2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Delay-dependent exponential stability analysis of fuzzy delayed Hopfield neural networks: a fuzzy Lyapunov-Krasovskii functional approach

Li Sheng and Huizhong Yang

Abstract— **This paper investigates the delay-dependent exponential stability problem of Takagi-Sugeno (TS) fuzzy Hopfield neural networks (HNNs) with time-varying delay. Based on a fuzzy Lyapunov-Krasovskii functional (LKF), some delay-dependent stability criteria guaranteeing the exponential stability of the fuzzy HNNs are devised by taking the relationship between the terms in the Leibniz-Newton formula into account. Since free weighting matrices are used to express this relationship and the appropriate ones are selected by means of linear matrix inequalities (LMIs), the criteria are less conservative than existing ones reported in the literature for delayed fuzzy neural networks. A simulation example is provided to illustrate the effectiveness of the developed method.**

I. INTRODUCTION

Hopfield neural networks (HNNs) and their various generalizations have been successfully employed in many areas such as pattern recognition, associate memory and knowledge acquisition [1],[2]. Such applications of neural networks heavily depend on the dynamical behaviors of the networks. Therefore, stability analysis for neural networks has been investigated and a great number of approaches have been proposed [3]-[6]. Since time delays as a source of instability and bad performance always appear in many neural networks owing to the finite speed of information processing, the stability analysis for the delayed neural networks has received considerable attention. The existing results can be classified into two types: delay-independent criteria [3],[4] and delay-dependent criteria [5],[6]. The former is irrespective of the size of the delay and the later is concerned with the size of the delay. It has been shown that the delay-dependent stability conditions are generally less conservative than the delay-independent ones, especially when the size of the delay is small.

In the past two decades, the fuzzy logic theory has provided an appealing and efficient approach to deal with the analysis and synthesis problems for complex nonlinear systems. Recently, the dynamic Takagi-Sugeno (TS) fuzzy model [7] has become a popular tool and has been employed in most model-based fuzzy analysis approaches. The main feature of the TS fuzzy model is that a nonlinear system can be approximated by a set of TS linear models. The overall fuzzy model of the system is achieved by fuzzy blending of the set of TS linear models. The stability issue of fuzzy control systems has been discussed in an extensive literature. Most of the existing results were obtained by using a single Lyapunov function (SLF) method [8]. However, the main drawback associated to this method is that an SLF must work for all linear models, which in general leads to a conservative result. To relax this conservatism, the fuzzy Lyapunov-Krasovskii functional (LKF), which directly includes the membership functions, has been proposed to derive the stabilization conditions for TS fuzzy control systems [9],[10].

Very recently, the TS fuzzy models are used to describe the delayed fuzzy neural networks [11]-[13]. In [11], the global exponential stability in the mean square for the stochastic fuzzy HNNs with time-varying delay was studied by using the Lyapunov-Krasovskii approach. In [12], the globally robustly asymptotically stable conditions were presented for the uncertain fuzzy bi-directional associative memories (BAM) neural networks with time-varying delays. In [13], the global exponential stability problem of TS fuzzy cellular neural networks with time-varying delay was investigated based on the Lyapunov functional theory and linear matrix inequality techniques. However, most of the existing results for the delayed fuzzy neural networks are dedicated to delay-independent conditions. Furthermore, the works in [11]-[13] are based on a single LKF which further increases the conservatism of the results.

Motivated by the above discussion, the aim of this paper is to study the delay-dependent exponential stability for the fuzzy HNNs with time-varying delay by using a fuzzy LKF approach. A free-weighting matrix method combining with LMI techniques is employed to derive some new delay-dependent exponential stability criteria for the fuzzy HNNs. In contrast to the existing methods [11],[12], the proposed method reduces the conservatism of the stability results from three main aspects. The first one is that a fuzzy LKF is employed to further reduce the conservatism of the results. The second is that neither any model transformation nor any bounding technique for bounding cross terms is needed in the derivation processes. The third is that the time derivative of time-varying delay must be smaller than one is released in the proposed scheme. Moreover, the derived conditions are expressed in terms of linear matrix inequalities (LMIs), which can be checked numerically very efficiently via the LMI toolbox.

Notations: Throughout this paper, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space, and $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices. I denotes the identity matrix with appropriate dimensions and $\mathrm{diag}(\cdot)$ denotes the diagonal matrix. X^T and X^{-1} denote respectively the transpose and

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This work is supported by the National Natural Science Foundation of China (No. 60674092), High-tech R & D Program of Jiangsu (Industry) (No. BG2006010) and Program for Innovative Research Team of Jiangnan University.

The authors are with School of Communication and Control Engineering, Jiangnan University, Wuxi, 214122, P.R.China victory8209@sina.com(L. Sheng); yhz_jn@163.com(H. Yang)

978-1-4244-4524-0/09/$25.00 ©2009 AACC

the inverse of any square matrix X. We use $X > 0$ ($X < 0$) to denote a positive- (negative-) definite matrix X. $\|\cdot\|$ is the Euclidean norm in $\mathcal{R}^n$. $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) denotes the maximum (minimum) eigenvalue of X. The symbol $*$ is used to denote a matrix which can be inferred by symmetry.

II. MODEL DESCRIPTION AND PRELIMINARIES

The model of Hopfield neural networks with time-varying delay can be expressed as follows:

$$\dot{u}_i(t) = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t - \tau_j(t))) + J_i, \tag{1}$$

where $u_i(t) (i = 1, 2, \ldots, n)$ is the state variable of the ith neuron at time t; $d_i > 0$ represents the passive decay rate; a_{ij} is the synaptic connection weight; $g_j(\cdot)$ is the activation function of the neuron; J_i denotes the external input; $\tau_j(t)$ represents the time-varying delay of neural networks satisfying $0 < \tau_j(t) \leq \tau$, and $\dot{\tau}_j(t) \leq \sigma$, where τ and σ are positive constants.

Throughout this paper, we assume that
(H) There exists a positive diagonal matrix $L = \mathrm{diag}(l_1, l_2, \ldots, l_n) > 0$ such that

$$|g_i(\xi_1) - g_i(\xi_2)| \leq l_i |\xi_1 - \xi_2|,$$

for all $\xi_1, \xi_2 \in \mathcal{R}$, $i = 1, 2, \ldots, n$.

It is reasonable to assume that the neural network (1) has only one equilibrium point [21], denoted by $u^* = [u_1^*, u_2^*, \cdots, u_n^*]^T$. We shift the equilibrium to the origin by the transformation $x(t) = u(t) - u^*$, which yields the following system

$$\frac{dx(t)}{dt} = -Dx(t) + Af(x(t - \tau(t))), \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathcal{R}^n$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $A = (a_{ij})_{n \times n}$, $f(x) = [f_1(x_1), f_2(x_2), \ldots, f_n(x_n)]^T \in \mathcal{R}^n$ with $f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$ $(i = 1, 2, \ldots, n)$. Under the assumption (H), it is easy to get $|f_i(x_i(t))| \leq l_i |x_i(t)|$ $(i = 1, 2, \ldots, n)$.

The continuous fuzzy system was proposed to represent a nonlinear system [11]. The system dynamics can be captured by a set of fuzzy rules which characterize local correlations in the state space. Each local dynamic described by the fuzzy **IF-THEN** rule has the property of linear input-output relation. In the following section, we will consider a general class of fuzzy Hopfield neural networks with time-varying delay based on the TS fuzzy model concept. The k th rule of this TS fuzzy model is of the following form:

Plant Rule k :
IF $\theta_1(t)$ **is** η_1^k **and** $\cdots$ **and** $\theta_p(t)$ **is** η_p^k
THEN

$$\dot{x}(t) = -D_k x(t) + A_k f(x(t - \tau(t))), \quad \forall t \geq 0, \tag{3}$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \tag{4}$$

where $k \in \mathcal{S} = \{1, 2, \ldots, r\}$, η_i^k $(i = 1, 2, \ldots, p)$ is the fuzzy set, $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]^T$ is the premise variable vector, r is the number of **IF-THEN** rules.

The final output of the fuzzy system is inferred as follows:

$$\dot{x}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) \big[- D_k x(t) + A_k f(x(t - \tau(t))) \big], \tag{5}$$

where

$$\mu_k(\theta(t)) = \frac{v_k(\theta(t))}{\sum_{j=1}^{r} v_j(\theta(t))}, \quad v_k(\theta(t)) = \prod_{j=1}^{p} \eta_j^k(\theta_j(t)),$$

in which $\eta_j^k(\theta_j(t))$ is the grade of membership of $\theta_j(t)$ in η_j^k. According to the theory of fuzzy sets, we have

$$v_k(\theta(t)) \geq 0, \quad k = 1, 2, \ldots, r, \quad \sum_{k=1}^{r} v_k(\theta(t)) > 0,$$

for all t. Therefore, it implies

$$\mu_k(\theta(t)) \geq 0, \quad k = 1, 2, \ldots, r, \quad \sum_{k=1}^{r} \mu_k(\theta(t)) = 1,$$

for all t.

Definition 1. The system (5) is said to be globally exponentially stable with a convergence α if there exist constants $\alpha > 0$ and $\nu \geq 1$ such that

$$\|x(t)\| \leq \nu \sup_{-\tau \leq \vartheta \leq 0} \|x(\vartheta)\| e^{-\alpha t} \quad \text{for all } t \geq 0.$$

In order to confirm that the origin of (5) is globally exponentially stable, let $\hat{x}(t) = e^{\alpha t} x(t)$ and the dynamics of (3) can be transformed into the following form:

Plant Rule k :
IF $\theta_1(t)$ **is** η_1^k **and** $\cdots$ **and** $\theta_p(t)$ **is** η_p^k
THEN

$$\begin{aligned} \dot{\hat{x}}(t) &= -(D_k - \alpha I)\hat{x}(t) + A_k \hat{f}(\hat{x}(t - \tau(t))) \\ &= -C_k \hat{x}(t) + A_k \hat{f}(\hat{x}(t - \tau(t))), \quad \forall t \geq 0, \end{aligned} \tag{6}$$

$$\hat{x}(t) = \hat{\phi}(t), \quad t \in [-\tau, 0], \tag{7}$$

where $C_k = D_k - \alpha I$, $\hat{f}(\hat{x}(t - \tau(t))) = e^{\alpha t} f(x(t - \tau(t)))$ and $\hat{\phi}(t) = e^{\alpha t}\phi(t), t \in [-\tau, 0]$. According to the assumption (H), we have

$$\begin{aligned} \hat{f}^T(\hat{x}(t - \tau(t)))\hat{f}(\hat{x}(t - \tau(t))) &= \|\hat{f}(\hat{x}(t - \tau(t)))\|^2 \\ &\leq e^{2\alpha\tau} \hat{x}^T(t - \tau(t)) L^2 \hat{x}(t - \tau(t)), \end{aligned} \tag{8}$$

where $L = \mathrm{diag}(l_1, l_2, \ldots, l_n) > 0$.

The final output of the fuzzy system is inferred as follows:

$$\dot{\hat{x}}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) \big[- C_k \hat{x}(t) + A_k \hat{f}(\hat{x}(t - \tau(t))) \big]. \tag{9}$$

For convenience, we set

$$\bar{C}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) C_k, \quad \bar{A}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) A_k,$$

then the system (9) can be rewritten as

$$\dot{\hat{x}}(t) = -\bar{C}(t)\hat{x}(t) + \bar{A}(t)\hat{f}(\hat{x}(t - \tau(t))). \tag{10}$$

To give our main results in the next section, we need the following lemmas.

Lemma 1. For any vectors $a, b \in \mathcal{R}^n$, the inequality

$$2a^T b \leq a^T X a + b^T X^{-1} b$$

holds, where X is any positive matrix (i.e., $X > 0$).

Lemma 2 (Schur Complement [26]). Given constant matrices $\Omega_1, \Omega_2, \Omega_3$, where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

III. MAIN RESULTS

In this section, we will derive some delay-dependent criteria for the global exponential stability of the delayed fuzzy system (10) based on a fuzzy LKF.

Theorem 1. For the system (10), suppose (H) hold. Given scalars τ and σ, the equilibrium point of system (10) is globally exponentially stable with a convergence rate $\alpha > 0$, if there exist a matrix $P = P^T > 0$, time-varying matrices $\bar{Q}(t) = \bar{Q}^T(t) \geq 0$, $\bar{R}(s) = \bar{R}^T(s) \geq 0, s \in [t - \tau(t), t]$, $\bar{X}_l(t)$, $\bar{Y}_l(t)$, $\bar{Z}_{lm}(t)$, $l \leq m = 1, 2, 3, 4$ and a positive constant η satisfying the following inequalities:

$$\begin{aligned} &\Psi(t) \\ &= \begin{bmatrix} \Omega_{11}(t) & \Omega_{12}(t) & \Omega_{13}(t) & \Omega_{14}(t) \\ * & \Omega_{22}(t) & \Omega_{23}(t) & \Omega_{24}(t) \\ * & * & \Omega_{33}(t) & \Omega_{34}(t) \\ * & * & * & \Omega_{44}(t) \end{bmatrix} \\ &+\tau \begin{bmatrix} \bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) \\ * & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) \\ * & * & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) \\ * & * & * & \bar{Z}_{44}(t) \end{bmatrix} < 0 \end{aligned} \quad (11)$$

$$\begin{aligned} &\Phi(t, s) \\ &= \begin{bmatrix} \bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) & \bar{X}_1(t) \\ * & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) & \bar{X}_2(t) \\ * & * & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) & \bar{X}_3(t) \\ * & * & * & \bar{Z}_{44}(t) & \bar{X}_4(t) \\ * & * & * & * & \bar{R}(s) \end{bmatrix} \geq 0, \\ &s \in [t - \tau(t), t], \end{aligned} \quad (12)$$

for all t, where

$$\begin{aligned} \Omega_{11}(t) &= \bar{Q}(t) + \bar{X}_1(t) + \bar{X}_1^T(t) - \bar{Y}_1(t)\bar{C}(t) \\ &\quad -\bar{C}(t)\bar{Y}_1^T(t), \\ \Omega_{12}(t) &= -\bar{X}_1(t) + \bar{X}_2^T(t) - \bar{C}(t)\bar{Y}_2^T(t), \\ \Omega_{13}(t) &= \bar{X}_3^T(t) + \bar{Y}_1(t)\bar{A}(t) - \bar{C}(t)\bar{Y}_3^T(t), \\ \Omega_{14}(t) &= P + \bar{X}_4^T(t) - \bar{Y}_1(t) - \bar{C}(t)\bar{Y}_4^T(t), \\ \Omega_{22}(t) &= -(1 - \sigma)Q(t - \tau(t)) - \bar{X}_2(t) - \bar{X}_2^T(t) \\ &\quad +\eta e^{2\alpha\tau} L^2, \\ \Omega_{23}(t) &= -\bar{X}_3^T(t) + \bar{Y}_2(t)\bar{A}(t), \\ \Omega_{24}(t) &= -\bar{X}_4^T(t) - \bar{Y}_2(t), \\ \Omega_{33}(t) &= -\eta I + \bar{Y}_3(t)\bar{A}(t) + \bar{A}^T(t)\bar{Y}_3^T(t), \\ \Omega_{34}(t) &= -\bar{Y}_3(t) + \bar{A}^T(t)\bar{Y}_4^T(t), \\ \Omega_{44}(t) &= \tau\bar{R}(t) - \bar{Y}_4(t) - \bar{Y}_4^T(t). \end{aligned}$$

Proof. Define the following free fuzzy weighting matrices:

$$\bar{X}_l(t) = \sum_{k=1}^{r} \mu_k(\theta(t))X_{lk}, \quad \bar{Y}_l(t) = \sum_{k=1}^{r} \mu_k(\theta(t))Y_{lk},$$

$$\bar{Z}_{lm}(t) = \sum_{k=1}^{r} \mu_k(\theta(t))Z_{lmk},$$

where $X_{lk} \in \mathcal{R}^{n\times n}$, $Y_{lk} \in \mathcal{R}^{n\times n}$ and $Z_{lmk} \in \mathcal{R}^{n\times n}$, $l \leq m = 1, 2, 3, 4,\ k \in \mathcal{S}$.

From Leibniz-Newton formula and (8), we have

$$\int_{t-\tau(t)}^{t} \dot{\hat{x}}(s)ds = \hat{x}(t) - \hat{x}(t - \tau(t)), \quad (13)$$

$$\begin{aligned} &e^{2\alpha\tau}\hat{x}^T(t - \tau(t))L^2\hat{x}(t - \tau(t)) \\ &\quad -\hat{f}^T(\hat{x}(t - \tau(t)))\hat{f}(\hat{x}(t - \tau(t))) \geq 0. \end{aligned} \quad (14)$$

Therefore, by considering (10) and (14), for some time-varying matrices $\bar{X}_l(t)$ and $\bar{Y}_l(t)$, $l = 1, 2, 3, 4$, we obtain

$$\begin{aligned} \gamma_1(t) &= 2\big[\hat{x}^T(t)\bar{X}_1(t) + \hat{x}^T(t - \tau(t))\bar{X}_2(t) \\ &\quad +\hat{f}^T(\hat{x}(t - \tau(t)))\bar{X}_3(t) + \dot{\hat{x}}^T(t)\bar{X}_4(t)\big] \\ &\quad \times \left[\hat{x}(t) - \hat{x}(t - \tau(t)) - \int_{t-\tau(t)}^{t} \dot{\hat{x}}(s)ds\right] \\ &\equiv 0, \end{aligned} \quad (15)$$

$$\begin{aligned} \gamma_2(t) &= 2\big[\hat{x}^T(t)\bar{Y}_1(t) + \hat{x}^T(t - \tau(t))\bar{Y}_2(t) \\ &\quad +\hat{f}^T(\hat{x}(t - \tau(t)))\bar{Y}_3(t) + \dot{\hat{x}}^T(t)\bar{Y}_4(t)\big] \\ &\quad \times \left[-\bar{C}(t)\hat{x}(t) + \bar{A}(t)\hat{f}(\hat{x}(t - \tau(t))) - \dot{\hat{x}}(t)\right] \\ &\equiv 0. \end{aligned} \quad (16)$$

Since (12) implies that

$$\bar{Z}(t) = \begin{bmatrix} \bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) \\ * & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) \\ * & * & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) \\ * & * & * & \bar{Z}_{44}(t) \end{bmatrix} \geq 0,$$

we have the following inequality:

$$\tau\xi^T(t)\bar{Z}(t)\xi(t) - \int_{t-\tau(t)}^{t} \xi^T(t)\bar{Z}(t)\xi(t)ds \geq 0, \quad (17)$$

where $\xi^T(t) = \left[\hat{x}^T(t)\ \hat{x}^T(t - \tau(t))\ \hat{f}^T(\hat{x}(t - \tau(t)))\ \dot{\hat{x}}^T(t)\right]$.

Consider the following fuzzy LKF for the system (10):

$$\begin{aligned} V(t) &= \hat{x}^T(t)P\hat{x}(t) + \int_{t-\tau(t)}^{t} \hat{x}^T(s)\bar{Q}(s)\hat{x}(s)ds \\ &\quad + \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{\hat{x}}^T(s)\bar{R}(s)\dot{\hat{x}}(s)dsd\beta, \end{aligned} \quad (18)$$

where $\bar{Q}(s) = \sum_{k=1}^{r} \mu_k(\theta(s))Q_k$ and $\bar{R}(s) = \sum_{k=1}^{r} \mu_k(\theta(s))R_k$ are fuzzy weighting matrices which include the membership functions, $0 < P = P^T \in \mathcal{R}^{n\times n}$, $0 < Q_k = Q_k^T \in \mathcal{R}^{n\times n}$ and $0 < R_k = R_k^T \in \mathcal{R}^{n\times n}, k \in \mathcal{S}$. As discussed in [10], a single matrix P is used in the first term of

(18) instead of a fuzzy weighting matrix $\sum_{k=1}^{r}\mu_k(\theta(t))P_k$. The reason of such construction is to avoid the difficulty of determining the upper bound of $|\dot{\mu}_k(\theta(t))|,\ k\in\mathcal{S}$ in the stability analysis.

With (14)-(17), the time derivative of $V(t)$ along the trajectory of (10) is computed as

$$\begin{aligned}\dot V(t) &= 2\hat x^T(t)P\dot{\hat x}(t)+\hat x^T(t)\bar Q(t)\hat x(t)\\ &\quad -(1-\dot\tau(t))\hat x^T(t-\tau(t))\bar Q(t-\tau(t))\hat x(t-\tau(t))\\ &\quad +\tau\dot{\hat x}^T(t)\bar R(t)\dot{\hat x}(t)-\int_{t-\tau}^{t}\dot{\hat x}^T(s)\bar R(s)\dot{\hat x}(s)ds\\ &\le 2\hat x^T(t)P\dot{\hat x}(t)+\hat x^T(t)\bar Q(t)\hat x(t)\\ &\quad -(1-\sigma)\hat x^T(t-\tau(t))\bar Q(t-\tau(t))\hat x(t-\tau(t))\\ &\quad +\tau\dot{\hat x}^T(t)\bar R(t)\dot{\hat x}(t)-\int_{t-\tau(t)}^{t}\dot{\hat x}^T(s)\bar R(s)\dot{\hat x}(s)ds\\ &\quad +\gamma_1(t)+\eta e^{2\alpha\tau}\hat x^T(t-\tau(t))L^2\hat x(t-\tau(t))\\ &\quad +\gamma_2(t)-\eta\hat f^T(\hat x(t-\tau(t)))\hat f(\hat x(t-\tau(t)))\\ &\quad +\tau\xi^T(t)\bar Z(t)\xi(t)-\int_{t-\tau(t)}^{t}\xi^T(t)\bar Z(t)\xi(t)ds\\ &= \xi^T(t)\Psi(t)\xi(t)-\int_{t-\tau(t)}^{t}\zeta^T(t,s)\Phi(t,s)\zeta(t,s)ds,\end{aligned}\tag{19}$$

where $\xi^T(t)$ is the same as in (18) and $\zeta^T(t,s)=\begin{bmatrix}\xi^T(t) & \dot{\hat x}^T(s)\end{bmatrix}$.

Obviously, $\Psi(t)<0$ and $\Phi(t,s)\ge 0$ as defined in (11) and (12) imply that $\dot V(t)<0$ for any $\hat x(t)\ne 0$. It follows that

$$V(t)\le V(0).\tag{20}$$

According to (18), we have

$$\begin{aligned}V(0) &= \hat x^T(0)P\hat x(0)+\int_{-\tau(0)}^{0}\hat x^T(s)\bar Q(s)\hat x(s)ds\\ &\quad +\int_{-\tau}^{0}\int_{\beta}^{0}\dot{\hat x}^T(s)\bar R(s)\dot{\hat x}(s)dsd\beta,\\ &\le \lambda_{\max}(P)\|\hat\phi\|^2+\lambda_{\max}(Q_M)\int_{-\tau(0)}^{0}\hat x^T(s)\hat x(s)ds\\ &\quad +\lambda_{\max}(R_M)\int_{-\tau}^{0}\int_{\beta}^{0}\dot{\hat x}^T(s)\dot{\hat x}(s)dsd\beta,\end{aligned}\tag{21}$$

where $Q_M=\sup_{-\tau\le s\le 0}\bar Q(s)$, $R_M=\sup_{-\tau\le s\le 0}\bar R(s)$ and $\hat\phi=\sup_{-\tau\le s\le 0}\|\hat x(s)\|$.

It follows from Lemma 1 that

$$\begin{aligned}&\dot{\hat x}^T(s)\dot{\hat x}(s)\\ &= \left[-\bar C(s)\hat x(s)+\bar A(s)\hat f(\hat x(s-\tau(s)))\right]^T\\ &\quad\times\left[-\bar C(s)\hat x(s)+\bar A(s)\hat f(\hat x(s-\tau(s)))\right]\\ &\le 2\left[\hat x^T(s)\bar C^T(s)\bar C(s)\hat x(s)\right.\\ &\quad\left.+\hat f^T(\hat x(s-\tau(s)))\bar A^T(s)\bar A(s)\hat f(\hat x(s-\tau(s)))\right]\\ &\le 2\left[\lambda_{\max}(C_M^TC_M)+\lambda_{\max}(A_M^TA_M)\lambda_{\max}(L^2)\right]\|\hat\phi\|^2,\end{aligned}\tag{22}$$

where $C_M=\sup_{-\tau\le s\le 0}\bar C(s)$ and $A_M=\sup_{-\tau\le s\le 0}\bar A(s)$.

Thus

$$\begin{aligned}V(0) &\le \lambda_{\max}(P)\|\hat\phi\|^2+\tau\lambda_{\max}(Q_M)\|\hat\phi\|^2\\ &\quad +2\tau^2\lambda_{\max}(R_M)\left[\lambda_{\max}(C_M^TC_M)\right.\\ &\quad\left.+\lambda_{\max}(A_M^TA_M)\lambda_{\max}(L^2)\right]\|\hat\phi\|^2\\ &= \Lambda\|\hat\phi\|^2,\end{aligned}\tag{23}$$

where $\Lambda=\lambda_{\max}(P)+\tau\lambda_{\max}(Q_M)+2\tau^2\lambda_{\max}(R_M)\left[\lambda_{\max}(C_M^TC_M)+\lambda_{\max}(A_M^TA_M)\lambda_{\max}(L^2)\right]$.

On the other hand

$$V(t)\ge\hat x^T(t)P\hat x(t)\ge\lambda_{\min}(P)\|\hat x(t)\|^2.\tag{24}$$

Therefore, we have

$$\lambda_{\min}(P)\|\hat x(t)\|^2\le\Lambda\|\hat\phi\|^2.\tag{25}$$

Furthermore, from $\hat x(t)=e^{\alpha t}x(t)$, we can conclude the following result:

$$\|x(t)\|\le\sqrt{\frac{\Lambda}{\lambda_{\min}(P)}}\|\phi\|e^{-\alpha t}.\tag{26}$$

where $\phi=\sup_{-\tau\le s\le 0}\|x(s)\|$. It is clear that $\sqrt{\frac{\Lambda}{\lambda_{\min}(P)}}\ge 1$ and by Definition 1, the system (10) is globally exponentially stable with convergence rate α. The proof is completed.

If we restrict $\bar R(s)>0,\ s\in[t-\tau(t),t]$, then the matrices $\bar Z_{lm}(t), l\le m=1,2,3,4$ and the condition (12) can be eliminated from Theorem 1 and we have the following corollary.

Corollary 1. For the system (10), suppose (H) hold. Given scalars τ and σ, the equilibrium point of system (10) is globally exponentially stable with a convergence rate $\alpha>0$, if there exist a matrix $P=P^T>0$, time-varying matrices $\bar Q(t)=\bar Q^T(t)\ge 0$, $\bar R(s)=\bar R^T(s)>0, s\in[t-\tau(t),t]$, $\bar X_l(t),\ \bar Y_l(t),\ l=1,2,3,4$ and a positive constant η satisfying the following inequality:

$$\begin{bmatrix}\Omega_{11}(t) & \Omega_{12}(t) & \Omega_{13}(t) & \Omega_{14}(t) & \tau\bar X_1(t)\\ * & \Omega_{22}(t) & \Omega_{23}(t) & \Omega_{24}(t) & \tau\bar X_2(t)\\ * & * & \Omega_{33}(t) & \Omega_{34}(t) & \tau\bar X_3(t)\\ * & * & * & \Omega_{44}(t) & \tau\bar X_4(t)\\ * & * & * & * & -\tau\bar R(s)\end{bmatrix}<0,$$
$$s\in[t-\tau(t),t],\tag{27}$$

for all t, where $\Omega_{ij}(t), i\le j=1,2,3,4$ are the same as in (11).

Proof. In the following, we take

$$\begin{bmatrix} \bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) \\ * & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) \\ * & * & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) \\ * & * & * & \bar{Z}_{44}(t) \end{bmatrix} = \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \\ \bar{X}_3(t) \\ \bar{X}_4(t) \end{bmatrix} \bar{R}^{-1}(s) \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \\ \bar{X}_3(t) \\ \bar{X}_4(t) \end{bmatrix}^T \geq 0. \quad (28)$$

Obviously, (28) ensures $\Phi(t,s) \geq 0$ because

$$\begin{bmatrix} \bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) & \bar{X}_1(t) \\ * & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) & \bar{X}_2(t) \\ * & * & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) & \bar{X}_3(t) \\ * & * & * & \bar{Z}_{44}(t) & \bar{X}_4(t) \\ * & * & * & * & \bar{R}(s) \end{bmatrix} = \begin{bmatrix} \bar{X}_1(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{X}_2(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{X}_3(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{X}_4(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{R}^{\frac{1}{2}}(s) \end{bmatrix} \begin{bmatrix} \bar{X}_1(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{X}_2(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{X}_3(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{X}_4(t)\bar{R}^{-\frac{1}{2}}(s) \\ \bar{R}^{\frac{1}{2}}(s) \end{bmatrix}^T \geq 0.$$

According to Lemma2, the inequality (11) with the choice of (28) is equivalent to (27). The proof is completed.

Remark 1. It is noted that restricting $\bar{R}(s) > 0$, $s \in [t-\tau(t), t]$ for the stability condition of Theorem 1 may increase the conservatism of the overall condition. However, we can obtain a LMI-based exponential stability condition with fewer number of involved decision variables which definitely accelerates computation from Corollary 1.

Theorem 2. For the system (10), suppose (H) hold. Given scalars τ and σ, the equilibrium point of system (10) is globally exponentially stable with a convergence rate $\alpha > 0$, if there exist matrices $P = P^T > 0$, $Q_k = Q_k^T \geq 0$, $R_k = R_k^T > 0$, X_{lk}, Y_{lk}, $l = 1,2,3,4$, $k \in \mathcal{S}$ and a positive constant η satisfying the following LMIs:

$$\Xi_{\rho\varrho kk} < 0, \quad \rho, \varrho, k \in \mathcal{S}, \quad (29)$$

$$\frac{1}{r-1}\Xi_{\rho\varrho kk} + \frac{1}{2}\big(\Xi_{\rho\varrho km} + \Xi_{\rho\varrho mk}\big) < 0, \\ \rho, \varrho, k, m \in \mathcal{S}, \quad k \neq m, \quad (30)$$

where

$$\Xi_{\rho\varrho km} = \begin{bmatrix} Q_k + \Gamma_{11,km} & \Gamma_{12,km} & \Gamma_{13,km} & P + \Gamma_{14,km} & \tau X_{1k} \\ * & -(1-\sigma)Q_\rho + \Gamma_{22,k} & \Gamma_{23,km} & \Gamma_{24,km} & \tau X_{2k} \\ * & * & -\eta I + \Gamma_{33,km} & \Gamma_{34,km} & \tau X_{3k} \\ * & * & * & \tau R_k + \Gamma_{44,m} & \tau X_{4k} \\ * & * & * & * & -\tau R_\varrho \end{bmatrix}, \quad (31)$$

and

$$\begin{aligned} \Gamma_{11,km} &= X_{1k} + X_{1k}^T - Y_{1m}C_k + C_k Y_{1m}^T, \\ \Gamma_{12,km} &= -X_{1k} + X_{2k}^T - C_k Y_{2m}^T, \\ \Gamma_{13,km} &= X_{3k}^T + Y_{1m}A_k - C_k Y_{3m}^T, \\ \Gamma_{14,km} &= X_{4k}^T - Y_{1m} - C_k Y_{4m}^T, \\ \Gamma_{22,k} &= -X_{2k} - X_{2k}^T + \eta e^{2\alpha\tau} L^2, \\ \Gamma_{23,km} &= -X_{3k}^T + Y_{2m}A_k, \\ \Gamma_{24,km} &= -X_{4k}^T - Y_{2m}, \\ \Gamma_{33,km} &= Y_{3m}A_k + A_k^T Y_{3m}^T, \\ \Gamma_{34,km} &= -Y_{3m} + A_k^T Y_{4m}^T, \\ \Gamma_{44,m} &= -Y_{4m} - Y_{4m}^T. \end{aligned}$$

Proof. The inequality (27) can be rewritten as

$$\sum_{\rho=1}^{r}\sum_{\varrho=1}^{r}\sum_{k=1}^{r}\sum_{m=1}^{r} \mu_\rho(\theta(t-\tau(t)))\mu_\varrho(\theta(s))\mu_k(\theta(t)) \\ \times \mu_m(\theta(t))\Xi_{\rho\varrho km} < 0, \quad s \in [t-\tau(t), t]. \quad (32)$$

According to the Theorem 2.2 in [8], with $\Xi_{\rho\varrho km}$ given in (31), if the conditions (29) and (30) hold, then (32) is fulfilled. Therefore, it follows from Corollary 1 that the equilibrium of the system (10) is globally exponentially stable with the convergence rate α.

Remark 2. The number of LMIs in Theorem 2 is $r^3 + r^3(r-1)$, which may be lead to a great computational effort. However, we can get a trade-off between the increase of conservatism and the reduction of number in LMIs by choosing certain matrices. For instance, by taking $Q_k = Q$ or $R_k = R$, $k \in \mathcal{S}$, the number of LMIs is $r^2 + r^2(r-1)$. If we take $Q_k = Q$ and $R_k = R$, $k \in \mathcal{S}$, then the number is $r + r(r-1)$, which would be greatly reduced. Therefore, we have the following corollary.

Corollary 2. For the system (10), suppose (H) hold. Given scalars τ and σ, the equilibrium point of system (10) is globally exponentially stable with a convergence rate $\alpha > 0$, if there exist matrices $P = P^T > 0$, $Q = Q^T \geq 0$, $R = R^T > 0$, X_{lk}, Y_{lk}, $l = 1,2,3,4$, $k \in \mathcal{S}$ and a positive constant η satisfying the following LMIs:

$$\Xi_{kk} < 0, \quad k \in \mathcal{S}, \quad (33)$$

$$\frac{1}{r-1}\Xi_{kk} + \frac{1}{2}\big(\Xi_{km} + \Xi_{mk}\big) < 0, \ k, m \in \mathcal{S}, \ k \neq m, \quad (34)$$

where

$$\Xi_{km} = \begin{bmatrix} Q + \Gamma_{11,km} & \Gamma_{12,km} & \Gamma_{13,km} & P + \Gamma_{14,km} & \tau X_{1k} \\ * & -(1-\sigma)Q + \Gamma_{22,k} & \Gamma_{23,km} & \Gamma_{24,km} & \tau X_{2k} \\ * & * & -\eta I + \Gamma_{33,km} & \Gamma_{34,km} & \tau X_{3k} \\ * & * & * & \tau R + \Gamma_{44,m} & \tau X_{4k} \\ * & * & * & * & -\tau R \end{bmatrix} \quad (35)$$

and $\Gamma_{ij,km}$, $i \leq j = 1,2,3,4$ are the same as in (31).

IV. A SIMULATION EXAMPLE

In this section, we present a numerical example to demonstrate the effectiveness of the proposed results.

Example 1. Let $r = 2$. Consider the following TS fuzzy HNN:

Plant Rule k :
IF $\theta_1(t)$ **is** η_1^k **and** $\cdots$ **and** $\theta_p(t)$ **is** η_p^k
THEN

$$\dot{x}(t) = -D_k x(t) + A_k f(x(t-\tau(t))),\ \forall t \geq 0, \quad (36)$$

where $\eta_i^k (i = 1, 2, \ldots, p)$ is the fuzzy set, $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]^T$ is the premise variable vector. The activation function is described by $f(x) = \tanh(x)$, and the time-varying delay is assumed as $\tau(t) = 0.5\sin(3t) + 1$ with $\tau = 1.5$ and $\sigma = 1.5$. Obviously, the assumption (H) is satisfied with $L = \mathrm{diag}(1, 1)$. The fuzzy HNN can be described by

$$\dot{\hat{x}}(t) = \sum_{k=1}^{2} \mu_k(\theta(t)) \big[-C_k \hat{x}(t) + A_k \hat{f}(\hat{x}(t-\tau(t))) \big], \quad (37)$$

where

$$C_1 = D_1 - \alpha I,\ D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.6 & -0.5 \\ 0.4 & 0.7 \end{bmatrix},$$

$$C_2 = D_2 - \alpha I,\ D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.5 & 0.2 \\ 0.8 & -0.5 \end{bmatrix}.$$

The fuzzy membership functions are selected as

$$\mu_1(\hat{x}_1, \hat{x}_2) = \frac{1}{\exp(-\hat{x}_1 - \hat{x}_2)}, \quad \mu_2 = 1 - \mu_1.$$

Applying the conditions in Corollary 2 with convergence rate $\alpha = 0.05$, we use the Matlab LMI Control Toolbox to solve the LMIs in (33) and (34), and obtain the feasible solution as follows:

$$\eta = 2.1360 \times 10^3,$$

$$P = 10^4 \times \begin{bmatrix} 1.2770 & 0 \\ 0 & 1.2770 \end{bmatrix},$$

$$Q = 10^3 \times \begin{bmatrix} 1.2588 & 0.0010 \\ 0.0010 & 1.2584 \end{bmatrix},$$

$$R = 10^4 \times \begin{bmatrix} 1.7254 & 0.0006 \\ 0.0006 & 1.7251 \end{bmatrix}.$$

Therefore, all the conditions of Corollary 2 in this paper are satisfied, which imply the solutions of system (36) are globally exponentially stable. The constraint $\dot{\tau}_j(t) \leq \sigma < 1$ is essential in [11],[12], but it is not necessary in our results. Fig. 1 shows the response of the previous fuzzy system with the initial condition $[-1.2, 0.4]^T$.

V. CONCLUSION

In this paper, some delay-dependent exponential stability criteria are established for fuzzy Hopfield neural networks with time-varying delay by using the free-weighting matrix approach, the Leibniz-Newton formula and the Lyapunov method. A fuzzy LKF instead of a single LKF is employed to derive the proposed delay-dependent results, which reduce the conservatism of the results. The stability conditions are presented in terms of linear matrix inequalities. Finally, a numerical example is provided to illustrate the effectiveness of the derived results.

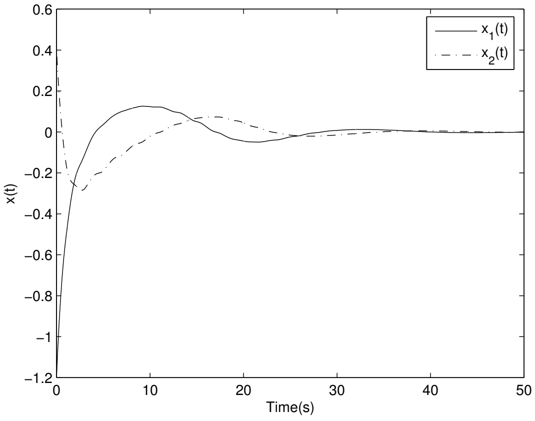

Fig.1. Responses of the state vector $x(t)$.

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