} Q A_m^{-1} B\right)\left[\left(\left\|\tilde{\Theta}^\top \Phi\right\| - \Delta_2\right)^2 - \Delta_2^2\right] \leq 0 \right\} \tag{27}$$

where

$$\Delta_1 = \frac{\lambda_{max}(P)\|B\|\|\delta_\varepsilon\|}{\lambda_{min}(Q)} \tag{28}$$

$$\Delta_2 = \frac{\sigma_{max}\left(B^\top P A_m^{-1} B\right)\|\varphi\|}{\lambda_{min}\left(B^\top A_m^{-\top} Q A_m^{-1} B\right)} \tag{29}$$

Proof: Choose a Lyapunov candidate function

$$V = e^\top P e + \operatorname{trace}\left(\tilde{\Theta}^\top \Gamma^{-1} \tilde{\Theta}\right) \tag{30}$$

Evaluating $\dot{V}$ yields

$$\dot{V} = -e^\top Q e + 2e^\top PB\left(\tilde{\Theta}^\top \Phi - \varepsilon\right) - 2e^\top PB\tilde{\Theta}^\top \Phi + 2\nu \Phi^\top \hat{\Theta} B^\top P A_m^{-1} B \tilde{\Theta}^\top \Phi \tag{31}$$

PA_m^{-1} can be decomposed into a symmetric part $M = \frac{1}{2}\left(PA_m^{-1} + A_m^{-\top}P\right) = -\frac{1}{2}A_m^{-\top} Q A_m^{-1}$ and an anti-symmetric part $N = \frac{1}{2}\left(PA_m^{-1} - A_m^{-\top}P\right)$. Since $M < 0$, then $PA_m^{-1} < 0$. Using the property $y^\top N y = 0$, $\dot{V}$ becomes

$$\dot{V} = -e^\top Q e - 2e^\top PB\varepsilon - \nu \Phi^\top \tilde{\Theta} B^\top A_m^{-\top} Q A_m^{-1} B \tilde{\Theta}^\top \Phi + 2\nu \Phi^\top \Theta B^\top P A_m^{-1} B \tilde{\Theta}^\top \Phi \tag{32}$$

By completing the squares, an upper bound of $\dot{V}$ is

$$\dot{V} \leq -\lambda_{min}(Q)\left[(\|e\| - \Delta_1)^2 - \Delta_1^2\right] - \nu \lambda_{min}\left(B^\top A_m^{-\top} Q A_m^{-1} B\right)\left[\left(\left\|\tilde{\Theta}^\top \Phi\right\| - \Delta_2\right)^2 - \Delta_2^2\right] \tag{33}$$

If $\mathscr{C}$ is a compact set defined in Eq. (27), then for bounded tracking error, $\dot{V} \leq 0$ outside the compact set $\mathscr{C}$, but $V(t)$ increases inside the compact set $\mathscr{C}$, which contains $e = 0$ and $\tilde{\Theta}$=0, whose trajectories will all stay inside $\mathscr{C}$. It follows by LaSalle's Invariance Principle that e and $\tilde{\Theta}$ are uniformly bounded.

The effect of the optimal control modification is to add a damping term to the weight update law, which depends on the persistent excitation.

Theorem 2: In the presence of fast adaptation, i.e., $\lambda_{min}(\Gamma) \gg 1$, the adaptive law (11) is robustly stable for $\nu = 1$ with all closed-loop poles having negative real values.

Proof: The adaptive law (11) can be written as

$$\frac{\Phi^\top \dot{\hat{\Theta}}}{\Phi^\top \Gamma \Phi} = -\left(e^\top P - \nu \Phi^\top \hat{\Theta} B^\top P A_m^{-1}\right) B \tag{34}$$

If $\Gamma \gg 1$ is large and the input is PE, then in the limit as $\Phi^\top \Gamma \Phi \to \infty$

$$B\hat{\Theta}^\top \Phi \to \frac{1}{\nu} P^{-1} A_m^\top P e \tag{35}$$

Hence, the closed-loop tracking error equation becomes

$$\dot{e} = -P^{-1}\left[\left(\frac{1+\nu}{2\nu}\right) Q - \left(\frac{1-\nu}{2\nu}\right) S\right] e - B\left(\Theta^\top \Phi + \varepsilon\right) \tag{36}$$

where $S = A_m^\top P - PA_m$.

For $\nu = 1$, the closed-loop poles are all real, negative values with $\operatorname{Re}[s] = -\lambda\left(P^{-1}Q\right)$. The system transfer function matrix $H(s) = \left(sI + P^{-1}Q\right)^{-1}$ is strictly positive real (SPR) since $H(j\omega) + H^\top(-j\omega) > 0$, and thus the system is minimum phase and dissipative [16]. The Nyquist plot of a strictly stable transfer function for a SISO system is strictly in the right half plane with a phase shift less than or equal to $\frac{\pi}{2}$ [16], corresponding to a phase margin of at least $\frac{\pi}{2}$. For a MIMO system, the diagonal elements of the system transfer function matrix exhibit a similar behavior.

Lemma 1: The equilibrium state $y = 0$ of the differential equation

$$\dot{y} = -\Phi^\top(t)\Gamma\Phi(t)y \tag{37}$$

where $y(t) : [0,\infty) \to \mathbb{R}$, $\Phi(t) \in \mathscr{L}_2 : [0,\infty) \to \mathbb{R}^n$ is a piecewise continuous and bounded function, and $\Gamma > 0 \in \mathbb{R}^{n\times n}$, is uniformly asymptotically stable, if there exists a constant $\gamma > 0$ such that

$$\frac{1}{T_0}\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\,d\tau \geq \gamma \tag{38}$$

which implies that y is locally bounded by the solution of a linear differential equation

$$\dot{z} = -\gamma z \tag{39}$$

for $t \in [t_i, t_i + T_0]$, where $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Proof: Choose a Lyapunov candidate function and evaluate its time derivative

$$V = \frac{1}{2}y^2 \tag{40}$$

$$\dot{V} = -\Phi^\top(t)\Gamma\Phi(t)y^2 = -2\Phi^\top(t)\Gamma\Phi(t)V \tag{41}$$

Then, there exists $\gamma > 0$ for which V is uniformly asymptotically stable since

$$V(t + T_o) = V(t)\exp\left(-2\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\,d\tau\right) \leq V(t)e^{-2\gamma T_0} \tag{42}$$

This implies that

$$\exp\left(-2\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\,d\tau\right) \leq e^{-2\gamma T_0} \tag{43}$$

Thus, the equilibrium $y = 0$ is uniformly asymptotically stable if

$$\frac{1}{T_0}\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\,d\tau \geq \gamma \tag{44}$$

provided $\Phi(t) \in \mathscr{L}_2$ is bounded.

Then $y(t) \in \mathscr{L}_2 \cap \mathscr{L}_\infty$ since

$$2\gamma \int_0^\infty y^2(t)\,dt \le V(0) - V(t \to \infty) = V(0)\left(1 - \lim_{t\to\infty} e^{-2\gamma t}\right) \tag{45}$$

It follows that

$$\dot{V} \le -2\gamma V \Rightarrow y\dot{y} \le -\gamma y^2 \tag{46}$$

which implies that the solution of Eq. (37) is bounded from above if $y \ge 0$ and from below if $y \le 0$ by the local solution of

$$\dot{z} = -\gamma z \tag{47}$$

for $t \in [t_i, t_i + T_0]$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1, 2, \ldots, n \to \infty$.

Equation (47) also applies for $\Phi = \Phi(y)$ since the condition $\Phi(y(t)) \in \mathscr{L}_2$ is identically satisfied given that $y \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. This is shown by evaluating $\dot{V}$ as

$$\dot{V} = \dot{y}\frac{dV}{dy} = -\Phi^\top(y)\Gamma\Phi(y)\, y\frac{dV}{dy} = -2\Phi^\top(y)\Gamma\Phi(y)V \tag{48}$$

Thus

$$\frac{dV}{V} = 2\frac{dy}{y} \tag{49}$$

Suppose there exists γ such that

$$\frac{dy}{y} \le -\gamma dt \tag{50}$$

Then multiplying both sides of Eq. (50) by y^2 and dividing by dt result in the same equation as Eq. (46). Thus, V is uniformly asymptotically stable and y is bounded by Eq. (47). Therefore, γ given by Eq. (38) satisfies Eq. (50).

Lemma 1 is a version of the well-known Comparison Lemma [17]. A different version of the proof is also provided by Nadrenda and Annaswamy [18].

Lemma 2: The solution of a linear differential equation

$$\dot{y} = Ay + g(t) \tag{51}$$

where $y(t) : [0, \infty) \to \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix, and $g(t) : [0, \infty) \to \mathbb{R}^n \in \mathscr{L}_\infty$ is a piecewise continuous, bounded function, is asymptotically stable and semi-globally bounded from above by the solution of a differential equation

$$\dot{z} = A\left(z - \alpha\left|A^{-1}c\right|\right) \tag{52}$$

where $\alpha \ge 1 \in \mathbb{R}$ and $c = \sup_t |g(t)|$.

Proof: For matching initial conditions $y(0) = z(0)$, the solutions of y and z are

$$y = e^{At}y(0) + \int_0^t e^{A(t-\tau)}g(\tau)\,d\tau \tag{53}$$

$$z = e^{At}y(0) - \int_0^t e^{A(t-\tau)}\alpha A\left|A^{-1}c\right|d\tau \tag{54}$$

If $A^{-1}c > 0$, then

$$y = z + \int_0^t e^{A(t-\tau)}\alpha c\,d\tau + \int_0^t e^{A(t-\tau)}g(\tau)\,d\tau = z + \int_0^t e^{A(t-\tau)}A\left[\alpha A^{-1}c + A^{-1}g(\tau)\right]d\tau \tag{55}$$

$\alpha \ge 1$ can be made large enough for $\alpha A^{-1}c + A^{-1}g(\tau) > 0$ because $A^{-1}c > 0$ and g is bounded, and since $\int_0^t e^{A(t-\tau)}A\,d\tau \le 0$, then

$$\int_0^t e^{A(t-\tau)}A\left[\alpha A^{-1}c + A^{-1}g(\tau)\right]d\tau \le 0 \tag{56}$$

Therefore, $y \le z$.

If $A^{-1}c < 0$, then

$$y = z - \int_0^t e^{A(t-\tau)}\alpha c\,d\tau + \int_0^t e^{A(t-\tau)}g(\tau)\,d\tau = z - \int_0^t e^{A(t-\tau)}A\left[\alpha A^{-1}c - A^{-1}g(\tau)\right]d\tau \tag{57}$$

α can be made large enough for $\alpha A^{-1}c - A^{-1}g(\tau) < 0$ because $A^{-1}c < 0$ and g is bounded, therefore $y \le z$. Thus, $y \le z$ for all $t \in [0, \infty)$ and some $\alpha \ge 1$.

Theorem 3: The steady state tracking error is bounded by

$$\lim_{t\to\infty}\sup_t \|e\| = \frac{\lambda_{max}(P)\|B\|}{\sigma_{min}(A_m^\top P + \nu P A_m)}\left[\nu\|\varphi\| + \nu\|A_m\|\left\|A_m^{-1}\right\|\|\delta_\varepsilon\| + \frac{1}{\gamma}\left\|\left(B^\top A_m^{-\top}PB\right)^{-1}\right\|\|\beta\|\right] \tag{58}$$

if there exists a constant $\gamma > 0$ such that $\gamma = \inf_t\left(\frac{1}{T_0}\int_t^{t+T_0}\Phi^\top\Gamma\Phi\,d\tau\right) > 0 \in \mathbb{R}$ and a constant vector $\beta > 0 \in \mathbb{R}^n$ where $\beta = \sup_t\left|\tilde{\Theta}^\top\dot{\Phi}\right|$.

Proof: Since $\tilde{\Theta}$ is bounded by the adaptive law (11) and $\lim_{t_f\to\infty}\left|d\left(\hat{\Theta}^\top\Phi\right)/dt\right| < \sigma_t$ exists which implies $d\left(\hat{\Theta}^\top\Phi\right)/dt$ is bounded, then $\beta = \sup_t\left|\tilde{\Theta}^\top\dot{\Phi}\right| \in \mathscr{L}_\infty$ is bounded.

Since $e \in \mathscr{L}_2$, $x \in \mathscr{L}_2$, and so $\Phi(x) \in \mathscr{L}_2$, then using Lemma 1, the adaptive law (11) can be written as

$$\frac{d}{dt}\left(\tilde{\Theta}^\top\Phi\right) = \dot{\tilde{\Theta}}^\top\Phi + \tilde{\Theta}^\top\dot{\Phi} \le -\gamma B^\top Pe + \gamma\nu B^\top A_m^{-\top}PB\left(\tilde{\Theta}^\top\Phi - \varphi - \left|\left(\gamma\nu B^\top A_m^{-\top}PB\right)^{-1}\beta\right|\right) \tag{59}$$

for $t \in [t_i, t_i + T_0]$, where $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Using Lemma 2 with $\alpha = 1$ for simplicity, we write

$$\dot{e} \le A_m\left(e - \left|A_m^{-1}B\delta_\varepsilon\right|\right) + B\tilde{\Theta}^\top\Phi \tag{60}$$

Differentiating Eq. (60) and upon substitution yields

$$\ddot{e} - \left(A_m + \gamma\nu BB^\top A_m^{-\top}P\right)\dot{e} + \left(\gamma BB^\top P + \gamma\nu BB^\top A_m^{-\top}PA_m\right)e \le -\gamma\nu BB^\top A_m^{-\top}PB\left[\varphi + \left|\left(\gamma\nu B^\top A_m^{-\top}PB\right)^{-1}\beta\right|\right] + \gamma\nu BB^\top A_m^{-\top}PA_m\left|A_m^{-1}B\delta_\varepsilon\right| \tag{61}$$

The steady state upper bound on the norm of $\tilde{\Theta}^\top \Phi$ is also obtained as

$$\limsup_{t\to\infty} \sup_t \left\|\tilde{\Theta}^\top \Phi\right\| = \frac{\lambda_{max}(P)\,\|A_m\|}{\sigma_{min}\left(A_m^\top P + \nu P A_m\right)} \left[\nu \left\|\Theta^\top \Phi\right\| + \|A_m\| \left\|A_m^{-1}\right\| \|\delta_\varepsilon\| + \frac{1}{\gamma} \left\|\left(B^\top A_m^{-\top} P B\right)^{-1}\right\| \|\beta\|\right] \quad (62)$$

The steady state upper bound on $\|e\|$ is solved from Eq. (61) which leads to Eq. (58). Thus for $\gamma \to \infty$, the last term on the RHS of Eq. (58) goes to zero, and $\|e\|$ is only dependent on ν. If, in addition, $\nu \to 0$, then $\|e\| \to 0$, but if $\nu \to \infty$, $e \in \mathscr{L}_\infty$ is finite and does not tend to zero. Thus, ν has to be selected small enough to provide a desired tracking performance, but large enough to provide sufficient robustness against time delay or unmodeled dynamics. A practical bound for ν is $0 < \nu < 1$ since any increase in ν beyond its optimal value $\nu = 1$ will actually reduce robustness as well as tracking performance. Both Theorems 2 and 3 provide a guidance in a trade-off design process for selecting a suitable value of ν to meet performance and robustness requirements.

III. Flight Control Example

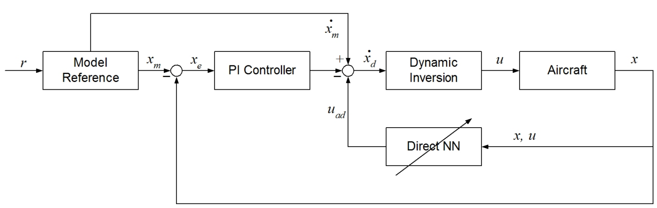

Fig. 1 - Direct Neural Network Adaptive Flight Control

Consider the following inner loop adaptive flight control architecture as shown in Fig. 1. The plant model is

$$\dot{x} = A_{11}x + A_{12}z + B_1 u + f_1(x,z) \quad (63)$$

$$\dot{z} = A_{21}x + A_{22}z + B_2 u + f_2(x,z) \quad (64)$$

where $x = \begin{bmatrix} p & q & r \end{bmatrix}^\top$ is a vector of roll, pitch, and yaw rates; $z = \begin{bmatrix} \phi & \alpha & \beta \end{bmatrix}^\top$ is a vector of bank angle, angle of attack, and sideslip angle; $u = \begin{bmatrix} \delta_a & \delta_e & \delta_r \end{bmatrix}^\top$ is a vector of aileron, elevator, and rudder deflections; and $f_i(x,z)$, $i = 1,2$ is an uncertainty

$$f_i(x,z) = C_{i1}x + C_{i2}z + \varepsilon = \Theta_i^\top \Phi + \varepsilon \quad (65)$$

The angular rates are designed to follow a second-order reference angular rate model specified as

$$\left(s + 2\zeta_i\omega_i + \frac{\omega_i^2}{s}\right) x_{m_i} = g_i \delta_i \quad (66)$$

where x_{m_i}, $i = 1,2,3$ corresponds to p, q, r, respectively; δ_i is the corresponding lateral stick input, longitudinal stick input, and rudder pedal input; $\zeta_i > 0$ is the corresponding damping ratio; and $\omega_i > 0$ is the corresponding frequency.

Assuming the pair (A_{11}, B_1) is controllable and z is stabilizable, the angular rate feedback control is given by

$$u = B_1^{-1}\left[-\left(K_p + \frac{K_i}{s}\right)x + Gr - A_{11}x - A_{12}z - \hat{\Theta}_1^\top \Phi\right] \quad (67)$$

where $K_p = \operatorname{diag}(2\zeta_1\omega_1, 2\zeta_2\omega_2, 2\zeta_3\omega_3)$, $K_i = \operatorname{diag}\left(\omega_1^2, \omega_2^2, \omega_3^2\right)$, $G = \operatorname{diag}(g_1, g_2, g_3)$, and $r = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 \end{bmatrix}^\top$.

Let $e = \begin{bmatrix} \int_0^t (x_m - x)\, d\tau & x_m - x \end{bmatrix}^\top$ be the tracking error, then the tracking error equation is given by Eq. (8) with

$$A_m = \begin{bmatrix} 0 & I \\ -K_i & -K_p \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (68)$$

Let $Q = 2I$, then it can shown that $B^\top P A_m^{-1} B = -K_i^{-2} < 0$, so the adaptive law (11) becomes

$$\dot{\hat{\Theta}}_1 = -\Gamma\Phi\left(e^\top P B + \nu \Phi^\top \hat{\Theta}_i K_i^{-2}\right) \quad (69)$$

The uncertainty is modeled as an airframe structural damage to the left wing of a generic transport aircraft as shown Fig. 2. The objective is to track a pitch rate doublet while regulating the rate responses in the roll and yaw axes.

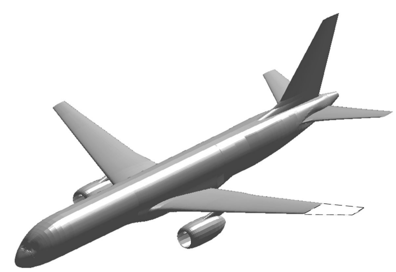

Fig. 2 - Damaged Generic Transport Aircraft

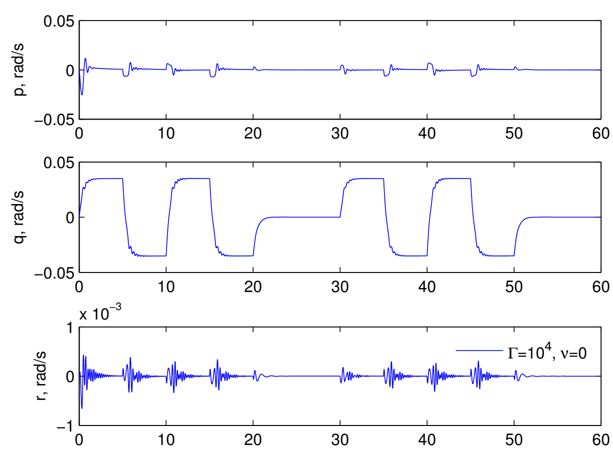

Fig. 3 - Aircraft Rate Responses with Standard MRAC

Figure 3 is a plot of the aircraft angular rates due to the standard direct MRAC ($\nu = 0$) using an adaptive gain $\Gamma = 10^4$. The tracking performance drastically improves in all axes. However, high-frequency oscillations can clearly be seen in the yaw rate response.

Figure 4 is the aircraft rate response with the optimal control modification with $\nu = 0.033$, which results in no observable high-frequency oscillation in spite of the fact that the adaptive gain is two orders of magnitude greater than that for the standard direct MRAC.

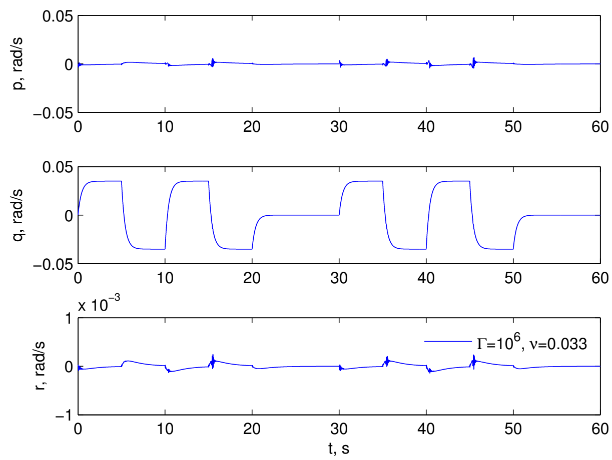

Fig. 4 - Aircraft Rate Responses with Optimal Control Modification

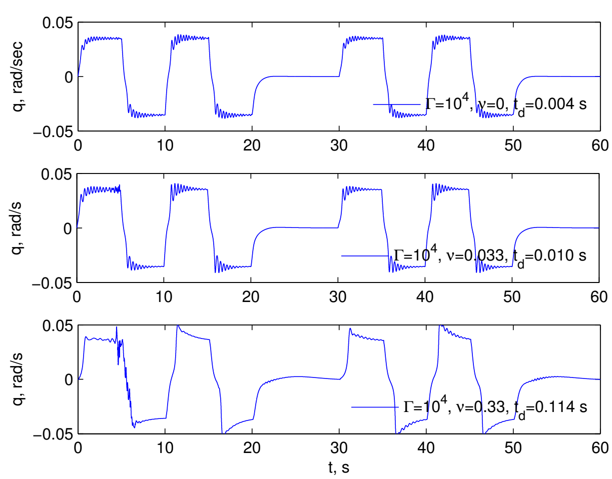

Fig. 5 - Pitch Rate Responses with Time Delay

Figure 5 illustrates the time delay effect on the adaptive optimal control. A time delay is introduced between the aircraft plant input and output to simulate destabilizing uncertainties. For the same adaptive gain $\Gamma = 10^4$, the standard MRAC can tolerate up to 0.004 s time delay before the adaptive law goes unstable. With the optimal control modification, the time delay margin increases to 0.010 s and 0.114 s for $\nu = 0.033$ and $\nu = 0.33$, respectively. This is consistent with the analysis that increasing ν results in improved robustness against time delay or unmodeled dynamics.

IV. Conclusions

This study presents a new adaptive optimal control modification that adds a damping term to the standard MRAC that is proportional to the persistent excitation. The optimal control modification can be tuned using a parameter ν to provide a trade-off between tracking performance and stability robustness. Increasing ν improves stability margins but reduces tracking performance. When ν approaches unity, the system is robustly stable with all closed-loop poles having negative real values. Simulations demonstrate that the optimal control modification achieves better tracking performance at a much larger adaptive gain than the standard MRAC, and can also tolerate a much greater time delay.

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