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Tracking and Set-Point VFO Control for an Articulated Mobile Robot with On-Axle Hitched Trailer

Maciej Michałek and Krzysztof Kozłowski

***Abstract*— The paper presents unified framework of feedback motion control design with the Vector Field(s) Orientation (VFO) method for the articulated tractor-semitrailer vehicle in a context of trajectory tracking as well as set-point control problems. The work is an extension of the VFO control method for the nonholonomic systems with higher-dimensional state vector. Obtained control algorithm has an intuitive geometrical interpretation of particular control components yielding smooth and *natural* vehicle motion quality together with the simple controller parametric synthesis. Theoretical considerations are confirmed by simulation results obtained for the tasks of backward pushing a trailer and parallel parking maneuvers.**

I. Introduction

The paper addresses the problem of motion control design for the articulated mobile vehicle with one on-axle hitched trailer depicted in Fig. 1, which was an object of investigation for example in [5], [4]. The Vector Field(s) Orientation (VFO) control approach presented in the sequel allows, in contrast to other solutions, to treat in a unified manner two basic control tasks like trajectory tracking and set-point regulation, which can be realized either in a forward or in a backward motion strategy. VFO control design concept (applied for the first time to a unicycle mobile robot and described in [2]) originates from simple geometrical interpretations and model decomposition related to the kinematics of the considered vehicle. VFO approach gives a simple control solution characterized by smooth transient behavior of a controlled vehicle and intuitive interpretation of particular control components with very simple practical tuning of design parameters. Proposed control concept relies on treating the trailer body as the unicycle system with fictitious inputs designed with the VFO methodology, which are expected next to be realized by physical control inputs of the differentially-driven tractor. Presented approach appears to be an extension of the VFO method for the higher-dimensional nonholonomic system.

A. Vehicle model

Let us consider the articulated vehicle presented in Fig. 1, where the unicycle-like tractor is driven by two control inputs: the angular velocity u_1 and the longitudinal (positive or negative) velocity u_2. The passive trailer is hitched on the tractor wheel-axle at the distance L from the guidance point P. The characteristic point $P = (x, y)$ of the trailer will attract our special attention in relation to the considered

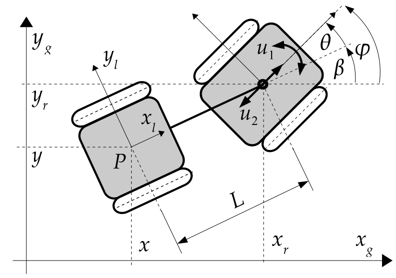

Fig. 1. Articulated vehicle with on-axle hitched trailer

control tasks. If one defines, according to Fig. 1, the vehicle configuration vector as $\boldsymbol{q} \triangleq [\theta\ \beta\ x\ y]^T$ the kinematic model describing the vehicle can be formulated as follows:

$$\begin{bmatrix}\dot{\theta}\\ \dot{\beta}\\ \dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} u_1 + \begin{bmatrix}-(1/L)\sin\theta\\ (1/L)\sin\theta\\ \cos\beta\cos\theta\\ \sin\beta\cos\theta\end{bmatrix} u_2 \tag{1}$$

where $x = x_r - L\cos\beta$, $y = y_r - L\sin\beta$, $L > 0$ is a known kinematic parameter and

$$\boldsymbol{q}^* \triangleq [\beta\ x\ y]^T \in \mathbb{R}\times\mathbb{R}^2, \qquad \theta \in [-\Theta, +\Theta], \quad \Theta \leqslant \frac{\pi}{2}. \tag{2}$$

II. Control problem statement

Let us define two types of the reference trajectory $\boldsymbol{q}_t$:

T1. the admissible reference trajectory $\boldsymbol{q}_t(\tau) \triangleq [\theta_t(\tau)\ \beta_t(\tau)\ x_t(\tau)\ y_t(\tau)]^T \in \mathcal{Q}$, $\mathcal{Q} = [-\Theta, \Theta]\times\mathbb{R}\times\mathbb{R}^2$, which is sufficiently smooth: $\dot{\boldsymbol{q}}_t(\tau), \ddot{\boldsymbol{q}}_t(\tau) \in \mathcal{L}_\infty$ and for all $\tau \geqslant 0$ satisfies the vehicle kinematics (1) for some reference inputs $u_{1t}(\tau) \in \mathbb{R}$ and $u_{2t}(\tau)$ such that:
$$\forall_{\tau\geqslant 0}\ u_{2t}(\tau) \neq 0, \tag{3}$$

T2. the constant (degenerated) reference trajectory $\boldsymbol{q}_t(\tau) \equiv \boldsymbol{q}_t \triangleq [\theta_t\ \beta_t\ x_t\ y_t]^T \in \mathcal{Q}$, with $\theta_t \triangleq 0$.

Condition (3) describes the so-called *persistency-of-excitation* condition strictly related to the VFO method, which involves persistent longitudinal motion of the reference tractor during the whole control time-horizon.

Introducing now the posture error:

$$\boldsymbol{e}(\tau) = \begin{bmatrix}e_\theta(\tau)\\ e_\beta(\tau)\\ e_x(\tau)\\ e_y(\tau)\end{bmatrix} \triangleq \begin{bmatrix}\theta_t(\tau) - \theta(\tau)\\ f_s\left(\beta_t(\tau) - \beta(\tau)\right)\\ x_t(\tau) - x(\tau)\\ y_t(\tau) - y(\tau)\end{bmatrix} \in [-\pi,\pi]^2\times\mathbb{R}^2, \tag{4}$$

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Both authors are with Chair of Control and Systems Engineering, Poznan University of Technology, Piotrowo 3A, 60-965 Poznań, Poland {maciej.michalek/krzysztof.kozlowski}@put.poznan.pl

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where $f_s(\cdot) : \mathbb{R} \mapsto [-\pi, \pi]$, one can state the control problem as follows.

Problem 1: For the reference trajectories T1 and T2 find a bounded feedback control law $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{q}_t, \boldsymbol{q}, \cdot)$, which applied to the kinematics (1) ensures boundedness and convergence of the posture error $\boldsymbol{e}(\tau)$ in the sense that:

$$\lim_{\tau\to\infty} \| \boldsymbol{e}(\tau)\| \leqslant \epsilon, \tag{5}$$

where $\epsilon \geqslant 0$ defines an assumed arbitrary small ultimate error envelope.

Taking $\epsilon = 0$ gives asymptotic convergence but $\epsilon > 0$ implies practical convergence (ultimate boundedness).

We would like to stress, that defining the vehicle model (1) in relation to the guidance point P connected to the trailer rather than to the tractor, the control tasks of tracking and set-point regulation are related mainly to the trailer body. The tractor body is considered only indirectly by the relative hitching angle θ. However, since we formulated the control problem for the whole state vector $\boldsymbol{q}$ of the articulated vehicle, the feedback controllers proposed in the next sections should ensure boundedness and convergence for the hitching angle too.

The rest of the paper will be devoted to solution of Problem 1 by derivation in a common framework of two separate VFO controllers for trajectories T1 and T2, respectively.

III. Control design approach

First, we will rewrite model (1) in a more suitable form in relation to the VFO method. In order to do this we introduce the auxiliary inputs:

$$w_1 := u_2(1/L)\sin\theta, \tag{6}$$

$$w_2 := u_2 \cos\theta, \tag{7}$$

$$w_3 := u_1 - u_2(1/L)\sin\theta, \tag{8}$$

which allow one to rewrite (1) in the new decomposed form of two subsystems:

$$\Sigma_\theta : \quad \dot{\theta} = w_3, \tag{9}$$

$$\overline{\Sigma} : \quad \begin{bmatrix} \dot{\beta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ \cos\beta \\ \sin\beta \end{bmatrix} w_2. \tag{10}$$

Although subsystems Σ_θ and $\overline{\Sigma}$ are not independent (they are related to each other by (6)-(8)), we can temporarily treat them in this manner for control design purposes. Now we are ready to explain the control design concept utilizing the special structure of (9)-(10).

Subsystem Σ_θ is very simple and it formulates the first-order dynamics of the hitching angle θ. This variable can be treated as an *additional variable* since it does not contribute to the trailer posture but only describes the relative angular displacement between the tractor and the trailer. Further, the subsystem $\overline{\Sigma}$ describes the trailer by the well known unicycle model with new inputs w_1 and w_2 as denoted in Fig. 2. From now on we assume that we can freely design these inputs. In next two sections we will show that they can be designed in a special unified manner according to the VFO control methodology, which turns out to be very efficient for the unicycle kinematics[1] (see [2]). Next design step answers how to realize physically previously designed inputs w_1 and w_2. It can be done using definitions (6)-(8) as follows. According to (6) and (7) – compare also Fig. 2 – one can find that w_1 and w_2 result from orthogonal projections of u_2 input in relation to the hitching angle θ. Hence, to realize w_1 and w_2 we have to compute the desired hitching angle θ and appropriately define the longitudinal velocity u_2. From (6) and (7) we first define the auxiliary (desired) hitching angle as follows:

$$\theta_a \triangleq \arctan\left(\frac{Lw_1}{w_2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{11}$$

which generally can not be realized immediately due to the integral relation in (9). Therefore we have to define properly the input signal w_3 from (8), which will make the auxiliary hitching error

$$e_{\theta a} \triangleq (\theta_a - \theta) \in [-\pi, \pi] \tag{12}$$

converge to zero at least at the limit for $\tau \to \infty$. We propose to take:

$$w_3 \triangleq k_\theta e_{\theta a} + \dot{\theta}_a, \qquad k_\theta > 0, \tag{13}$$

where $\dot{\theta}_a$ is a time derivative of the auxiliary hitching angle θ_a, and k_θ is a design parameter. We would like to emphasize that (13) is not the only possibility. One can propose here any other *auxiliary controller* formula which applied to (9) ensures the asymptotic convergence of $e_{\theta a}$ to zero.

In the last design step we have to formulate the equations, which allow to compute physically realizable control inputs u_2 and u_1. Using (6)-(7) we propose to take:

$$u_2 \triangleq \sigma\frac{w_2}{\cos\theta} + (1-\sigma)\frac{Lw_1}{\sin\theta} \tag{14}$$

with

$$\sigma \triangleq \begin{cases} 1 & \text{for} \quad |\theta| \in [0, \frac{\pi}{4}) \\ 0 & \text{for} \quad |\theta| \in [\frac{\pi}{4}, \Theta] \end{cases}. \tag{15}$$

Note, that the switching strategy defined by (15) does not cause any discontinuity in (14), since $w_2/\cos\theta = Lw_1/\sin\theta$ for $|\theta| = \frac{\pi}{4}$ (compare Fig. 2). Now from (8) one can directly obtain:

$$u_1 = w_3 + u_2(1/L)\sin\theta, \tag{16}$$

where w_3 and u_2 results from (13) and (14) ,respectively.

At this moment we have to return to the issue concerning the design of auxiliary inputs w_1 and w_2 for subsystem (10). Since (10) represents the unicycle model, we can directly apply to it the VFO methodology described in [2]. Due to space limitations we will not derive the VFO control algorithm in details but only give all important formulas with few explanatory comments.

[1]The unicycle is an archetypical nonholonomic system example in the sense of the VFO method utilization.

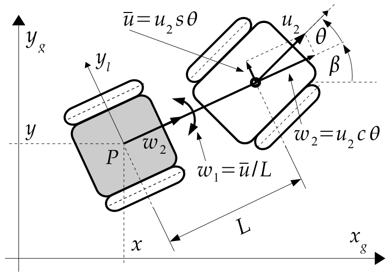

Fig. 2. Trailer treated as the unicycle with inputs w_1, w_2

A. *VFO control strategy for the trailer model* $\overline{\Sigma}$

The VFO method originates from decomposition of the subsystem (10) into two simpler systems:

$$\Sigma_\beta : \quad \dot{\beta} = w_1, \tag{17}$$

$$\Sigma^* : \quad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix} w_2 \quad \Rightarrow \quad \dot{\boldsymbol{q}}^* = \boldsymbol{g}_2^*(\beta) w_2 \tag{18}$$

with $\boldsymbol{q}^* = [x\ y]^T \in \mathbb{R}^2$ being the position sub-vector of the state $\overline{\boldsymbol{q}} = [\beta\ x\ y]^T \in \mathbb{R}\times\mathbb{R}^2$. It can be seen from (18) that direction of velocity $\dot{\boldsymbol{q}}^*$ is related to the direction of $\boldsymbol{g}_2^*(\beta)$ (its orientation depends also on $\mathrm{sgn}(w_2)$), which in turn can be freely controlled by the input w_1 since it drives the first state variable β (eq. (17)). Moreover, the second input w_2 drives (or in other words it *pushes*) the sub-state $\boldsymbol{q}^*$ along the current direction of $\boldsymbol{g}_2^*(\beta)$. According to these simple geometrical interpretations we can imagine the following control strategy. Let us assume that in every state point $\overline{\boldsymbol{q}}$ we can determine the vector $\boldsymbol{h}(\overline{\boldsymbol{q}}, \overline{\boldsymbol{q}}_t, \cdot) = [h_\beta\ \boldsymbol{h}^{*T}]^T \in \mathbb{R}^3$ defining the convergence direction (and orientation) to the reference trajectory $\overline{\boldsymbol{q}}_t = [\beta_t\ x_t\ y_t]^T$ (the set of such vectors for all $\overline{\boldsymbol{q}} \in \mathbb{R}\times\mathbb{R}^2$ can be called as the *convergence vector field*). Using previous interpretations we can propose partitioning of the control process into the *orienting* one and the *pushing* one. The first subprocess is responsible for reorientation of $\boldsymbol{g}_2^*(\beta)$ (by input w_1) to put its direction on the direction defined by $\boldsymbol{h}^*$, while the second one should push the sub-state (with input w_2) toward the reference position trajectory $\boldsymbol{q}_t^*$. Appropriate definition of $\boldsymbol{h}$, which takes into account the nonholonomic nature and the differential flatness property (see [7]) of (10) should imply, apart from smooth and non-oscillatory convergence $\boldsymbol{q}^* \to \boldsymbol{q}_t^*$, also the ultimate asymptotic convergence for β state variable, which temporarily is allowed to diverge from β_t due to its auxiliary (but crucial) role which it plays during the orienting control subprocess. Let us thus introduce the *convergence vector field* as follows:

$$\boldsymbol{h} = \begin{bmatrix}h_\beta\\ h_x\\ h_y\end{bmatrix} = \begin{bmatrix}h_\beta\\ \boldsymbol{h}^*\end{bmatrix} \in \mathbb{R}^3, \tag{19}$$

where, according to the VFO strategy, we propose to take:

$$h_\beta \triangleq k_\beta e_{\beta a} + \dot{\beta}_a = k_\beta(\beta_a - \beta) + \dot{\beta}_a \tag{20}$$

$$\boldsymbol{h}^* \triangleq k_p \boldsymbol{e}^* + \boldsymbol{v}^* = k_p(\boldsymbol{q}_t^* - \boldsymbol{q}^*) + \boldsymbol{v}^* \tag{21}$$

with $\boldsymbol{v}^* \in \mathbb{R}^2$ being the feed-forward velocity term described in the sequel, $\boldsymbol{q}_t^* = [x_t\ y_t]^T$, $\boldsymbol{q}^* = [x\ y]^T$, $k_\beta, k_p > 0$ being two design parameters and respectively:

$$e_{\beta a} \triangleq (\beta_a - \beta) \in \mathbb{R}, \tag{22}$$

$$\beta_a \triangleq \mathrm{Atan2c}\left(\mathrm{sgn}(f) h_y, \mathrm{sgn}(f) h_x\right) \in \mathbb{R}, \tag{23}$$

$$\dot{\beta}_a = \frac{\dot{h}_y h_x - h_y \dot{h}_x}{h_x^2 + h_y^2} \quad \text{for} \quad h_x^2 + h_y^2 \neq 0, \tag{24}$$

where f is a nonzero scalar function (designed later), and $\mathrm{Atan2c}(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ is a continuous version of the discontinuous function $\mathrm{Atan2}(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \mapsto [-\pi, \pi)$. Note, that h_β component of $\boldsymbol{h}$ is designed in (20) with relation to the auxiliary variable β_a, not to the reference one β_t. Particular definitions for terms $\boldsymbol{v}^*$ and f in (21) and (23) related to the flatness property of $\overline{\Sigma}$ (see next sections) will guarantee however that the following important relation is met (at least in $\mathbb{S}^1$): $\lim_{\boldsymbol{e}^* \to \boldsymbol{0}} [\beta_a(\boldsymbol{h}^*(\boldsymbol{e}^*, \cdot), \cdot) - \beta_t] = 0$. It will preserve asymptotic convergence for the orientation error $e_\beta = (\beta_t - \beta)$ to zero in the neighborhood of $\boldsymbol{q}_t^*$. Utilizing introduced formulas one can propose the VFO control law for the unicycle kinematics (10) as follows:

$$w_1 \triangleq h_\beta \quad \Rightarrow \quad w_1 = k_\beta(\beta_a - \beta) + \dot{\beta}_a, \tag{25}$$

$$w_2 \triangleq \boldsymbol{g}_2^{*T}(\beta)\boldsymbol{h}^* \quad \Rightarrow \quad w_2 = h_x \cos\beta + h_y \sin\beta, \tag{26}$$

where (25) denotes the *orienting control input* and (26) the *pushing control input*. It is evident that (25) applied into (17) gives exponential convergence of β to β_a during the orienting subprocess. Definition (26) expresses the *cautious pushing* heuristic concept, where the pushing intensity (understood as $|w_2|$) is proportional to orthogonal projection of $\boldsymbol{h}^*$ onto the current direction of vector $\boldsymbol{g}_2^*(\beta)$. More details concerning the VFO control method for the unicycle kinematics can be found in [2].

It is important to note, that VFO control law (25)-(26) with formulas (20)-(24) are applicable without changes in a unified framework to both control tasks: to trajectory tracking as well as to set-point regulation. The only differences come from particular definitions of two terms: the velocity component $\boldsymbol{v}^*$ in (21) and the function f in (23). In the next sections we introduce both definitions and comment the particular choices.

IV. VFO TRACKING CONTROL

Since in the trajectory tracking case $\dot{\overline{\boldsymbol{q}}}_t = [\dot{\beta}_t\ \dot{x}_t\ \dot{y}_t]^T \not\equiv \boldsymbol{0}$ we propose to choose the term $\boldsymbol{v}^*$ from (21) and function f from (23) as follows:

$$\boldsymbol{v}^*(\tau) \triangleq \dot{\boldsymbol{q}}_t^*(\tau), \qquad \text{where} \qquad \dot{\boldsymbol{q}}_t^*(\tau) = \begin{bmatrix}\dot{x}_t(\tau)\\ \dot{y}_t(\tau)\end{bmatrix}, \tag{27}$$

$$f(\tau) \triangleq w_{2t}(\tau) \quad \overset{(7)}{\Longrightarrow} \quad f(\tau) \triangleq u_{2t}(\tau)\cos\theta_t(\tau), \tag{28}$$

where $w_{2t}(\tau)$ is a reference pushing input for kinematics (18) and $u_{2t}(\tau)$ is a real reference input of the controlled vehicle (1). Definition (27) has an interpretation of the feed-forward component in the linear combination (21), which together with (28) ensures that at the limit for $\boldsymbol{e}^* \to \mathbf{0}$ holds $\beta_a|_{\boldsymbol{e}^*=\mathbf{0}} = \text{Atan2c}\left(\text{sgn}(w_{2t})\dot{y}_t, \text{sgn}(w_{2t})\dot{x}_t\right)$, hence

$$\beta_a|_{\boldsymbol{e}^*=\mathbf{0}} \mod 2\pi = \beta_t \tag{29}$$

and

$$\dot{\beta}_a\Big|_{\boldsymbol{e}^*=\mathbf{0}} = \dot{\beta}_t = w_{1t} \overset{(6)}{=} u_{2t}(1/L)\sin\theta_t. \tag{30}$$

Note, that $\text{sgn}(w_{2t}) \equiv \text{sgn}(u_{2t}\cos\theta_t)$ is constant and equal to $+1$ or -1 in the whole control time-horizon due to assumptions (3) and (2). Moreover, using (28) in (23) guarantees that the controlled vehicle (1) will move with proper motion strategy (forward/backward) not only on the reference trajectory but also during a transient stage. Having determined all needed components we are ready to formulate the first proposition.

Proposition 1: (VFO tracking controller) *Let us define the set in the position error space:* $\mathcal{E}^* \triangleq \mathbb{R}^2 \setminus \{\boldsymbol{e}^* : \boldsymbol{e}^* = -k_p^{-1}\dot{\boldsymbol{q}}_t^*\}$. *Assuming that*

$$\forall_{\tau\geqslant 0}\boldsymbol{e}^*(\tau) \in \mathcal{E}^* \quad \overset{(21,27)}{\Longrightarrow} \quad \forall_{\tau\geqslant 0}\,\|\,\boldsymbol{h}^*\| \neq 0 \tag{31}$$

the VFO feedback control law defined by (14) and (16) with auxiliary inputs defined by (25), (26), and (13) together with (20)-(24) and (27)-(28) applied to the articulated vehicle (1) solves the Problem 1 with $\epsilon = 0$ *for a given reference trajectory* $\boldsymbol{q}_t(\tau)$ *of T1 type.*

Proof: (Sketch) The proof will consist of the five steps: S1 to S5. In step S1 it can be seen that applying (16) with (13) into (1) gives the exponential convergence of $e_{\theta a}(\tau)$ to zero as $\tau \to \infty$. Using (11) one can write the following useful relation:

$$\lim_{\theta\to\theta_a}\left[\tan\theta - \frac{Lw_1}{w_2}\right] = 0, \tag{32}$$

which will be utilized next. In step S2 we show the convergence of $e_{\beta a}(\tau)$. To do this let us combine the second equation of (1) with (14) to write, that for $\theta \to \theta_a$ (according to S1) we have:

$$\dot{\beta}\Big|_{\theta=\theta_a} = \begin{cases} (1/L)\sin\theta_a(w_2/\cos\theta_a) & \text{for} \quad |\theta_a| \in [0, \frac{\pi}{4}) \\ (1/L)\sin\theta_a(Lw_1/\sin\theta_a) & \text{for} \quad |\theta_a| \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases},$$

which with (32) gives $\dot{\beta} = w_1 \overset{(25)}{=} k_\beta e_{\beta a} + \dot{\beta}_a$ for all $|\theta_a| \in [0, \frac{\pi}{2}]$. As a consequence $e_{\beta a}(\tau) \to 0$ for $\tau \to \infty$. Step S3 relates to behavior of position error $\boldsymbol{e}^* = \boldsymbol{q}_t^* - \boldsymbol{q}^*$. It can be simply shown that substituting (14) into (1) implies for $\theta \to \theta_a$ that $\dot{x} = \cos\beta w_2$ and $\dot{y} = \sin\beta w_2$ with w_2 defined in (26). Hence we can consider now the subsystem (18) with the fictitious input w_2 from (26). Since $\dot{\boldsymbol{e}}^* = \dot{\boldsymbol{q}}_t^* - \dot{\boldsymbol{q}}^*$ and from (21) and (27) we have $\dot{\boldsymbol{q}}_t^* = \boldsymbol{h}^* - k_p\boldsymbol{e}^*$, one can write the auxiliary equation of perturbed position-error dynamics as follows:

$$\dot{\boldsymbol{e}}^* = -k_p\boldsymbol{e}^* + \boldsymbol{r}(\boldsymbol{e}^*, e_{\beta a}, \cdot), \tag{33}$$

where (according to (18) and (26))

$$\boldsymbol{r}(\boldsymbol{e}^*, e_{\beta a}, \cdot) = \boldsymbol{h}^*(\boldsymbol{e}^*, \cdot) - \boldsymbol{g}_2^*(\beta)\,\|\,\boldsymbol{h}^*(\boldsymbol{e}^*, \cdot)\|\cos\alpha(e_{\beta a}), \tag{34}$$

with: $\alpha(e_{\beta a}) = e_{\beta a} \pm \kappa\pi$ ($\kappa = 0$ for $w_{2t} > 0$ $\kappa = 1$ for $w_{2t} < 0$) and $\|\,\boldsymbol{r}\|^2 = \|\,\boldsymbol{h}^*\|^2(1 - \cos^2 e_{\beta a})$. It is evident that $\lim_{e_{\beta a}\to 0}\|\,\boldsymbol{r}(e_{\beta a})\| = 0$ and due to corollary from step S2 we have: $\lim_{\tau\to\infty}\|\,\boldsymbol{r}(\tau)\| = 0$. Using now the lemmas from [3] related to the stability of perturbed systems (pages 350-355) one can conclude about boundedness and asymptotic convergence of $\boldsymbol{e}^*(\tau)$ to zero for $\tau \to \infty$. In step S4 we utilize (29) and corollary from step S2 to conclude that $\lim_{\tau\to\infty} e_\beta(\tau) = 0$. The last step S5 relates to behavior of the hitching angle error $e_\theta(\tau)$. Using (11) with w_1 and w_2 from (25) and (26) expressed for $e_{\beta a} = 0$ (due to S2), $\boldsymbol{e}^* = \mathbf{0}$ (due to S3), $\beta_a \mod 2\pi = \beta_t$ and $\dot{\beta}_a = \dot{\beta}_t$ (due to (29) and (30)) we obtain:

$$\theta_a = \arctan\left(\frac{L\dot{\beta}_t}{\dot{x}_t c\beta_t + \dot{y}_t s\beta_t}\right) \overset{(1)}{=} \arctan\left(\frac{L\frac{1}{L}s\theta_t u_{2t}}{u_{2t}c\theta_t}\right) = \\ = \arctan(\tan\theta_t) = \theta_t.$$

where we used the short notation $c\theta \equiv \cos\theta$ and $s\theta \equiv \sin\theta$. Above expression together with the convergence result from step S1 allow to conclude that $\lim_{\tau\to\infty} e_\theta(\tau) = 0$. ■

Remark 1: Fictitious input (25) and, as a consequence, real inputs (14) and (16) generally have discontinuous nature. It results from (23), which is not determined for $\boldsymbol{h}^* = \mathbf{0}$ *(justification of assumption (31)). Equality* $\boldsymbol{h}^* = \mathbf{0}$ *relates to the case when* $\boldsymbol{e}^* = -k_p^{-1}\dot{\boldsymbol{q}}_t^*$. *In geometrical interpretation it requires from the trailer of motion exactly along the same direction as the reference trailer but with the opposite orientation. As a consequence, the controlled system naturally converges to the reference one (such a conclusion also results from (34) and (33): since* $\boldsymbol{h}^* = \mathbf{0}$ *we have* $\boldsymbol{r} = \mathbf{0}$ *and by (33) it follows that exponential convergence of* $\boldsymbol{e}^*$ *holds). Therefore determination of auxiliary variable* β_a *and its time-derivative for* $\boldsymbol{h}^* = \mathbf{0}$ *requires our attention. To obtain a well-defined input (25) for all* $\boldsymbol{e}^* \in \mathbb{R}^2$ *we propose to introduce additional definitions valid in a small* ε*-vicinity of point* $\boldsymbol{h}^* = \mathbf{0}$*:*

$$\beta_a \triangleq \beta_a(\tau_-), \qquad \dot{\beta}_a \triangleq 0 \quad \textit{for} \quad \|\,\boldsymbol{h}^*\| \leqslant \varepsilon, \tag{35}$$

where $0 < \varepsilon < \inf_\tau |u_{2t}(\tau)\cos\theta_t(\tau)|$ *and* τ_- *denotes the time instant when the system reaches* ε*-vicinity. Above definitions allows the general form of the input (25) to remain unchanged. Note that the considered case when* $\boldsymbol{h}^* = \mathbf{0}$ *may occure only during a transient stage and is non-persistent and unlikely in practice.*

Remark 2: The time-derivative $\dot{\theta}_a$ *involved in definition (13) results from formal differentiation of (11) as follows:* $\dot{\theta}_a = L(\dot{w}_1 w_2 - w_1\dot{w}_2)/(w_2^2 + L^2 w_1^2)$. *However, due to the second time-derivatives* $\ddot{x}$ *and* $\ddot{y}$ *appearing in the final form of* $\dot{w}_1$, *practical implementation of above formula seems to be problematic. Instead, we propose to utilize the exact robust differentiator described in [6], which guarantees finite-time convergence of the estimate* $\hat{\dot{\theta}}_a$ *to* $\dot{\theta}_a$ *using* θ_a *from (11) as an input signal.*

V. VFO SET-POINT CONTROL

In the case of the set-point regulation we can not use (27) since $\dot{\boldsymbol{q}}_t^* \equiv \mathbf{0}$. But if we note that $\dot{\boldsymbol{q}}_t^* = \boldsymbol{g}_2^*(\beta_t)u_{2t}\cos\theta_t$ for $u_{2t} \equiv 0$, one can utilize the nonzero vector field $\boldsymbol{g}_2^*(\beta_t)$, which defines the reference direction and orientation of the trailer at the set-point $\boldsymbol{q}_t^* = [x_t \; y_t]^T$. Let us utilize it and introduce the so-called *virtual reference velocity* $\dot{\boldsymbol{q}}_{vt}^*$ and by analogy to (27) write:

$$\boldsymbol{v}^*(\tau) \triangleq \dot{\boldsymbol{q}}_{vt}^*(\tau), \quad \text{where} \quad \dot{\boldsymbol{q}}_{vt}^*(\tau) \triangleq \delta(\boldsymbol{e}^*(\tau))\boldsymbol{g}_2^*(\beta_t) \quad (36)$$

with

$$\delta(\boldsymbol{e}^*(\tau)) \triangleq -\eta\,\mathrm{sgn}(f)\,\|\,\boldsymbol{e}^*(\tau)\|\,, \quad f \triangleq e_{x0} = e_x(0), \quad (37)$$

where $0 < \eta < k_p$ is an additional design parameter. Although, the virtual reference velocity vanishes for $\boldsymbol{e}^* \to \mathbf{0}$ but it turns out to be very helpful during the transient stage playing the role of directing the trailer near the reference point $\boldsymbol{q}_t^*$. Since the function f is now equal to the initial value of $e_x(\tau)$ component, the sign of f is constant during the whole control time-horizon. Initial value of e_x determines the motion strategy (forward/backward), in which the vehicle will reach the set point. It is worth to note, that f can be treated here as a *decision variable* which can be freely designed according to practical needs of motion strategy selection. Now, we can formulate the second proposition.

Proposition 2: Let us define the set in the positional error space: $\mathcal{E}^* \triangleq \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Assuming that

$$\forall_{0\leqslant\tau<\infty}\boldsymbol{e}^*(\tau) \in \mathcal{E}^* \quad \overset{(21,36,37)}{\Longrightarrow} \quad \forall_{0\leqslant\tau<\infty}\,\|\,\boldsymbol{h}^*\| \neq 0 \quad (38)$$

the VFO feedback control law defined by (14) and (16) with auxiliary inputs defined by (25), (26), and (13) together with (20)-(24) and (36)-(37) applied to the articulated vehicle (1) solves the Problem 1 with $\epsilon = 0$ for a given reference (degenerated) trajectory $\boldsymbol{q}_t$ of T2 type.

Proof: (Sketch) The proof will be presented by analogy to the proof of Proposition 1. First two steps S1-S2 are the same with the same convergence conclusions for $e_{\theta a}$ and $e_{\beta a}$ errors. In step S3 one can (similarly to considerations in the former proof) derive the following auxiliary equation of positional error dynamics:

$$\dot{\boldsymbol{e}}^* = -k_p\boldsymbol{e}^* + \boldsymbol{r}(\boldsymbol{e}^*, e_{\beta a}, \cdot) - \dot{\boldsymbol{q}}_{vt}^* \quad (39)$$

where disturbing term $\boldsymbol{r}$ is defined as in (34) and $\|\,\boldsymbol{r}\|^2 = \|\,\boldsymbol{h}^*\|^2\,(1 - \cos^2 e_{\beta a})$. Now defining the positive definite function $V \triangleq (1/2)\boldsymbol{e}^{*T}\boldsymbol{e}^*$ one can show that its time-derivative along the solution of (39) satisfies:

$$\dot{V} \leqslant -[k_p(1-\gamma) - \eta(1+\gamma)]\,\|\,\boldsymbol{e}^*\|^2 = -\zeta(\gamma)\,\|\,\boldsymbol{e}^*\|^2\,, \quad (40)$$

where $\gamma = \gamma(e_{\beta a}) := \sqrt{1-\cos^2 e_{\beta a}}$ and the function $\zeta(\gamma)$ is positive definite if $\gamma(e_{\beta a}) < (k_p - \eta)/(k_p + \eta)$. Since $0 < \eta < k_p$ (from assumption), $\gamma(e_{\beta a}) \in [0, 1]$ and $e_{\beta a}(\tau) \to 0$ (according to S2) the last inequality is met for all $\tau \geqslant \tau_\gamma$, where τ_γ is a some finite time instant, which always exists. As a consequence, one can conclude about boundedness and asymptotic convergence of $\boldsymbol{e}^*(\tau)$ to zero for $\tau \to \infty$. In step S4 one can find from (34) that for $e_{\beta a} \to 0$ (step S2) we have $\dot{\boldsymbol{q}}^* \to \boldsymbol{h}^*$. Hence, according to model (1) we can write:

$$\tan\beta = \left.\frac{\dot{y}}{\dot{x}}\right|_{e_{\beta a}=0} = \frac{\frac{-k_p e_y}{\eta \mathrm{sgn}(e_{x0})\|\,\boldsymbol{e}^*\|} + \sin\beta_t}{\frac{-k_p e_x}{\eta \mathrm{sgn}(e_{x0})\|\,\boldsymbol{e}^*\|} + \cos\beta_t}. \quad (41)$$

Above formula implies that for $\boldsymbol{e}^* \to \mathbf{0}$ it holds: $\tan\beta \to \tan\beta_t$. Since for $\mathrm{sgn}(e_{x0}) = +1$ the vehicle moves forward and for $\mathrm{sgn}(e_{x0}) = -1$ it moves backward, it can be recognized that for both strategies $\beta(\tau) \to \beta_t$ as $\tau \to \infty$. Step S5 considers the hitching angle behavior. For $e_{\beta a} = 0$ we have $\beta = \beta_a$ and one can rewrite (11) as follows: $\theta_a = \arctan\left(L\dot{\beta}/\pm\,\|\,\boldsymbol{h}^*\|\right)$. Since $\beta(\tau) \to \beta_t$ asymptotically and $\dot{\beta}_t \equiv 0$, the time derivative $\dot{\beta} \to 0$. Moreover, convergence rate of $\dot{\beta}$ is higher than convergence rate of $\boldsymbol{h}^*$ due to subsequent behavior of errors $e_{\beta a}$ and $\boldsymbol{e}^*$ as it has been stated in steps S2 to S3. As a consequence, the nominator of arctan $\left(L\dot{\beta}/\pm\,\|\,\boldsymbol{h}^*\|\right)$ goes to zero faster than the denominator and we obtain that $\theta_a(\tau) \to 0$ for $\tau \to \infty$. Finally, the corollary from step S1 allows to conclude: $\lim_{\tau\to\infty}\theta(\tau) = 0$. ■

Proposition 2 excludes the stabilized point from the domain of proper determination of the VFO controller due to definitions (23), (24), and (11), which are not defined for $\boldsymbol{e}^* = \mathbf{0}$. Since this point is reachable only in infinity, the proposed controller belongs to the so-called *almost stabilizers*, [1]. In practice however, we need a well defined solution also at this point. Therefore we have to determine the ϵ-vicinity of $\overline{\boldsymbol{e}} = [e_\beta \; \boldsymbol{e}^{*T}]^T = \mathbf{0}$ and introduce auxiliary definitions of the following signals:

$$\beta_a \triangleq \beta_a(\tau_-), \quad \dot{\beta}_a \triangleq 0, \quad \theta_a \triangleq \theta_t, \quad \dot{\theta}_a \triangleq 0, \quad u_2 \triangleq 0,$$

which can be activated inside the ϵ-vicinity. In this case the VFO stabilizer solves the Problem 1 for the assumed non-zero ϵ-envelope.

VI. SIMULATION RESULTS

Two kinds of simulation test have been performed: A) tracking of the persistently exciting trajectory of type T1 in a backward motion strategy assuming a non-zero initial tracking errors, B) set-point regulation as the parallel parking maneuver in a forward motion strategy. Kinematic parameter of the articulated vehicle have been assumed to be $L = 0.2$ m. Values of initial conditions, reference signals and VFO controller parameters are described below:
A) $\boldsymbol{q}(0) = [-\frac{\pi}{4} \;\; -\frac{\pi}{4} \;\; 0.2 \;\; 0.2]^T$, $\boldsymbol{q}_t(0) = \mathbf{0}$ and $u_{1t}(\tau) = -0.025 + 0.2\sin 2\tau$, $u_{2t} = -0.04$, $k_\theta = 10$, $k_\beta = 5$, $k_p = 1$,
B) $\boldsymbol{q}(0) = [0\;0\;0\;-1]^T$, $\boldsymbol{q}_t = \mathbf{0}$ and $k_\theta = 10$, $k_\beta = 5$, $k_p = 1$, $\eta = 0.7$.
Both tests have been conducted with utilization of the second-order exact robust differentiator (see [6]) for computations of $\dot{\theta}_a$ signal involved in (13). During implementation of the differentiator we chosen the Lipschitz constant $C_L = 100$ and we used the Euler-method integration procedure

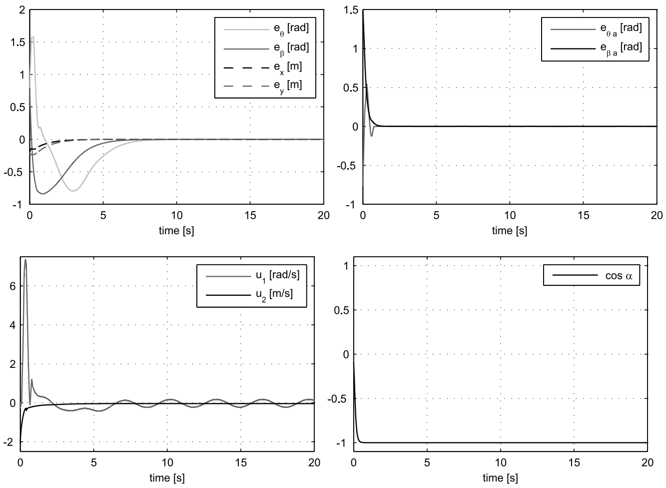

Fig. 3. Time-plots of signals for the trajectory tracking task A

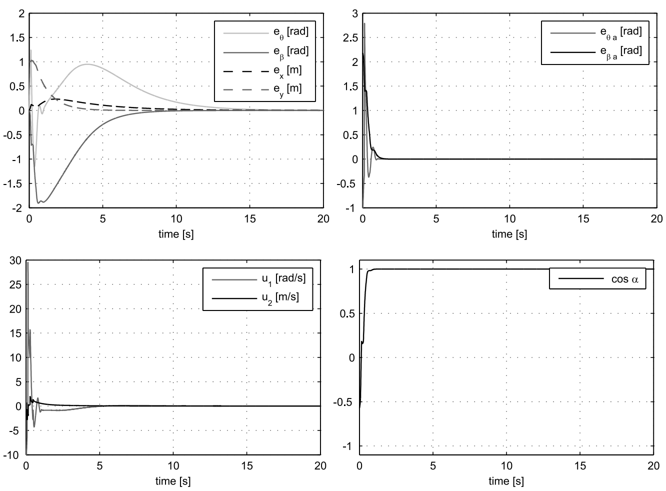

Fig. 4. Time-plots of signals for the set-point regulation task B

with the fixed sampling time $T_p = 0.005$ s. Obtained results are illustrated in Figs. 3 to 5. It is worth to note fast errors convergence together with smooth and non-oscillatory trailer motion during transient stages. It can be seen also the *sequential* behavior of particular error components (in relation to considerations included in the proofs). High effectiveness of the orienting process, characteristic for VFO control strategy, has been presented by the time-plots of $\cos\alpha = \cos(e_{\beta a} \pm \kappa\pi)$, where $\alpha = \angle(\boldsymbol{g}_2^*(\beta), \boldsymbol{h}^*)$. *Natural* motion character of the vehicle (understood as comparable with human-driven vehicle) obtained in both tests with the VFO strategy is presented in Fig. 5, where the tractor is represented by a triangle and the trailer by a rectangle.

VII. Conclusions and Future Works

The novel motion control algorithm utilizing the VFO control strategy for an articulated vehicle has been presented and numerically tested. Proposed method relies on vehicle model decomposition and treating the trailer body as a unicycle system with fictitious inputs. These inputs are freely

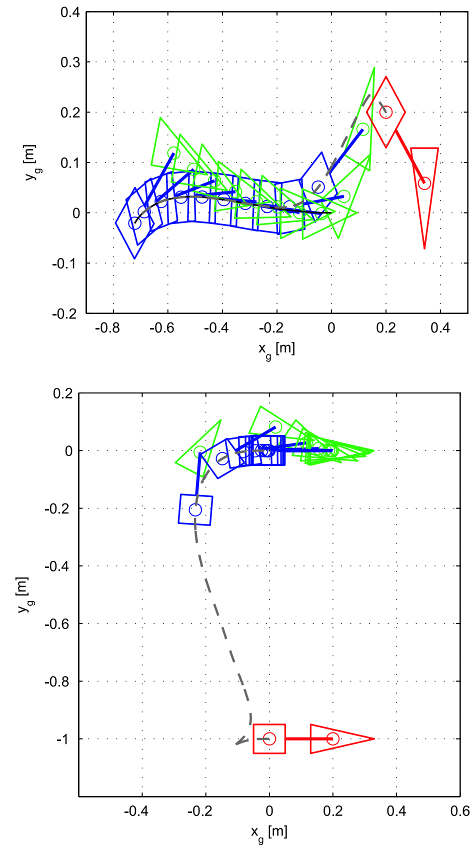

Fig. 5. Robot paths and stroboscopic view of vehicle motion on a plane $\{x_g, y_g\}$ for tracking task A (top) and set-point regulation B (bottom)

designed according to the VFO method – particularly efficient for the unicycle kinematics. Next, the real control inputs of an articulated vehicle are designed to realize previously defined trailer body fictitious controls. Simple geometrical interpretations of the VFO strategy allow to design kinematic controllers for tracking as well as for set-point regulation in a unified manner. Future extensions of the proposed method will be related to the case of an articulated vehicle equipped with an off-axle trailer, and also to the case of multi-trailer nonholonomic systems.

References

[1] A. Astolfi. Exponential stabilization of a wheeled mobile robot via discontinuous control. *Nonlinear Control System Design Symposium*, pages 741–746, 1995.

[2] M. Michałek and K. Kozłowski. Vector Field Orientation feedback control method for a differentially-driven vehicle. *Accepted for publication in IEEE Transactions on Control Systems Technology*.

[3] H. K. Khalil. *Nonlinear systems. 3rd Edition*. Prentice-Hall, Upper Saddle River, New Jersey, 2002.

[4] D. Kim and J. Oh. Globally asymptotically stable tracking control for a trailer system. *Journal of Robotic Systems*, 19(5):199–205, 2002.

[5] F. Lamiraux, S. Sekhavat, and J. Laumond. Motion planning and control for Hilare pulling a trailer. *IEEE Transactions on Robotics and Automation*, 15(4):640–652, 1999.

[6] A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, 76(9/10):924–941, 2003.

[7] P. Rouchon, M. Fliess, J. Levine, and P. Martin. Flatness, motion planning and trailer systems. In *Proceedings of the 32nd Conference on Decision and Control*, pages 2700–2705, San Antonio, Texas, 1993.