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Robust Delay-Dependent Stability of Polytopic Systems with Interval Time-Varying Delay

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Abstract—**This paper concerns the problem of robust stability analysis for a polytopic system with interval time- varying delay via parameter-dependent Lyapunov functions. By a relaxation approach with slack matrices and a descriptor model transformation of the system, the product between Lyapunov variables and the system matrices is eliminated. With this feature, a delay-dependent robust stability criterion is expressed as a set of linear matrix inequalities (LMIs). Compared with the results based on parameter -independent Lyapunov functions, this is promising for less conservatism. Two numerical examples are included to illustrate the effectiveness of the proposed method.**

I. Introduction

THE phenomena of time-delay are often encountered in many practical systems, such as aircraft systems, neural network, nuclear reactor, chemical engineering systems, population dynamic models, inferred grinding model, and manual control [1], [2]. In many systems, the models of systems are described by functional differential equations of polytopic type, where system matrices belong to a convex combination of the polytope vertices. Physical examples for polytopic systems have VTOL helicopter systems, satellite systems, missile systems, etc [3]-[5]. On the other hand, time-delay is a source of performance degradation and instability in many cases. Hence, the stability problem of time-delay systems is of theoretical and practical importance. Several results on robust control of time-delay systems subject to polytopic uncertainties have been reported in [6]-[8]. However, the model considered in these papers requires that the range of time-varying delay is from zero to an upper bound. So if the practical systems do not satisfy this assumption, the results without taking into account the information of the lower bound of delay are conservative. In practice, time delay in many systems usually varies in a range for which the lower bound is not restricted to be zero. Consequently, stability of systems with interval time-varying delays has been an attractive topic in theory analysis and practical application [9]-[11]. One typical example of such systems is the networked control system (NCS) which uses a data network in a control loop. In [9], the delay-dependent BRL and the stabilization criterion for H_∞ control of uncertain systems with time-varying delay in a range are formulated in the linear matrix inequality (LMI) form. In [10], the problem of delay-dependent robust stability for a class of uncertain linear systems with interval time-varying delay is investigated based on the Lyapunov–Krasovskii functional approach. In [11], an appropriate type of Lyapunov functional is constructed to study the problem of stability analysis for systems with interval time-varying delay.

Depending on whether the criterion itself contains the sizes of time delays, the criteria for time-delay systems can be classified into two categories, namely delay-independent criteria and delay-dependent criteria. Generally speaking, the latter ones are less conservative than the former ones. To the best of authors' knowledge, for the case where the lower bound of delay is greater than zero, there has been no result available for robust delay-dependent stability of polytopic systems with interval time-varying delay. Undoubtedly, the Lyapunov theory is one of the main approaches to deal with polytopic systems. However, the quadratic stability, which uses a single or parameter-independent Lyapunov function for testing the stability over the whole uncertain domain, may lead to conservative results in the case where the uncertain parameters are time-invariant. Motivated by this fact, Lyapunov functions depending on the uncertain parameters have been proposed to reduce quadratic stability conservatism [12]-[14].

Recently, a descriptor system approach is proposed for time-delayed systems. It reduces significantly the over-design compared to the traditional methods due to the facts of being based on a transformed model, equivalent to the original system, and fewer terms needed to be bounded in the derivation [15].

So, in this paper, the problem of robust stability for polytopic systems with time-varying delay in a range is investigated by means of parameter-dependent Lyapunov functions. With the introduction of slack variables, a descriptor system approach is adopted to obtain a delay-dependent stability criterion in terms of LMIs, which reduces the conservatism occurring in the stability problem with a fixed Lyapunov function. It is also shown that this criterion includes the delay-dependent/rate-independent criterion and the delay-independent/rate-dependent criterion as special cases. In the derivative of the Lyapunov functional, with the introduction of the augmented vector

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$\xi(t) \triangleq \left[x^{\mathrm{T}}(t) \ \ \dot{x}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, the term $\dot{x}^{\mathrm{T}}(t)(\tau_{\mathrm{m}}^2 P_3(\lambda) + \tau_{\Delta}^2 P_4(\lambda))\dot{x}(t)$ is formulated as $\xi^{\mathrm{T}}(t)\mathrm{diag}\{0, \tau_{\mathrm{m}}^2 P_3(\lambda) + \tau_{\Delta}^2 P_4(\lambda)\}\xi(t)$, which avoids replacing $\dot{x}(t)$ in $\dot{x}^{\mathrm{T}}(t)(\tau_{\mathrm{m}}^2 P_3(\lambda) + \tau_{\Delta}^2 P_4(\lambda))\dot{x}(t)$ with the state equation. In consequence, the Lyapunov matrices $P_3(\lambda)$ and $P_4(\lambda)$ which handle time delay are not involved in any product terms with the system matrices A and A_d, which allows Lyapunov matrices to be different for different vertices of the polytope. Finally, the applicability and less conservatism of our result are demonstrated through simulation studies.

II. Problem Statement and Main Results

Consider the following polytopic system with interval time-varying delay

$$\begin{aligned} &\dot{x}(t) = A(\lambda)x(t) + A_{\mathrm{d}}(\lambda)x(t-\tau(t)) \\ &x(t) = \phi(t), t \in [-h, 0] \end{aligned}, \tag{1}$$

where $x(t) \in R^n$ is the state vector and the initial vector ϕ is a continuously differentiable function from $[-h, 0]$ to R^n. We assume that $\tau(t)$ is a differentiable function satisfying

$$0 \le \tau_m \le \tau(t) \le \tau_M, \ \dot{\tau}(t) \le d, \ t \ge 0. \tag{2}$$

The unknown system matrices $A(\lambda)$ and $A_d(\lambda)$ are assumed to be uncertain but belonging to a known convex compact set of polytope type, i.e.,

$$\begin{aligned} &(A(\lambda), A_d(\lambda)) \in S \triangleq \{(A(\lambda), A_d(\lambda)) \mid (A(\lambda), A_d(\lambda)) \\ &= \sum_{i=1}^{N} \lambda_i (A_i, A_{di}), \lambda_i \ge 0, \sum_{i=1}^{N} \lambda_i = 1\}, \end{aligned} \tag{3}$$

where (A_i, A_{di}) is vertices of the above convex polytope and $\lambda \triangleq [\lambda_1, \cdots, \lambda_N]^{\mathrm{T}} \in R^N$ denotes a vector of uncertain and time-invariant real parameters.

The following lemma will be used to prove our results.

Lemma 1. For any constant matrix $P > 0$ and differentiable vector function $x(t)$ with appropriate dimensions, we have

$$[\int_{t-\tau_m}^{t} \dot{x}(s)\mathrm{d}s]^{\mathrm{T}} P [\int_{t-\tau_m}^{t} \dot{x}(s)\mathrm{d}s] \le \tau_m \cdot \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s, \tag{4}$$

$$\begin{aligned} &[\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)\mathrm{d}s]^{\mathrm{T}} P [\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)\mathrm{d}s] \le (\tau(t) - \tau_m) \\ &\times \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s \le (\tau_M - \tau_m) \cdot \int_{t-\tau_M}^{t-\tau_m} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s, \end{aligned} \tag{5}$$

where $0 \le \tau_m \le \tau(t) \le \tau_M$, $t \ge 0$.

Proof: For any scalar ρ, vector φ and $P > 0$, we have

$$\int_{t-\tau_m}^{t} [\dot{x}(s) - \rho\varphi]^{\mathrm{T}} P [\dot{x}(s) - \rho\varphi]\mathrm{d}s \ge 0. \tag{6}$$

From (6), it follows that

$$\rho^2 \tau_m \varphi^{\mathrm{T}} P \varphi - 2\rho \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{d}s P \varphi + \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s \ge 0. \tag{7}$$

Since (7) is satisfied for any scalar ρ, the inequality

$$\left[2\int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{d}s P \varphi\right]^2 - 4\tau_m \varphi^{\mathrm{T}} P \varphi \cdot \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s \le 0 \tag{8}$$

is derived. Setting $\varphi = \int_{t-\tau_m}^{t} \dot{x}(s)\mathrm{d}s$, we obtain

$$\left[\varphi^{\mathrm{T}} P \varphi\right]^2 - \tau_m \varphi^{\mathrm{T}} P \varphi \cdot \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s \le 0. \tag{9}$$

Since scalar $\varphi^{\mathrm{T}} P \varphi > 0$, (9) is equivalent to

$$\varphi^{\mathrm{T}} P \varphi - \tau_m \cdot \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s) P \dot{x}(s)\mathrm{d}s \le 0, \tag{10}$$

which indicates that (4) is satisfied. By replacing $\int_{t-\tau_m}^{t} \dot{x}(s)\mathrm{d}s$ with $\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)\mathrm{d}s$, the first inequality of (5) can be derived similarly. Since the second inequality of (5) is satisfied obviously, the proof of (5) is completed.

Using the method in [15], we represent system (1) in the equivalent descriptor form

$$\begin{aligned} &\dot{x}(t) = \eta(t) \\ &\eta(t) = A(\lambda)x(t) + A_{\mathrm{d}}(\lambda)x(t-\tau(t)) \end{aligned}. \tag{11}$$

The following theorem presents a delay-dependent and rate-dependent robust stability analysis result based on parameter-dependent Lyapunov functional.

Theorem 1. System (1) with parameter uncertainty (3) and interval time-varying delay $\tau(t)$ satisfying (2) is robustly asymptotically stable if there exist symmetric positive definite matrices P_{0i}, P_{1i}, P_{2i}, P_{3i}, P_{4i}, $i = 1, \cdots, N$, and matrices P_5, P_6, Q_1, Q_2, Q_3 such that

$$\Xi_i = \begin{bmatrix} \Xi_{i11} & \Xi_{i12} & \Xi_{i13} & \Xi_{i14} \\ * & \Xi_{i22} & \Xi_{i23} & 0 \\ * & * & \Xi_{i33} & \Xi_{i34} \\ * & * & * & \Xi_{i44} \end{bmatrix} < 0, \tag{12}$$

where * represents the symmetric form in the matrix and

$$\begin{aligned} &\Xi_{i11} = P_5^{\mathrm{T}} A_i + A_i^{\mathrm{T}} P_5 + P_{1i} + P_{2i} - P_{3i} + Q_2^{\mathrm{T}} A_i + A_i^{\mathrm{T}} Q_2, \\ &\Xi_{i12} = P_{0i} - P_5^{\mathrm{T}} + A_i^{\mathrm{T}} P_6 - Q_2^{\mathrm{T}} + A_i^{\mathrm{T}} Q_1, \\ &\Xi_{i13} = P_5^{\mathrm{T}} A_{di} + Q_2^{\mathrm{T}} A_{di} + A_i^{\mathrm{T}} Q_3, \Xi_{i14} = P_{3i}, \\ &\Xi_{i22} = \tau_m^2 P_{3i} + \tau_\Delta^2 P_{4i} - P_6 - P_6^{\mathrm{T}} - Q_1 - Q_1^{\mathrm{T}}, \tau_\Delta = \tau_M - \tau_m, \\ &\Xi_{i23} = P_6^{\mathrm{T}} A_{di} + Q_1^{\mathrm{T}} A_{di} - Q_3, \Xi_{i34} = P_{4i}, \bar{d} = 1 - d, \\ &\Xi_{i33} = -\bar{d} P_{2i} - P_{4i} + Q_3^{\mathrm{T}} A_{di} + A_{di}^{\mathrm{T}} Q_3, \Xi_{i44} = -P_{1i} - P_{3i} - P_{4i}. \end{aligned}$$

Proof: Define the Lyapunov–Krasovskii functional

$$V(t, \lambda) = V_1 + V_2 + V_3 + V_4 + V_5 + V_6, \tag{13}$$

where

$$V_1 = x^{\mathrm{T}}(t) P_0(\lambda) x(t), \tag{14}$$

$$V_2 = \int_{t-\tau_m}^{t} x^{\mathrm{T}}(s) P_1(\lambda) x(s)\mathrm{d}s, \tag{15}$$

$$V_3 = \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s) P_2(\lambda) x(s)\mathrm{d}s, \tag{16}$$

$$V_4 = \tau_m \cdot \int_{t-\tau_m}^{t} (s - (t - \tau_m)) \dot{x}^{\mathrm{T}}(s) P_3(\lambda) \dot{x}(s)\mathrm{d}s, \tag{17}$$

$$V_5 = (\tau_M - \tau_m) \cdot \int_{t-\tau_M}^{t-\tau_m} (s - (t - \tau_M)) \dot{x}^{\mathrm{T}}(s) P_4(\lambda) \dot{x}(s)\mathrm{d}s, \tag{18}$$

$$V_6 = (\tau_M - \tau_m)^2 \cdot \int_{t-\tau_m}^{t} \dot{x}^{\mathrm{T}}(s) P_4(\lambda) \dot{x}(s)\mathrm{d}s, \tag{19}$$

$$P_j(\lambda)=\sum_{i=1}^{N}\lambda_i P_{ji}>0\,,\; j=0,1,2,3,4\,, \tag{20}$$

and P_{ji}, $j=0,\cdots,4$), are matrices to be determined. Then, the time derivative of $V(t,\lambda)$ is given by

$$\begin{aligned}
&\dot V(t,\lambda)\\
&=x^{\mathrm T}(t)P_0(\lambda)\dot x(t)+\dot x^{\mathrm T}(t)P_0(\lambda)x(t)+x^{\mathrm T}(t)P_1(\lambda)x(t)\\
&\quad -x^{\mathrm T}(t-\tau_m)P_1(\lambda)x(t-\tau_m)+x^{\mathrm T}(t)P_2(\lambda)x(t)\\
&\quad -(1-\dot\tau(t))x^{\mathrm T}(t-\tau(t))P_2(\lambda)x(t-\tau(t))+\tau_m^2\dot x^{\mathrm T}(t)P_3(\lambda)\dot x(t)\\
&\quad -\tau_m\cdot\int_{t-\tau_m}^{t}\dot x^{\mathrm T}(s)P_3(\lambda)\dot x(s)\mathrm{d}s+(\tau_M-\tau_m)^2\dot x^{\mathrm T}(t)P_4(\lambda)\dot x(t)\\
&\quad -(\tau_M-\tau_m)\cdot\int_{t-\tau_M}^{t-\tau_m}\dot x^{\mathrm T}(s)P_4(\lambda)\,\dot x(s)\mathrm{d}s.
\end{aligned} \tag{21}$$

By lemma 1 and Leibniz–Newton formula, we have

$$\begin{aligned}
&-\tau_m\cdot\int_{t-\tau_m}^{t}\dot x^{\mathrm T}(s)P_3(\lambda)\dot x(s)\mathrm{d}s\\
&\le-[\int_{t-\tau_m}^{t}\dot x(s)\mathrm{d}s]^{\mathrm T}P_3(\lambda)[\int_{t-\tau_m}^{t}\dot x(s)\mathrm{d}s]\\
&=-(x(t)-x(t-\tau_m))^{\mathrm T}P_3(\lambda)(x(t)-x(t-\tau_m)),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-(\tau_M-\tau_m)\cdot\int_{t-\tau_M}^{t-\tau_m}\dot x^{\mathrm T}(s)P_4(\lambda)\,\dot x(s)\mathrm{d}s\\
&\le-[\int_{t-\tau(t)}^{t-\tau_m}\dot x(s)\mathrm{d}s]^{\mathrm T}P_4(\lambda)[\int_{t-\tau(t)}^{t-\tau_m}\dot x(s)\mathrm{d}s]\\
&=-(x(t-\tau_m)-x(t-\tau(t)))^{\mathrm T}P_4(\lambda)(x(t-\tau_m)-x(t-\tau(t))).
\end{aligned} \tag{23}$$

Note that one can obtain

$$\begin{bmatrix}\eta(t)\\ -\eta(t)+A(\lambda)x(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ A(\lambda) & -1\end{bmatrix}\begin{bmatrix}x(t)\\ \eta(t)\end{bmatrix}. \tag{24}$$

By virtue of the descriptor form (11) and (24), we have

$$\begin{aligned}
&x^{\mathrm T}(t)P_0(\lambda)\dot x(t)\\
&=\begin{bmatrix}x(t)\\ \dot x(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}P_0(\lambda) & P_5^{\mathrm T}\\ 0 & P_6^{\mathrm T}\end{bmatrix}\begin{bmatrix}\dot x(t)\\ 0\end{bmatrix}\\
&=\begin{bmatrix}x(t)\\ \eta(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}P_0(\lambda) & P_5^{\mathrm T}\\ 0 & P_6^{\mathrm T}\end{bmatrix}\begin{bmatrix}\eta(t)\\ -\eta(t)+A(\lambda)x(t)\end{bmatrix}\\
&\quad+\begin{bmatrix}x(t)\\ \eta(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}P_0(\lambda) & P_5^{\mathrm T}\\ 0 & P_6^{\mathrm T}\end{bmatrix}\begin{bmatrix}0\\ A_d(\lambda)x(t-\tau(t))\end{bmatrix}\\
&=\begin{bmatrix}x(t)\\ \eta(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}P_5^{\mathrm T}A(\lambda) & P_0(\lambda)-P_5^{\mathrm T}\\ P_6^{\mathrm T}A(\lambda) & -P_6^{\mathrm T}\end{bmatrix}\begin{bmatrix}x(t)\\ \eta(t)\end{bmatrix}\\
&\quad+\begin{bmatrix}x(t)\\ \eta(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}P_5^{\mathrm T}A_d(\lambda)\\ P_6^{\mathrm T}A_d(\lambda)\end{bmatrix}x(t-\tau(t)).
\end{aligned} \tag{25}$$

From (22), (23) and (25), we have

$$\begin{aligned}
&\dot V(t,\lambda)\\
&\le\xi^{\mathrm T}(t)\begin{bmatrix}P_5^{\mathrm T}A(\lambda)+A^{\mathrm T}(\lambda)P_5 & P_0(\lambda)-P_5^{\mathrm T}+A^{\mathrm T}(\lambda)P_6\\ P_0(\lambda)-P_5+P_6^{\mathrm T}A(\lambda) & -P_6-P_6^{\mathrm T}\end{bmatrix}\xi(t)\\
&\quad+\xi^{\mathrm T}(t)\begin{bmatrix}P_5^{\mathrm T}A_d(\lambda)\\ P_6^{\mathrm T}A_d(\lambda)\end{bmatrix}x(t-\tau(t))+x^{\mathrm T}(t-\tau(t))\begin{bmatrix}P_3^{\mathrm T}A_d(\lambda)\\ P_3^{\mathrm T}A_d(\lambda)\end{bmatrix}^{\mathrm T}\xi(t)\\
&\quad+x^{\mathrm T}(t)(P_1(\lambda)+P_2(\lambda)-P_3(\lambda))x(t)+x^{\mathrm T}(t-\tau_m)\\
&\quad\times(-P_1(\lambda)-P_3(\lambda)-P_4(\lambda))x(t-\tau_m)+x^{\mathrm T}(t-\tau(t))\\
&\quad\times(-\bar dP_2(\lambda)-P_4(\lambda))x(t-\tau(t))+x^{\mathrm T}(t)P_3(\lambda)x(t-\tau_m)\\
&\quad+x^{\mathrm T}(t-\tau_m)P_3(\lambda)x(t)+x^{\mathrm T}(t-\tau_m)P_4(\lambda)x(t-\tau(t))\\
&\quad+x^{\mathrm T}(t-\tau(t))P_4(\lambda)x(t-\tau_m)+\dot x^{\mathrm T}(t)(\tau_m^2P_3(\lambda)+\tau_\Delta^2P_4(\lambda))\dot x(t)\\
&=\bar\xi^{\mathrm T}(t)\Psi(\lambda)\bar\xi(t),
\end{aligned} \tag{26}$$

where

$$\xi(t)\triangleq\begin{bmatrix}x^{\mathrm T}(t) & \eta^{\mathrm T}(t)\end{bmatrix}^{\mathrm T}, \tag{27}$$

$$\bar\xi(t)=\begin{bmatrix}\xi^{\mathrm T}(t) & x^{\mathrm T}(t-\tau(t)) & x^{\mathrm T}(t-\tau_m)\end{bmatrix}^{\mathrm T}, \tag{28}$$

$$\Psi(\lambda)=\begin{bmatrix}\Gamma(\lambda) & \begin{bmatrix}A_d^{\mathrm T}(\lambda)P_5 & A_d^{\mathrm T}(\lambda)P_6\end{bmatrix}^{\mathrm T} & \begin{bmatrix}P_3(\lambda) & 0\end{bmatrix}^{\mathrm T}\\ * & -\bar dP_2(\lambda)-P_4(\lambda) & P_4(\lambda)\\ * & * & \Psi_{33}(\lambda)\end{bmatrix}, \tag{29}$$

$$\Gamma(\lambda)=\begin{bmatrix}\Gamma_{11}(\lambda) & P_0(\lambda)-P_5^{\mathrm T}+A^{\mathrm T}(\lambda)P_6\\ * & -P_6-P_6^{\mathrm T}+\tau_m^2P_3(\lambda)+\tau_\Delta^2P_4(\lambda)\end{bmatrix}, \tag{30}$$

$$\Gamma_{11}(\lambda)=P_5^{\mathrm T}A(\lambda)+A^{\mathrm T}(\lambda)P_5+P_1(\lambda)+P_2(\lambda)-P_3(\lambda)\,, \tag{31}$$

$$\Psi_{33}(\lambda)=-P_1(\lambda)-P_3(\lambda)-P_4(\lambda)\,. \tag{32}$$

Using (11), we have

$$\begin{aligned}
&[\eta^{\mathrm T}(t)Q_1^{\mathrm T}+x^{\mathrm T}(t)Q_2^{\mathrm T}+x^{\mathrm T}(t-\tau(t))Q_3^{\mathrm T}]\cdot[-\eta(t)+A(\lambda)x(t)\\
&+A_{\mathrm d}(\lambda)x(t-\tau(t))]+[-\eta(t)+A(\lambda)x(t)+A_{\mathrm d}(\lambda)x(t-\tau(t))]^{\mathrm T}\\
&\times[Q_1\eta(t)+Q_2x(t)+Q_3x(t-\tau(t))]=0.
\end{aligned} \tag{33}$$

Adding the left side of (33) into (26) yields

$$\dot V(t,\lambda)\le\bar\xi^{\mathrm T}(t)\Xi(\lambda)\bar\xi(t)\,, \tag{34}$$

where

$$\Xi(\lambda)=\begin{bmatrix}\Xi_1(\lambda) & \Xi_2(\lambda) & \begin{bmatrix}P_3(\lambda) & 0\end{bmatrix}^{\mathrm T}\\ * & \Xi_3(\lambda) & P_4(\lambda)\\ * & * & \Psi_{33}(\lambda)\end{bmatrix}, \tag{35}$$

$$\Xi_1(\lambda)=\begin{bmatrix}\Xi_{11}(\lambda) & P_0(\lambda)-P_5^{\mathrm T}+A^{\mathrm T}(\lambda)P_6-Q_2^{\mathrm T}+A^{\mathrm T}(\lambda)Q_1\\ * & -P_6-P_6^{\mathrm T}+\tau_m^2P_3(\lambda)+\tau_\Delta^2P_4(\lambda)-Q_1-Q_1^{\mathrm T}\end{bmatrix}, \tag{36}$$

$$\Xi_{11}(\lambda)=\Gamma_{11}(\lambda)+Q_2^{\mathrm T}A(\lambda)+A^{\mathrm T}(\lambda)Q_2\,, \tag{37}$$

$$\Xi_2(\lambda)=\begin{bmatrix}A_d^{\mathrm T}(\lambda)(P_5+Q_2)+Q_3^{\mathrm T}A(\lambda) & A_d^{\mathrm T}(\lambda)(P_6+Q_1)-Q_3^{\mathrm T}\end{bmatrix}^{\mathrm T}, \tag{38}$$

$$\Xi_3(\lambda)=-\bar dP_2(\lambda)-P_4(\lambda)+Q_3^{\mathrm T}A_{\mathrm d}(\lambda)+A_{\mathrm d}^{\mathrm T}(\lambda)Q_3\,, \tag{39}$$

According to (12), we have

$$\Xi(\lambda)=\sum_{i=1}^{N}\lambda_i\Xi_i<0\,. \tag{40}$$

From (34) and (40), we get $\dot V(t,\lambda)<0$. Then, according to the Lyapunov theory, polytopic system (1) with interval time-varying delay is robustly stable. This proof is completed.

Remark 1. In Theorem 1, with the introduction of the slack variables P_5, P_6 and the corresponding augmented vector $\xi(t)\triangleq\begin{bmatrix}x^{\mathrm T}(t) & \eta^{\mathrm T}(t)\end{bmatrix}^{\mathrm T}$, delay-dependent stability criterion (12) does not involve the product between the Lyapunov matrix $P_0(\lambda)$ and system matrices $A(\lambda)$ and $A_{\mathrm d}(\lambda)$. Hence, for

(12), P_{0i} are not required to be the same, but the slack variables P_5 and P_6 are. So, it is expected to lead to a less conservative stability condition, as there are no other constraints imposed on P_5 and P_6. Besides, the augmented vector can be used to formulate $\dot{x}^{\mathrm{T}}(t)(\tau_{\mathrm{m}}^2 P_3(\lambda)+\tau_{\Delta}^2 P_4(\lambda))\dot{x}(t)$ as $\xi^{\mathrm{T}}(t)\mathrm{diag}\{0,\tau_{\mathrm{m}}^2 P_3(\lambda)+\tau_{\Delta}^2 P_4(\lambda)\}\xi(t)$, which avoids replacing $\dot{x}(t)$ in the term $\dot{x}^{\mathrm{T}}(t)(\tau_{\mathrm{m}}^2 P_3(\lambda)+\tau_{\Delta}^2 P_4(\lambda))\dot{x}(t)$ with the state equation and so eliminates the multiplication relation between the Lyapunov matrix $P_3(\lambda), P_4(\lambda)$ and system matrices $A(\lambda)$, $A_{\mathrm{d}}(\lambda)$.

Remark 2. Another feature for our method is that the delay-dependent result can be obtained without using bounded inequality with respect to the cross terms, which reduces the conservatism further. It is also worth mentioning that the slow variation constraint $\dot{\tau}(t)<1$ is not necessary for all the results in this paper.

In many cases, the information of the derivative of delay is unknown. Regarding this circumstance, a delay-dependent/rate-independent robust stability condition is derived as follows by choosing $P_{2i}=0$ in Theorem 1.

Corollary 1. Polytopic system (1) with $\tau(t)$ satisfying $0\le\tau_m\le\tau(t)\le\tau_M$ is robustly asymptotically stable if there exist symmetric positive definite matrices P_{0i}, P_{1i}, P_{3i}, P_{4i}, $i=1,\cdots,N$, and matrices P_5, P_6, Q_1, Q_2, Q_3 such that

$$\bar{\Xi}_i=\begin{bmatrix}\bar{\Xi}_{i11} & \Xi_{i12} & \Xi_{i13} & \Xi_{i14}\\ * & \Xi_{i22} & \Xi_{i23} & 0\\ * & * & \bar{\Xi}_{i33} & \Xi_{i34}\\ * & * & * & \Xi_{i44}\end{bmatrix}<0,$$

where Ξ_{i12}, Ξ_{i13}, Ξ_{i14}, Ξ_{i22}, Ξ_{i23}, Ξ_{i34}, Ξ_{i44} are defined in Theorem 1, and

$$\bar{\Xi}_{i11}=P_5^{\mathrm{T}}A_i+A_i^{\mathrm{T}}P_5+P_{1i}-P_{3i}+Q_2^{\mathrm{T}}A_i+A_i^{\mathrm{T}}Q_2,$$
$$\bar{\Xi}_{i33}=-P_{4i}+Q_3^{\mathrm{T}}A_{di}+A_{di}^{\mathrm{T}}Q_3.$$

With $P_3(\lambda)=P_4(\lambda)=0$ in Theorem 1, a delay-independent robust stability criterion of polytopic system (1) with $\tau(t)$ satisfying $\dot{\tau}(t)\le d$ is obtained easily and thus is omitted here.

As a special case, if we choose single Lyapunov functions $P_j(\lambda)=P_j>0\ (j=0,\cdots,4)$, the parameter-independent criterion follows.

Corollary 2. System (1) with parameter uncertainty (3) and interval time-varying delay $\tau(t)$ satisfying (2) is robustly asymptotically stable if there exist symmetric positive definite matrices P_0, P_1, P_2, P_3, P_4, and matrices Q_1, Q_2, Q_3 such that

$$\tilde{\Xi}_i=\begin{bmatrix}\tilde{\Xi}_{i11} & \tilde{\Xi}_{i12} & \tilde{\Xi}_{i13} & \tilde{\Xi}_{i14}\\ * & \tilde{\Xi}_{i22} & \tilde{\Xi}_{i23} & 0\\ * & * & \tilde{\Xi}_{i33} & \tilde{\Xi}_{i34}\\ * & * & * & \tilde{\Xi}_{i44}\end{bmatrix}<0,\ i=1,\cdots,N, \quad (41)$$

where * denotes the symmetric terms in the matrix and

$$\tilde{\Xi}_{i11}=P_0A_i+A_i^{\mathrm{T}}P_0+P_1+P_2-P_3+Q_2^{\mathrm{T}}A_i+A_i^{\mathrm{T}}Q_2,$$
$$\tilde{\Xi}_{i12}=A_i^{\mathrm{T}}Q_1-Q_2^{\mathrm{T}},\ \tilde{\Xi}_{i13}=P_0A_{di}+Q_2^{\mathrm{T}}A_{di}+A_i^{\mathrm{T}}Q_3,$$
$$\tilde{\Xi}_{i14}=P_3,\ \tilde{\Xi}_{i22}=\tau_m^2P_3+\tau_\Delta^2P_4-Q_1-Q_1^{\mathrm{T}},$$
$$\tilde{\Xi}_{i23}=Q_1^{\mathrm{T}}A_{di}-Q_3,\ \tilde{\Xi}_{i33}=-\bar{d}P_2-P_4+Q_3^{\mathrm{T}}A_{di}+A_{di}^{\mathrm{T}}Q_3,$$
$$\tilde{\Xi}_{i34}=P_4,\ \tilde{\Xi}_{i44}=-P_1-P_3-P_4,\ \bar{d}=1-d,\ \tau_\Delta=\tau_M-\tau_m.$$

Proof: The Lyapunov–Krasovskii functional $V(t,\lambda)$ is chosen as (13) with $P_j(\lambda)=P_j>0\ (j=0,\cdots,4)$. Differentiating $V(t,\lambda)$ with respect to t gives

$$\begin{aligned}&\dot{V}(t,\lambda)\\&=x^{\mathrm{T}}(t)(P_0A(\lambda)+A^{\mathrm{T}}(\lambda)P_0)x(t)+2x^{\mathrm{T}}(t)P_0A_{\mathrm{d}}^{\mathrm{T}}(\lambda)x(t-\tau(t))\\&\quad+x^{\mathrm{T}}(t)P_1x(t)-x^{\mathrm{T}}(t-\tau_m)P_1x(t-\tau_m)+x^{\mathrm{T}}(t)P_2x(t)\\&\quad-(1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))P_2x(t-\tau(t))+\tau_m^2\dot{x}^{\mathrm{T}}(t)P_3\dot{x}(t)\\&\quad-\tau_m\cdot\int_{t-\tau_m}^{t}\dot{x}^{\mathrm{T}}(s)P_3\dot{x}(s)\mathrm{d}s+(\tau_M-\tau_m)^2\dot{x}^{\mathrm{T}}(t)P_4\dot{x}(t)\\&\quad-(\tau_M-\tau_m)\cdot\int_{t-\tau_M}^{t-\tau_m}\dot{x}^{\mathrm{T}}(s)P_4\dot{x}(s)\mathrm{d}s.\end{aligned}\quad (42)$$

From (22), (23) with $P_j(\lambda)=P_j$ and (33), we have

$$\dot{V}(t,\lambda)\le\bar{\xi}^{\mathrm{T}}(t)\tilde{\Xi}(\lambda)\bar{\xi}(t), \quad (43)$$

where $\bar{\xi}(t)$ is defined in (28) and

$$\tilde{\Xi}(\lambda)=\begin{bmatrix}\tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14}\\ * & \tilde{\Xi}_{22} & \tilde{\Xi}_{23} & 0\\ * & * & \tilde{\Xi}_{33} & \tilde{\Xi}_{34}\\ * & * & * & \tilde{\Xi}_{44}\end{bmatrix}, \quad (44)$$

$$\tilde{\Xi}_{11}=P_0A(\lambda)+A^{\mathrm{T}}(\lambda)P_0+P_1+P_2-P_3+Q_2^{\mathrm{T}}A(\lambda)+A^{\mathrm{T}}(\lambda)Q_2,$$
$$\tilde{\Xi}_{12}=A^{\mathrm{T}}(\lambda)Q_1-Q_2^{\mathrm{T}},\ \tilde{\Xi}_{22}=\tau_m^2P_3+\tau_\Delta^2P_4-Q_1-Q_1^{\mathrm{T}},$$
$$\tilde{\Xi}_{13}=P_0A_d(\lambda)+Q_2^{\mathrm{T}}A_d(\lambda)+A^{\mathrm{T}}(\lambda)Q_3,\ \tilde{\Xi}_{23}=Q_1^{\mathrm{T}}A_d(\lambda)-Q_3,$$
$$\tilde{\Xi}_{33}=-\bar{d}P_2-P_4+Q_3^{\mathrm{T}}A_d(\lambda)+A_d^{\mathrm{T}}(\lambda)Q_3,\ \tilde{\Xi}_{14}=P_3$$
$$\tilde{\Xi}_{34}=P_4,\ \tilde{\Xi}_{44}=-P_1-P_3-P_4,\ \bar{d}=1-d,\ \tau_\Delta=\tau_M-\tau_m. \quad (45)$$

Taking (41) into account, we have

$$\tilde{\Xi}(\lambda)=\sum_{i=1}^{N}\lambda_i\tilde{\Xi}_i<0. \quad (46)$$

From (43) and (46), we get $\dot{V}(t,\lambda)<0$. Hence, by the Lyapunov theory, the existence of $V(t,\lambda)>0$ such that $\dot{V}(t,\lambda)<0$ guarantees robust asymptotic stability of polytopic system given in (1). This completes the proof.

Remark 3. It should be pointed that, for polytopic system (1) with interval time-varying delay satisfying (2), the parameter-independent stability criterion (41) in Corollary 2 can also be obtained by parameter-dependent stability criterion (12) in Theorem 1 with the following transformation $P_{0i}=P_0$, $P_{1i}=P_1$, $P_{2i}=P_2$, $P_{3i}=P_3$, $P_{4i}=P_4$, $P_5=P_0$, $P_6=0$. It is obvious that this treatment usually causes conservatism inevitably.

With $P_2 = 0$ in Corollary 2, delay-dependent/rate-independent robust stability criterion of polytopic system (1) with $\tau(t)$ satisfying $0 \le \tau_m \le \tau(t) \le \tau_M$ can be obtained by using parameter-independent Lyapunov functions.

Corollary 3. Polytopic system (1) with interval time-varying delay $\tau(t)$ satisfying $0 \le \tau_m \le \tau(t) \le \tau_M$ is robustly asymptotically stable if there exist symmetric positive definite matrices P_0, P_1, P_3, P_4 and matrices Q_1, Q_2, Q_3 such that

$$\hat{\Xi}_i = \begin{bmatrix} \hat{\Xi}_{i11} & \tilde{\Xi}_{i12} & \tilde{\Xi}_{i13} & \tilde{\Xi}_{i14} \\ * & \tilde{\Xi}_{i22} & \tilde{\Xi}_{i23} & 0 \\ * & * & \hat{\Xi}_{i33} & \tilde{\Xi}_{i34} \\ * & * & * & \tilde{\Xi}_{i44} \end{bmatrix} < 0, i = 1, \cdots, N,$$

where $\tilde{\Xi}_{i12}$, $\tilde{\Xi}_{i13}$, $\tilde{\Xi}_{i14}$, $\tilde{\Xi}_{i22}$, $\tilde{\Xi}_{i23}$, $\tilde{\Xi}_{i34}$, $\tilde{\Xi}_{i44}$ are defined in Corollary 2 and

$$\hat{\Xi}_{i11} = P_0 A_i + A_i^{\mathrm{T}} P_0 + P_1 - P_3 + Q_2^{\mathrm{T}} A_i + A_i^{\mathrm{T}} Q_2,$$
$$\hat{\Xi}_{i33} = -P_4 + Q_3^{\mathrm{T}} A_{di} + A_{di}^{\mathrm{T}} Q_3, \tau_\Delta = \tau_M - \tau_m.$$

III. Illustrative Example And Simulation Results

In this section, computer simulations are carried out to show the effectiveness and less conservativeness of parameter-dependent results than parameter-independent ones.

Example1. Consider system (1) with the following matrices borrowed from [8]

$$A_1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, A_3 = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_{d3} = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}.$$

Our purpose is to determine the upper bound τ_M of interval time-varying delay $\tau(t)$ for which the system is robustly asymptotically stable. For comparison purposes, parameter-dependent and parameter-independent stability criteria are applied to this system, respectively. For the time-varying delay with $d = 0.3$ and $\tau_m = 0.4$, the upper bound τ_M of $\tau(t)$ is found to be 0.9434 by Corollary 2. According to Theorem 1, however, it is found that the system is robustly asymptotically stable for $\tau_M = 2.3997$, which shows that parameter-dependent Lyapunov function (Theorem 1) yields less conservative stability criterion than parameter-independent one (Corollary 2). To provide relatively complete information, we calculate the upper bound τ_M for different time-varying cases with given lower bound τ_m, listed in Tables I-II, where the acronyms have the following meaning

Th1	stability criterion in Theorem 1
Cr1	stability criterion in Corollary 1
Cr2	stability criterion in Corollary 2
Cr3	stability criterion in Corollary 3.

TABLE I

Upper bounds τ_M with given τ_m for different d

τ_m	0	0.4	1.0	2.4
Th1($d=0.3$)	2.3992	2.3997	2.3997	2.5965
Cr2($d=0.3$)	0.9434	0.9434	1.0391	-
Th1($d=1.2$)	0.9084	1.0650	1.4608	2.5964
Cr2($d=1.2$)	0.6923	0.7970	1.0391	-

TABLE II

Upper bounds τ_M with given τ_m for unknown d

τ_m	0	0.2	0.4	0.6	0.8
Cr1	0.9095	0.9664	1.0650	1.1848	1.3181
Cr3	0.6923	0.7376	0.7970	0.8686	0.9500

The notation"-"indicates that no upper bound can be obtained.

To illustrate the advantage of Theorem 1 further, another example is considered.

Example2. Consider the following system with a time-varying delay:

$$A = \begin{bmatrix} 0 & -0.12 + 12\rho \\ 1 & -0.465 - \rho \end{bmatrix} \quad A_d = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}$$

and $|\rho| \le 0.035$ [8].

If we let $\rho_m = 0.035$ and set

$$A_1 = \begin{bmatrix} 0 & -0.12 + 12\rho_m \\ 1 & -0.465 - \rho_m \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -0.12 - 12\rho_m \\ 1 & -0.465 + \rho_m \end{bmatrix},$$
$$A_{d1} = A_{d2} = A_d = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix},$$

then the system is described by (1).

It is noted that, for given lower bound $\tau_m = 0.1$ and $d = 0.1$, Corollary 2 is not able to conclude robust stability even for $\tau_M = 0.100000000001$. According to Theorem 1, however, it is demonstrated that this system is robustly stable for $\tau_M = 0.7863$.

IV. Conclusion

This paper has presented a strategy using a descriptor system approach and parameter-dependent Lyapunov functions to deal with robust stability analysis for a polytopic system with interval time-varying delay. The time delay allows a range for which the lower bound is greater than zero. By introducing slack matrices, a delay-dependent robust stability criterion is derived in the framework of LMIs, which determines the interval bound guaranteeing the asymptotic stability for the considered systems. Some numerical examples are provided to demonstrate that the proposed method significantly improves the allowed delay bounds outperforming parameter-independent approaches.

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