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Certainty-Equivalence Design for Output Regulation of a Nonlinear Benchmark System

Fabio Celani

Abstract— **This paper deals with the design of a feedback controller that solves an output regulation problem for the nonlinear benchmark system known as TORA. The controller uses only measurements of the of the rotational position and is obtained through a certainty-equivalence approach.**

I. INTRODUCTION

The problem of controlling a nonlinear benchmark system called TORA (Translational Oscillator with a Rotational Actuator) and also known as RTAC (Rotational-Translational ACtuator) was introduced in [1]. The latter control problem has received a considerable amount of attention from several researchers. Most of them focused on global stabilization (see [1]–[5] and references therein). Some authors presented results concerning more general control problems such as output tracking and disturbance rejection (see e.g. [6] and [7]). Controlling the system so as to achieve asymptotic disturbance rejection has been investigated in [8], [9] and [10]. In those works results on nonlinear output regulation theory (see [11] and [12]) have been employed. In [8] and [10], using standard tecniques (see [11]), the authors design full-information regulators, i.e. regulators that use measurements of both the states of the plant and of the exosystem. On the other hand, the compensators presented in [9] use only position measurements, and one of them is robust with respect to sufficiently small uncertainties of the plant parameters.

In this work a regulator that uses *only measurements of the rotational position* is presented; in such scenario the translational displacement, which represents the regulated output, is not available for feedback. As a consequence, the resulting output regulation problem departs from the standard formulation and fits into the framework described in [13, Chapter 5] and [14]. However, the author is not able to apply the design technique in [13, Section 5.4.2] to the problem at hand since the conditions on quadratic stabilazibility and detectability are not easy to verify. On the other hand, it will be shown in the sequel that a solution can be found applying the certainty-equivalence design proposed in [14] .

The rest of the paper is organized as follows. In Section II the nonlinear output regulation problem under investigation is formulated. In Section III the general framework for certainty-equivalence design in nonlinear output regulation is recalled. In Section IV a full-information regulator is obtained; in Section V it is shown that the system under consideration satisfies an appropriate observability property; in Section VI a measurement feedback regulator based on certainty-equivalence is derived. Simulation results are presented in Section VII.

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Fabio Celani is with the Dipartimento di Informatica e Sistemistica Antonio Ruberti, Sapienza Università di Roma, 00185 Roma, Italy celani@dis.uniroma1.it

Notation. For $x \in \mathbb{R}^n$, $|x|$ denotes the Eucledian norm of x. For $A \subseteq \mathbb{R}^n$, $\omega(A)$ denotes the *ω-limit set of A* (see [15, pp. 7-8] for a definition of ω-limit set of a set). $L_f h(x)$ denotes the (Lie) derivative at x of h along f (see [16, p. 8]).

II. PROBLEM FORMULATION

The equations of motion of the TORA system after appropriate normalization and time scaling (see [1]) are given by

$$\begin{aligned} \ddot{x}_d + x_d &= \epsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + w_1 \\ \ddot{\theta} &= u - \epsilon \ddot{x}_d \cos\theta \ . \end{aligned} \tag{1}$$

In (1) x_d is the translational displacement, θ is the rotational position, u is the control input, w_1 is an harmonic disturbance input of known angular frequency ω, and ϵ is a known plant parameter with $0 < \epsilon < 1$. Setting $x \triangleq (x_d \ \dot{x}_d \ \theta \ \dot{\theta})^T$, equations (1) can be put in the following state-space form

$$\dot{x} = f(x) + g(x)u + p(x)w_1 \tag{2}$$

where

$$\begin{aligned} f(x) &\triangleq \begin{pmatrix} x_2 \\ \dfrac{-x_1 + \epsilon x_4^2 \sin x_3}{1-\epsilon^2\cos^2 x_3} \\ x_4 \\ \dfrac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1-\epsilon^2\cos^2 x_3} \end{pmatrix} \\ g(x) &\triangleq \frac{1}{1-\epsilon^2\cos^2 x_3} \begin{pmatrix} 0 \\ -\epsilon\cos x_3 \\ 0 \\ 1 \end{pmatrix} \\ p(x) &\triangleq \frac{1}{1-\epsilon^2\cos^2 x_3} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\epsilon\cos x_3 \end{pmatrix} . \end{aligned} \tag{3}$$

Disturbance w_1 can be seen as generated by the following exosystem

$$\dot{w} = Sw \tag{4}$$

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with $w \triangleq (w_1 \ w_2)^T$ and

$$S \triangleq \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} . \qquad (5)$$

The goal of this work is designing a feedback controller that achieves asymptotic disturbance rejection to zero of the translational displacement; thus, in terms of output regulation, the regulated error e is given by

$$e = x_1 . \qquad (6)$$

Here it is assumed that the only magnitude available for feedback is given by the rotational position; then, the measurable output y is given by

$$y = x_3 . \qquad (7)$$

The precise formulation of the output regulation problem here considered will be given in the forthcoming section.

Note that we are dealing with a nonlinear output regulation problem in which the regulated error e is not measurable. This makes the problem depart from standard framework of output regulation (see [11] and [12]). In [14] a general certainty-equivalence result for solving output regulation problems of this type was given; in the next section the latter result is recalled, and in the following sections it will be applied to the problem at hand.

III. GENERAL FRAMEWORK FOR CERTAINTY EQUIVALENCE DESIGN

In this section the certainty-equivalence design proposed in [14] is recalled; the latter design can be useful for finding a solution to nonlinear output regulation problems with the peculiar feature that the regulated error is unmeasurable. The design procedure presented here is slightly modified with respect to [14] so that it will be easier to apply it to the problem formulated in Section II.

Consider an output regulation problem for a sufficiently smooth system

$$\begin{array}{rcl} \dot{x} & = & \bar{f}(x,u,w) \\ e & = & h(x,w) \\ y & = & k(x,w) \end{array} \qquad (8)$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}^d$ is the exogenus input, $e \in \mathbb{R}$ is the regulated error, and $y \in \mathbb{R}$ is the measured output. The initial state of (8) $x(0)$ is unknown but ranges in a known compact set $X \subseteq \mathbb{R}^n$. The exogenus input w is supposed to be generated by a sufficiently smooth exosystem

$$\dot{w} = s(w) \qquad (9)$$

whose initial state $w(0)$ is unknown but ranges in a known compact and *invariant* set $W \subseteq \mathbb{R}^d$.

The objective is finding a regulator modeled by equations of the form

$$\begin{array}{rcl} \dot{\chi} & = & \varphi(\chi, y) \\ u & = & \rho(\chi, y) \end{array} \qquad (10)$$

with φ and ρ locally Lipschitz, and a set Δ of initial states of (10), such that the interconnection of systems (8), (9), and (10) possesses the following property; all the trajectories that start from $X \times W \times \Delta$ are bounded and are such that $\lim_{t\to\infty} e(t) = 0$.

A solution to the output regulation problem formulated above may be determined through the certainty-equivalence approach described in the sequel.

Assume that a memoryless full-information regulator $u = u^*(x,w)$ is available; more precisely, make the following assumption.

Assumption 1: There exists a sufficiently smooth function $u^*(x,w)$ such that system

$$\begin{array}{rcl} \dot{x} & = & \bar{f}(x, u^*(x,w), w) \\ \dot{w} & = & s(w) \end{array} \qquad (11)$$

restricted to the locally invariant cylinder $\mathbb{R}^n \times W$ satisfies the following

- all the trajectories that start from $X \times W$ are bounded;
- let $\mathcal{A} = \omega(X \times W)$; then, $\mathcal{A} \subseteq \{(x,w) : h(x,w) = 0\}$ and $\mathcal{A}$ is locally asymptotically stable.

In additon, assume that the plant augmented with the exosystem and observed through the measured output can be trasformed into a system in Gauthier-Kupka's observability canonical form (see [17, p. 22]); the importance of such assumption lies on the fact that for such systems it is known how to design an asymptotic state observer (see [17, pp. 95-101]). More in detail, assume what follows.

Assumption 2: There exists a sufficiently smooth global diffeomorphism

$$z = \phi(x,w) \quad z \in \mathbb{R}^{\tilde{n}} \qquad (12)$$

where $\tilde{n} = n + d$ that carries system

$$\begin{array}{rcl} \dot{x} & = & \bar{f}(x,u,w) \\ \dot{w} & = & s(w) \\ y & = & k(x,w) \end{array} \qquad (13)$$

into the following Gauthier-Kupka's observability canonical form

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{\tilde{n}-1} \\ \dot{z}_{\tilde{n}} \end{pmatrix} = \begin{pmatrix} F_1(z_1,z_2,u) \\ F_2(z_1,z_2,z_3,u) \\ \vdots \\ F_{\tilde{n}-1}(z_1,z_2,\ldots,z_{\tilde{n}},u) \\ F_{\tilde{n}}(z_1,z_2,\ldots,z_{\tilde{n}},u) \end{pmatrix}$$

$$= F(z,u)$$

$$y = K(z_1) , \qquad (14)$$

with F_i's such that

$$\frac{\partial F_i}{\partial z_{i+1}}(z_1,z_2,\ldots,z_{i+1},u) \neq 0$$
$$\forall (z_1,z_2,\ldots,z_{i+1},u) \in \mathbb{R}^{i+2} \quad i = 1,\ldots,\tilde{n}-1 , \qquad (15)$$

and with K such that

$$\frac{\partial K}{\partial z_1}(z_1) \neq 0 \quad \forall z_1 \in \mathbb{R} \, . \tag{16}$$

If both Assumption 1 and Assumption 2 are fulfilled, a certainty equivalence regulator can be obtained as follows.

Let $z_{max} > 0$ and $l > 0$ be design parameters, and define

$$\Theta \triangleq \{z \in \mathbb{R}^{\tilde{n}} \mid |z| \leq z_{max}\} \tag{17}$$

and

$$U \triangleq \{u \in \mathbb{R} : |u| \leq l\} \, . \tag{18}$$

Find a sufficiently smooth function $F^{gl} : \mathbb{R}^{\tilde{n}} \times \mathbb{R} \to \mathbb{R}^{\tilde{n}}$ of the form

$$F^{gl}(z,u) = \begin{pmatrix} F_1^{gl}(z_1, z_2, u) \\ F_2^{gl}(z_1, z_2, z_3, u) \\ \vdots \\ F_{\tilde{n}-1}^{gl}(z_1, z_2, \dots, z_{\tilde{n}}, u) \\ F_{\tilde{n}}^{gl}(z_1, z_2, \dots, z_{\tilde{n}}, u) \end{pmatrix} , \tag{19}$$

and a sufficiently smooth function $K^{gl} : \mathbb{R} \to \mathbb{R}$ such that the following properties hold

P1: $F^{gl}(z,u) = F(z,u)$ and $K^{gl}(z_1) = K(z_1) \quad \forall (z,u) \in \Theta \times U$;

P2: denote by $\underline{z}_i'$ the vector $(z_1', \dots, z_i') \in \mathbb{R}^i$ and by $\underline{z}_i''$ the vector $(z_1'', \dots, z_i'') \in \mathbb{R}^i$, $i = 1, \dots, \tilde{n}$; then, $\exists L > 0$ such that

$$|F_i^{gl}(\underline{z}_i', z_{i+1}', u) - F_i^{gl}(\underline{z}_i'', z_{i+1}'', u)| \leq L|\underline{z}_i' - \underline{z}_i''| \\ \forall (\underline{z}_i', z_{i+1}', \underline{z}_i'', z_{i+1}'', u) \in \mathbb{R}^{2i+3} \quad i = 1, \dots, \tilde{n}-1 \tag{20}$$

and

$$|F_{\tilde{n}}^{gl}(\underline{z}_{\tilde{n}}', u) - F_{\tilde{n}}^{gl}(\underline{z}_{\tilde{n}}'', u)| \leq L|\underline{z}_{\tilde{n}}' - \underline{z}_{\tilde{n}}''| \\ \forall (\underline{z}_{\tilde{n}}', \underline{z}_{\tilde{n}}'', u) \in \mathbb{R}^{2\tilde{n}+1} \; ; \tag{21}$$

P3: $\exists \, \alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$ such that

$$\alpha \leq \left| \frac{\partial F_i^{gl}}{\partial z_{i+1}}(z_1, z_2, \dots, z_{i+1}, u) \right| \leq \beta \\ \forall (z_1, z_2, \dots, z_{i+1}, u) \in \mathbb{R}^{i+2} \quad i = 1, \dots, \tilde{n}-1, \tag{22}$$

and

$$\alpha \leq \left| \frac{\partial K^{gl}}{\partial z_1}(z_1) \right| \leq \beta \;\; \forall z_1 \in \mathbb{R} \, . \tag{23}$$

Regarding on how to find F^{gl} see e.g. [17, p. 96].

Let

$$\overline{a}_i(z,u) = \frac{\partial F_i^{gl}}{\partial z_{i+1}}(z_1, z_2, \dots, z_{i+1}, u) \;\; i = 1, \dots, \tilde{n}-1$$

and define

$$A(z,u) \triangleq \begin{pmatrix} 0 & \overline{a}_1(z,u) & 0 & \cdots & 0 \\ 0 & 0 & \overline{a}_2(z,u) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \overline{a}_{\tilde{n}-1}(z,u) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{24}$$

In addition, define $C(z_1) \in \mathbb{R}^{1 \times \tilde{n}}$ as

$$C(z_1) \triangleq \begin{pmatrix} \frac{\partial K^{gl}}{\partial z_1}(z_1) & 0 & \cdots & 0 \end{pmatrix} . \tag{25}$$

Determine $N \in \mathbb{R}^{\tilde{n}}$ such that the following property holds.

P4: There exist $\lambda > 0$ and $S \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, with S symmetric and positive definite that satisfy the following inequality

$$(A(z,u) - NC(z_1))^T S + S(A(z,u) - NC(z_1)) \leq -\lambda I \\ \forall (z,u) \in \mathbb{R}^{\tilde{n}} \times \mathbb{R} \, .$$

To determine N it can be useful to see [17, p. 96].

Given $s > 0$, let σ_s be a saturation function defined by

$$\sigma_s(r) = \begin{cases} r & \text{if } |r| \leq s \\ \text{sgn}(r)s & \text{if } |r| > s \, . \end{cases}$$

Denote by

$$(x,w) = \phi^{-1}(z)$$

the inverse map of (12). Then, the measurement feedback regulator is described by the following equations

$$\begin{aligned} \dot{\hat{z}} &= F^{gl}(\hat{z},u) + G(y - K^{gl}(\hat{z}_1)) \\ u &= \sigma_l(u^*(\phi^{-1}(\hat{z})) \end{aligned} \tag{26}$$

where $G = D_g N$, $D_g = \text{diag}(g, g^2, \dots, g^{\tilde{n}})$, and $g \in \mathbb{R}$ is a design parameter.

The initial state of (26) $\hat{z}(0)$ is assumed to range on a fixed (but arbitrary) compact set $\hat{Z} \subset \mathbb{R}^{\tilde{n}}$.

The effectiveness of the above regulator is shown by the following result (see [14, Proposition 1]).

Proposition 1: If first $z_{max} > 0$ is picked sufficiently large, then $l > 0$ is chosen large enough, and finally the same is done with $g > 0$, then (26) solves the given output regulation problem.

Remark 2: Regulator (26) embeds an asymptotic state observer of system (14) which is given by its $\hat{z}$ dynamics (see [17, pp. 95-101]). Then, $\phi^{-1}(\hat{z})$ provides an estimate of $(x \;\, w)$, and clearly the proposed regulator is based on certainty-equivalence.

Remark 3: From the proof of [14, Proposition 1] it can be seen that the design parameter z_{max} should be picked sufficiently large so that the following holds. Consider any solution $(z(t) \;\, \hat{z}(t))$ of (14) controlled by (26) with $(z(0) \;\, \hat{z}(0)) \in Z \times \hat{Z}$; then, it must be $|z(t)| \leq z_{max} \forall t \geq 0$.

A method to determine z_{max} follows from the proof of [14, Proposition 1] and is described here. Let Z be the image through ϕ of $X \times W$, and consider the system

$$\dot{z} = F(z, \phi^{-1}(z)) \ . \tag{27}$$

In [14] it is shown that there exists a Lyapunov level set Ω_b that contains all solutions of (27) that originate in Z; moreover, it is shown that if the design parameters l and g are large enough, then Ω_{b+1} contains the z component of all trajectories of (14)-(26) that that originate in $Z \times \hat{Z}$; consequently, it would suffice to pick the design parameter z_{max} so that $\Theta \supseteq \Omega_{b+1}$ where Θ was defined in (17); however, if it is too difficult to compute Ω_{b+1}, as it will be the case of the forthcoming application, we can try to find an appropriate value of z_{max} by trial and error using simulations. In this regard, it can be useful to determine a value z^*_{max} with the following property: all solutions $z(t)$ of (27) that originate in Z satisfies the inequality $|z(t)| \leq z^*_{max}$ $\forall t \geq 0$; in fact, a good guess would be picking $z_{max} \geq z^*_{max}$.

The proof of [14, Proposition 1] provides also some thoretical lower bounds for l and g; however, if it is too complicated to compute those bounds, as it will be the case of the forthcoming application, again we can try to find appropriate values of l and g by trial and error using simulations.

In the next sections the general design method here presented is applied to the output regulation problem formulated in Section II.

IV. FULL-INFORMATION REGULATOR

The goal of this section is designing a memoryless full-information regulator $u = u^*(x, w)$ for system (2) driven by exosystem (4) that makes Assumption 1 of Section III fulfilled.

Similarly to [1], apply the feedback

$$\begin{aligned} u \ &= \ (1 - \epsilon^2 \cos^2 x_3) v - (\epsilon x_1 \cos x_3 \\ &\quad -\epsilon^2 x_4^2 \cos x_3 \sin x_3) + \epsilon \cos(x_3) w_1 \ , \end{aligned} \tag{28}$$

where v is a resudual input, and make the following change of coordinates

$$\begin{aligned} \xi_1 &\triangleq x_1 + \epsilon \sin x_3 \\ \xi_2 &\triangleq x_2 + \epsilon x_4 \cos x_3 \\ \xi_3 &\triangleq x_3 \\ \xi_4 &\triangleq x_4 \ . \end{aligned} \tag{29}$$

Then, (2) and (6) read as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\xi_1 + \epsilon \sin \xi_3 + w_1 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= v \\ e &= \xi_1 - \epsilon \sin \xi_3 \ . \end{aligned} \tag{30}$$

A solution to the regulator equations (see [11, Equation (2.10)]) associated to (30) and (4) is given by (see [8, Section V])

$$\begin{aligned} \pi_1(w) &= -\frac{w_1}{\omega^2} \\ \pi_2(w) &= -\frac{w_2}{\omega} \\ \pi_3(w) &= -\arcsin\left(\frac{w_1}{\epsilon\omega^2}\right) \\ \pi_4(w) &= -\frac{\omega w_2}{(\epsilon^2\omega^4 - w_1^2)^{\frac{1}{2}}} \\ c(w) &= \frac{\omega^2 w_1(\epsilon^2\omega^4 - w_1^2 - w_2^2)}{(\epsilon^2\omega^4 - w_1^2)^{\frac{3}{2}}} \ . \end{aligned} \tag{31}$$

Note that π and c are defined for $|w_1| < \epsilon\omega^2$; consequently, from now on we will consider only exogenus inputs $w(t)$ such that $|w_1(t)| < \epsilon\omega^2$ $\forall t \geq 0$. This is obtained enforcing that the initial state of (4) belongs to the following compact and invariant set

$$W = \{w \in \mathbb{R}^2 \mid |w| \leq w_{max}\}$$

with

$$0 < w_{max} < \epsilon\omega^2 \ . \tag{32}$$

Next, make the additional coordinate transformation

$$\tilde{\xi}_i \triangleq \xi_i - \pi_i(w) \quad i = 1, \ldots, 4 \tag{33}$$

and apply the feedback

$$v = c(w) + \tilde{v} \tag{34}$$

where $\tilde{v}$ is an additional residual input. Then, (30) reads as

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 &= -\tilde{\xi}_1 + \epsilon \sin\left(\tilde{\xi}_3 - \arcsin\left(\frac{w_1}{\epsilon\omega^2}\right)\right) + \frac{w_1}{\omega^2} \\ \dot{\tilde{\xi}}_3 &= \tilde{\xi}_4 \\ \dot{\tilde{\xi}}_4 &= \tilde{v} \\ e &= \tilde{\xi}_1 - \frac{w_1}{\omega^2} - \epsilon \sin\left(\tilde{\xi}_3 - \arcsin\left(\frac{w_1}{\epsilon\omega^2}\right)\right) \ . \end{aligned} \tag{35}$$

Note that if we are able to design a feedback law that makes $\tilde{\xi}$ converge to zero, then e converges to zero, too. We will achieve this objective using a backstepping design which is obtained through an adaptation of the design proposed in [6] and that can be summarized as follows. Consider the new coordinates

$$\begin{aligned} \zeta_1 &\triangleq \tilde{\xi}_1 \\ \zeta_2 &\triangleq \tilde{\xi}_2 + a_1\zeta_1 \\ \zeta_3 &\triangleq \epsilon \sin\left(\tilde{\xi}_3 - \arcsin\left(\tfrac{w_1}{\epsilon\omega^2}\right)\right) + \tfrac{w_1}{\omega^2} \\ &\quad -(a_1^2\zeta_1 - (a_1 + a_2)\zeta_2) \\ \zeta_4 &\triangleq \epsilon \cos\left(\tilde{\xi}_3 - \arcsin\left(\frac{w_1}{\epsilon\omega^2}\right)\right)\left(\tilde{\xi}_4 - \frac{\omega w_2}{(\epsilon^2\omega^4 - w_1^2)^{\frac{1}{2}}}\right) \\ &\quad +\frac{w_2}{\omega} - [a_1(1 - a_1^2) + a_2]\zeta_1 \\ &\quad -[a_1(a_2 + a_1) + a_2^2 - 1]\zeta_2 \\ &\quad +(a_1 + a_2 + a_3)\zeta_3 \end{aligned} \tag{36}$$

where $a_i > 0$ $i = 1, 2, 3$ are design parameters. Then, consider the Lyapunov function $V = \frac{1}{2}(\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2)$ and enforce that $\dot{V} = -a_1\zeta_1^2 - a_2\zeta_2^2 - a_3\zeta_3^2 - a_4\zeta_3^2$ where $a_4 > 0$ is a design parameter; it follows that

$$\tilde{v} = \frac{\beta(\tilde{\xi}, w, \epsilon, a_1, a_2, a_3, a_4)}{\epsilon \cos\left(\tilde{\xi}_3 - \arcsin\left(\frac{w_1}{\epsilon\omega^2}\right)\right)}, \tag{37}$$

where β is an appropiate smooth function. Thus, a memoryless full-information feedback $u = u^*(x, w)$ is obtained. However, the latter control law is well defined only if in the resulting closed-loop system it occurs that

$$\cos\left(\tilde{\xi}_3(t) - \arcsin\left(\frac{w_1(t)}{\epsilon\omega^2}\right)\right) \neq 0 \quad \forall t \geq 0 . \tag{38}$$

Using standard arguments, it can be shown that this occurs if $(x(0), w(0))$ belongs to an appropriate set $X \times W \subset \mathbb{R}^6$ which is contained in the the region of convergence of the full-information regulator.

Remark 4: A shorter procedure for designing a full-information regulator for the TORA system that fulfills Assumption 1 is adopted in [8, Section V] and in [10]; however, as it will be discussed in Section VI, with the design carried out here it is easier to determine the parameter z^*_{max} introduced in Remark 3; in addition, the approach here followed allows us to determine an estimate $X \times W$ of the region of convergence of the full-information regulator more simply than employing the method in [8].

V. GAUTHIER-KUPKA'S OBSERVABILITY CANONICAL FORM

In the present section it will be shown that plant (2) augmented with exosystem (4) and observed through the measured output (7) can be transormed into Gauthier-Kupka's observability canonical form.

Rewrite systems (2) and (4) with output (7) as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}, u) \\ y &= \tilde{h}(\tilde{x}) \end{aligned} \tag{39}$$

where

$$\begin{aligned} \tilde{x} &= (x\ w)^T \\ \tilde{f}(\tilde{x}, u) &= (f(x) + g(x)u + p(x)w_1\ Sw)^T \\ \tilde{h}(\tilde{x}) &= x_3 . \end{aligned} \tag{40}$$

Define

$$\tilde{f}_0(\tilde{x}) = \tilde{f}(\tilde{x}, 0) \tag{41}$$

and, as in [16, p. 463], trasform the $\tilde{x}$ coordinates of (39) into

$$z_i = \phi_i(\tilde{x}) = L^{i-1}_{\tilde{f}_0}\tilde{h}(\tilde{x}) \quad i = 1, \dots, 6 . \tag{42}$$

It can be verified that (42) defines a global diffeomorphism that transforms system (39) into a system of the form

$$\begin{aligned} \dot{z} &= F(z, u) \\ y &= z_1 , \end{aligned} \tag{44}$$

with

$$\frac{\partial F_i}{\partial z_{i+1}}(z_1, z_2, \dots, z_{i+1}, u) = 1$$
$$\forall (z_1, z_2, \dots, z_{i+1}, u) \in \mathbb{R}^{i+2} \quad i = 1, \dots, 5 .$$

Then, it can be concluded that (39) is transformed through the diffeomorphism (42) into Gauthier-Kupka's observability canonical form; consequently, Assumption 2 of Section III is fulfilled.

VI. MEASUREMENT FEEDBACK REGULATOR

Since Assumption 1 and Assumption 2 are satisfied, we can proceed with the design of a measurement feedback regulator as indicated in Section III. As discussed in Remark 3 it is useful to determine a positive scalar denoted by z^*_{max} which has the following property. Given system (2) and (4) controlled by the full-information regulator derived in Section IV, consider its trajectories that originate in $X \times W$, with X and W determined in the same section; express those trajectories in the z coordinates defined by (42); then, z^*_{max} must be such that $|z(t)| \leq z^*_{max}$ $\forall t \geq 0$. It is easy to see that z^*_{max} can be determined through standard calculations using the expressions (36) of the coordinate transformations previously introduced.

Next, for designing the regulator as in Section III, given system (44), it is useful to determine functions F^{gl} and K^{gl} that satisfy properties P1, P2, and P3 of Section III. To reach this objective, it is helpful to note through appropriate computations that for system (44) the following holds

$$\frac{\partial^2 F_i}{\partial z_{i+1}\partial z_j}(z_1, z_2, \dots, z_{i+1}, u) = 0$$
$$\forall (z_1, z_2, \dots, z_{i+1}, u) \in \mathbb{R}^{i+2}$$
$$i = 1, \dots, 5 \quad j = 1, \dots, i . \tag{45}$$

Then, by (45) and [18, Lemma 2] it is easy to prove that the functions defined by

$$F^{gl}_i(z, u) = F_i(\sigma_{z_{max}}(z_1), \dots, \sigma_{z_{max}}(z_i), z_{i+1}, \sigma_l(u)) \quad i = 1, \dots, 5$$

$$\begin{aligned} F^{gl}_6(z, u) &= F_6(\sigma_{z_{max}}(z_1), \dots, \sigma_{z_{max}}(z_6), \sigma_l(u)) \\ K^{gl}(z_1) &= z_1 \end{aligned}$$

do satisfy properties P1, P2, and P3 of Section III. The corresponding matricies A and C defined in (24) and (25) are given by

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} . \end{aligned} \tag{46}$$

Then, a vector $N \in \mathbb{R}^6$ that satisfies property P4 can be determined picking any N such that $A - NC$ is Hurwitz.

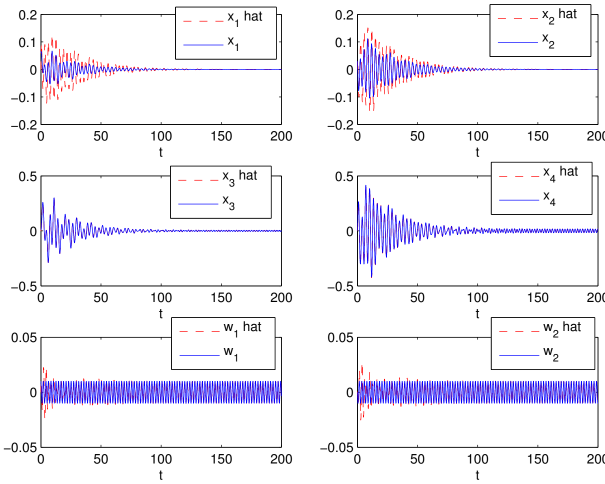

Fig. 1. Time behaviors

Finally, the measurement feedback regulator is given by (26) with initial state constrained to range on an arbitrarily fixed compact set $\hat{Z} \subset \mathbb{R}^6$, and with the parameters z_{max}, l, and g chosen as in Proposition 1. The latter controller guarantees output regulation if $(x(0)\ w(0)\ \hat{z}(0)) \in X \times W \times \hat{Z}$.

VII. SIMULATION

For simulation it has been set $\epsilon = 0.2$, $\omega = 3$, and $a_i = 0.05$ $i = 1, \ldots, 4$. The sets X and W have been determined as said in Section IV. The value $z^*_{max} = 1.08 \cdot 10^8$ has been determined following the method sketched in Section VI, and it has been set $z_{max} = z^*_{max}$. Vector N has been fixed imposing that the characteristic polynomial of $A - NC$, with A and C given in (46), is equal to the 6^{th} order Butterworth polynomial. The remaining design parameters have been selected as $l = g = 1$.

Set initial states as follows $x_1(0) = 0.06$, $x_2(0) = 0.03$, $x_2(0) = 0.009$, $x_4(0) = 0.002$, $w_1(0) = 0.01$, $w_2(0) = 0$, and $\hat{z}(0) = 0$; the corresponding time behaviors of the states of the plant, of the states of the exosystem, and of $(\hat{x}, \hat{w}) = \phi^{-1}(\hat{z})$ are plotted in Fig. 1. Note that the regulated variable x_1 is steered to zero as desired, and that the observation error $(x - \hat{x}\ w - \hat{w})$ converges to zero.

VIII. CONCLUSIONS

In the present work a feedback controller that solves an output regulation problem for the nonlinear benchmark system known as TORA is designed. The novelty of the controller with respect to previous works is that output regulation is achieved using only measurements of the rotational position. The design is carried out following a certainty-equivalence approach.

The compensator presented here leave open the question of robustness with respect to uncertainties in the plant and in the exosystem.

IX. ACKNOWLEDGMENTS

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REFERENCES

[1] C.-J. Wan, D. S. Bernstein, and V. T. Coppola, "Global stabilization of the oscillating eccentric rotor," in *Proceedings of the 33rd IEEE Conference on Decision and Control*, vol. 4, 1994, pp. 4024–4029.
[2] R. T. Bupp, D. S. Bernstein, and V. T. Coppola, "A benchmark problem for nonlinear control design: problem statement, experimental testbed, and passive nonlinear compensation," in *Proceedings of the 1995 American Control Conference*, vol. 6, 1995, pp. 4363–4367.
[3] M. Jankovic, D. Fontaine, and P. V. Kokotovic, "TORA example: cascade- and passivity-based control designs," *IEEE Transactions on Control Systems Technology*, vol. 4, no. 3, pp. 292–297, 1996.
[4] G. Escobar, R. Ortega, and H. Sira-Ramirez, "Output-feedback global stabilization of a nonlinear benchmark system using a saturated passivity-based controller," *IEEE Transactions on Control Systems Technology*, vol. 7, no. 2, pp. 289–293, 1999.
[5] D. Karagiannis, Z. P. Jiang, R. Ortega, and A. Astolfi, "Output-feedback stabilization of a class of uncertain non-minimum-phase nonlinear systems," *Automatica*, vol. 41, no. 9, pp. 1609–1615, 2005.
[6] J. Zhao and I. Kanellakopoulos, "Flexible backstepping design for tracking and disturbance attenuation," *International Journal of Robust and Nonlinear Control*, vol. 8, no. 4-5, pp. 331–348, 1998.
[7] Z. P. Jiang and I. Kanellakopoulos, "Global output-feedback tracking for a benchmark nonlinear system," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 1023–1027, 2000.
[8] A. Pavlov, N. van de Wouw, and H. Nijmeijer, "The local output regulation problem: convergence region estimates," *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 814–819, 2004.
[9] J. Huang and G. Hu, "Control design for the nonlinear benchmark problem via the output regulation method," *Journal of Control Theory and Applications*, vol. 2, no. 1, pp. 11–19, 2004.
[10] A. Pavlov, B. Janssen, N. van de Wouw, and H. Nijmeijer, "Experimental output regulation for a nonlinear benchmark system," *IEEE Transactions on Control Systems Technology*, vol. 15, no. 4, pp. 786–793, 2007.
[11] C. I. Byrnes, F. Delli Priscoli, and A. Isidori, *Output Regulation of Uncertain Nonlinear Systems*. Birkhauser, 1997.
[12] J. Huang, *Nonlinear Output Regulation: Theory and Applications*. SIAM, 2004.
[13] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: a Convergent Dynamics Approach*. Birkhauser, 2006.
[14] F. Celani, "Certainty equivalence in nonlinear output regulation with unmeasurable regulated error," in *Proceedings of the 17th IFAC World Congress*, Seoul, South Korea, 2008, pp. 12 147–12 152.
[15] J. Hale, L. Magalhaes, and W. Oliva, *Dynamics in Infinite Dimensions*. New York: Springer Verlag, 2002.
[16] A. Isidori, *Nonlinear Control Systems*. Springer-Verlag, 1995.
[17] J. Gauthier and I. Kupka, *Deterministic Observation Theory and Applications*. Cambridge University Press, 2001.
[18] H. Shim, Y. Son, and J. Seo, "Semi-global observer for multi-output nonlinear systems," *Systems and Control Letters*, vol. 42, no. 3, pp. 233–44, 2001.