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Implementation of FIR Control for H_∞ Output Feedback Stabilization of Linear Systems

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***Abstract*— In this paper, finite impulse response (FIR) control is addressed for H_∞ output feedback stabilization of linear systems. The problem we deal with is the construction of an output feedback controller with a certain finite impulse response structure such that the resulting closed-loop system is asymptotically stable and a prescribed H_∞ norm bound constraint is guaranteed. Some solvability conditions are suggested in this paper. Sufficient conditions are derived to obtain a suboptimal solution of the H_∞ FIR control problem via convex optimization. Also, an equivalent condition for the existence of H_∞ FIR control is presented in the set of linear matrix inequalities and a reciprocal matrices equality constraint. An effective computational algorithm involving linear matrix inequalities is suggested to solve a concave minimization problem characterizing a local optimal solution of the H_∞ FIR control problem. Numerical examples demonstrate the validity of the proposed H_∞ FIR control and the numerical efficiency of the proposed algorithm for FIR control.**

I. INTRODUCTION

In real control applications, it often happens that the states of the plant are not completely available and output feedback control has to be synthesized in many cases. In general, output feedback control often consists of a state feedback controller and an observer or a filter that estimates all the states from measured inputs and outputs [1]. It is noted that the filters are conventionally of the infinite impulse response (IIR) and hence output feedback control has the IIR structure of the form

$$u(k) = \sum_{i=0}^{k} H_{k-i} y(i) \tag{1}$$

for some gain function H_{k-i} and the filter initial state $\hat{x}(k_0) = 0$. That is, output feedback control of the form (1) uses all past measurements $[k_0, k]$ to compute the current control signal. Recently, by means of valuable numerical tools and semidefinite programming in particular [2], output feedback control is directly computed in the form of *full-order* or *reduced-order dynamic output feedback* without lunching of a filter [3]–[5]. Note that these existing dynamic output feedback controllers presented in the literature are also reduced to the IIR structure of the form (1).

Finite impulse response (FIR) estimators or filters have been exploited with much attention in signal processing, image processing, estimation and control areas due to the following advantages [6]–[9]: FIR filters make use of finite measurements on the most recent time interval to avoid long calculation that arises from large data sets as time goes. It is generally accepted in the literature that the FIR structure in filtering leads to more robustness to temporary modeling uncertainties and computational round-off errors than the IIR type of structure owing to the inherent properties of the FIR structure [6], [7], [9]. Likewise, the authors in [10] have recently applied ideas from the FIR filtering to the synthesis of control. That is, output feedback control using finite recent measured inputs and outputs, which is called receding horizon finite memory control, was constructed under the separation principle for a receding horizon LQG criterion. Note that this method needs a large number of both input and output measurements for least square FIR filtering; thus it requires storing a large amount of measurements in the memory at each sample time. Our work is similar in sprit to the previous work; however, this paper is more general in scope and focused on H_∞ control above all, which has drawn much attention in the literature. A more flexible framework for a finite memory control solution, which can also deal with important issues in control, such as robustness for uncertain systems, time-delay in systems, multi-objective constraints, and so forth [3], [5], [11], [12], is provided based on semidefinite programming. Only output measurements are required and the number of required measurements is not necessarily larger than the dimension of the system in most cases since the proposed method does not involve state construction. It can be shown that the proposed output feedback controller can be more robust against temporary modeling uncertainties than finite memory control in [10]. In particular, unlike the paper [10], a priori knowledge on the state feedback optimal control solution and the optimal FIR filter structure are not required in the design to build output feedback control in this present paper.

An output feedback control scheme with the finite impulse response structure, called FIR control in this paper,

$$u(k) = \sum_{i=k-N+1}^{k} H_{k-i} y(i), \tag{2}$$

is addressed as a counterpart of the existing *full-order* or *reduced-order dynamic control* of the IIR type for H_∞ output feedback stabilization of linear discrete-time systems in the present paper. That is, the FIR type of control will be obtained without lunching of any filter or state estimator, while the output feedback controller in [10] is not in the form of (2) and needs both input and output measurements and is constructed under the separation principle, which involves least square state estimation. In addition to the new type of control, the objective of this paper is to present an efficient design approach for construction of an output feedback controller with a certain finite memory structure

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specified *a priori* such that the resulting closed-loop system is asymptotically stable and a prescribed H_∞ norm bound constraint is guaranteed. Sufficient conditions are derived to obtain a suboptimal solution of the H_∞ FIR control problem via convex optimization involving linear matrix constraints. Also, a necessary and sufficient condition for the existence of a desired H_∞ FIR controller are presented through the set of linear matrix inequalities (LMIs) and a reciprocal matrices equality constraint, which is nonconvex. Effective iterative algorithms involving LMIs are suggested to solve the nonconvex matrix inequalities characterizing a solution of the H_∞ FIR control problem. The validity of the proposed H_∞ FIR control will be demonstrated by comparisons with some existing output feedback control schemes, such as H_∞ full-order control and H_∞ PID control. Also the numerical efficiency of the proposed algorithm will be illustrated by comparisons with some existing optimization solvers such as PENBMI in [13]. The approach in this paper can be directly applied not only to different types of performance criteria, such as LQG and mixed H_2/H_∞, but also many types of systems, such as uncertain systems, time-delay systems, and Markovian systems without difficulty, whereas such an extension is limited under the framework of [10].

The outline of this paper is as follows. A new type of H_∞ FIR control is proposed for output feedback stabilization of linear systems in Section II. Explicit LMI relaxations for H_∞ FIR control are presented in Section III. A necessary and sufficient condition for H_∞ FIR control is presented together with efficient computational local search algorithms in Section IV. Numerical examples are included to demonstrate the advantages of the proposed FIR control scheme and the numerical efficiency of the proposed algorithm for the implementation of FIR control in Section V. Finally, conclusions are given in Section VI.

II. H_∞ Design of FIR Control

Consider a linear system described by

$$\begin{aligned}\Sigma:\ x(k+1) &= Ax(k) + B_w w(k) + B_u u(k),\\ z(k) &= C_z x(k) + D_{zw} w(k) + D_{zu} u(k),\\ y(k) &= C_y x(k) + D_{yw} w(k),\end{aligned} \tag{3}$$

where $x \in \mathbf{R}^{n_x}$ is the state of the plant, $u \in \mathbf{R}^{n_u}$ is the control signal, $w \in \mathbf{R}^{n_w}$ is the exogenous input, $y \in \mathbf{R}^{n_y}$ is the measured output, and $z \in \mathbf{R}^{n_z}$ is the controlled output. The problem considered here is the design of an output feedback controller of the following finite impulse response (FIR) structure

$$\begin{aligned}\Sigma_c:\ u(k) &= H_0 y(k) + H_1 y(k-1) + H_2 y(k-2) + \cdots\\ &+ H_{N-1} y(k-N+1)\\ &= \sum_{i=k-N+1}^{k} H_{k-i} y(i)\end{aligned} \tag{4}$$

for the system Σ, where $N \in \mathbf{N}$ is called the horizon and assumed to be an arbitrary finite number. Let us define H, $Y_{k-1,N-1}$ as

$$H = \begin{pmatrix} H_1 & H_2 & \dots & H_{N-1}\end{pmatrix},\quad Y_{k-1,N-1} = \begin{pmatrix} y(k-1)\\ y(k-2)\\ \vdots \\ y(k-N+1)\end{pmatrix}.$$

Note that $Y_{k-1,N-1}$ can be implemented by the memory stack of the following form

$$\psi(k+1) = A_\psi \psi(k) + B_\psi y(k), \tag{5}$$

where $\psi \in \mathbf{R}^{n_\psi}$ is the state of the memory, n_ψ is a preassigned size of the memory by $n_\psi = n_y(N-1)$, and A_ψ, B_ψ are given by

$$A_\psi = \begin{pmatrix} 0 & 0\\ \mathrm{diag}(\overbrace{I_{n_y},\dots,I_{n_y}}^{N-2}) & 0\end{pmatrix},\quad B_\psi = \begin{pmatrix} I_{n_y}\\ 0\end{pmatrix}. \tag{6}$$

The controller of the FIR structure (4), thus, can be rewritten as

$$u(k) = H_0 y(k) + HY_{k-1,N-1} = H_0 y(k) + H\psi(k). \tag{7}$$

If the memory stack order $n_\psi = 0$, namely, $N = 1$, Σ_c is a memoryless output feedback controller, also called static output feedback controller, which is the simplest feedback control scheme that can be implemented with a minimum hardware cost in practice. Notice that N is an arbitrary finite number and not necessarily to be larger than the dimension of the system in this paper. Let us define a system matrix $\mathscr{K}$ of the FIR controller Σ_c by

$$\mathscr{K} := \begin{pmatrix} A_\psi & B_\psi\\ \begin{pmatrix} H_1 & H_2 & \dots & H_{N-1}\end{pmatrix} & H_0\end{pmatrix}. \tag{8}$$

The closed-loop system is then given by the following state equations

$$\begin{aligned}\Sigma_{cl}:\ x_{cl}(k+1) &= A_{cl} x_{cl}(k) + B_{cl} w(k),\\ z(k) &= C_{cl} x_{cl}(k) + D_{cl} w(k),\end{aligned} \tag{9}$$

where $x_{cl} = \begin{pmatrix} x^T & \psi^T\end{pmatrix}^T$ and the system matrix data of the closed-loop system Σ_{cl} is given by

$$\begin{aligned}\left(\begin{array}{c|c} A_{cl} & B_{cl}\\ \hline C_{cl} & D_{cl}\end{array}\right) &= \left(\begin{array}{cc|c} A & 0 & B_w\\ 0 & 0 & 0\\ \hline C_z & 0 & D_{zw}\end{array}\right)\\ &\quad + \left(\begin{array}{c|c} 0 & B_u\\ I_{n_\psi} & 0\\ \hline 0 & D_{zu}\end{array}\right)\mathscr{K}\left(\begin{array}{cc|c} 0 & I_{n_\psi} & 0\\ C_y & 0 & D_{yw}\end{array}\right)\\ &=: \left(\begin{array}{c|c} \tilde{A} & \tilde{B}_w\\ \hline \tilde{C}_z & \tilde{D}_{zw}\end{array}\right) + \left(\begin{array}{c} \tilde{B}_u\\ \hline \tilde{D}_{zu}\end{array}\right)\mathscr{K}\left(\begin{array}{c|c} \tilde{C}_y & \tilde{D}_{yw}\end{array}\right).\end{aligned} \tag{10}$$

Let T_{wz} be the closed-loop transfer function from w to z. Then, from the bounded real lemma [4], the closed-loop system Σ_{cl} is stable and the H_∞-norm of T_{wz} is smaller than γ, i.e., $\|T_{wz}\|_\infty < \gamma$, if and only if there exist a symmetric matrix $\mathcal{P} \in \mathbf{R}^{(n_x+n_\psi)\times(n_x+n_\psi)}$ and an FIR control system matrix $\mathscr{K} \in \mathbf{R}^{(n_\psi+n_u)\times(n_\psi+n_y)}$ satisfying the following inequality

$$\begin{pmatrix} -\mathcal{P} & \mathcal{P}A_{cl} & \mathcal{P}B_{cl} & 0 \\ A_{cl}^T\mathcal{P} & -\mathcal{P} & 0 & C_{cl}^T \\ B_{cl}^T\mathcal{P} & 0 & -\gamma I_{n_w} & D_{cl}^T \\ 0 & C_{cl} & D_{cl} & -\gamma I_{n_z} \end{pmatrix} < 0. \tag{11}$$

Note that (11) with the FIR control system matrix $\mathscr{K}$ includes a memoryless output feedback law as a special case. Also note that A_{cl}, B_{cl}, C_{cl}, and D_{cl} are affine transforms of $\mathscr{K}$ as shown in (10). Hence the matrix inequality (11) is a biaffine matrix inequality (BMI) on the variables $\mathcal{P}$ and $\mathscr{K}$. The BMI problem is, however, nonconvex and known to be NP-hard [14]. Moreover, notice that the finite memory structure built in $\mathscr{K}$, namely, the FIR control structure (4), is an enormously difficult issue and there is no instant solution to it unfortunately, which will be dealt with in this paper. One may try to use a BMI optimization solver, such as PENBMI [13], to find an FIR controller in (11). The numerical efficiency of PENBMI will be discussed with comparisons in Section V.

This paper approaches these issues by converting the problem to finding a controller of the structure built in the following form

$$\mathscr{K} = U + V \underbrace{\mathrm{diag}(H_0, H_1, H_2, \ldots, H_{N-1})}_{\mathscr{K}_{str}} W, \tag{12}$$

where $\mathscr{K}_{str}$ is a block diagonal matrix variable to be determined, and U, V, W are constant matrices given by

$$U = \left(\begin{array}{c:c} A_\psi & B_\psi \\ \hdashline 0 & 0 \end{array}\right), V = \left(\begin{array}{ccccc} 0 & 0 & 0 & \cdots & 0 \\ \hdashline I_{n_u} & I_{n_u} & I_{n_u} & \cdots & I_{n_u} \end{array}\right),$$

$$W = \left(\begin{array}{c:c} 0 & I_{n_y} \\ \mathrm{diag}(\overbrace{I_{n_y}, \ldots, I_{n_y}}^{N-1}) & 0 \end{array}\right).$$

Sufficient solvability conditions and computational local search algorithms for obtaining the FIR control gains in the structured variable $\mathscr{K}_{str}$ will be suggested in the following sections.

III. FIR Control Sufficient LMI Conditions

Lemma 1: Let symmetric matrices $\mathscr{A} \in \mathbf{R}^{n\times n}$ and matrices $\mathscr{B}$ and $\mathscr{C}$ of the row dimension n be given. If there exists a structured instrumental variable X_{str} such that the following inequality holds

$$\mathscr{A} + \mathscr{B}X_{str}\mathscr{C}^T + \mathscr{C}X_{str}^T\mathscr{B}^T < 0, \tag{13}$$

then the following projection inequalities are satisfied

$$\mathscr{B}^\perp \mathscr{A}\mathscr{B}^{\perp T} < 0, \quad \mathscr{C}^\perp \mathscr{A}\mathscr{C}^{\perp T} < 0, \tag{14}$$

where $\mathscr{B}^\perp$ and $\mathscr{C}^\perp$ denote any matrices whose columns form bases of the null spaces of $\mathscr{B}^T$ and $\mathscr{C}^T$, respectively.

Proof: (13) is only sufficient to (14). So note that the reverse does not hold. The sufficiency is given clearly from the Projection Lemma [15], [16]. ■

Note that Lemma 1 offers not only a sufficient condition but also a new degree of freedom with structure, namely, X_{str}, which is called a structured instrumental variable in this paper, for the matrix inequalities (14). Sufficient solvability conditions for H_∞ FIR control will be presented by applying Lemma 1 in the following.

Theorem 1: Assume that D_{zu} is a null matrix and B_u is a full column rank matrix. The system Σ is stabilizable by the FIR controller Σ_c and $\|T_{wz}\|_\infty < \gamma$ if there exists a solution $\{\mathcal{P}, Y_{str}, R, Z, \gamma\}$ to the following LMI Problem 1. A suitable H_∞ FIR controller is given by $\mathscr{K}_{str} = R^{-1}Z$, i.e., $H_i = R_i^{-1}Z_i$, for all $i = 0, 1, \ldots, N-1$.

Problem 1:

$$\text{Minimize}_{\{\mathcal{P}, Y_{str}, R, Z, \gamma\}}\ \gamma$$

subject to

$$\begin{pmatrix} \mathcal{P} - (Y_{str} + Y_{str}^T) & * & * & * \\ \begin{pmatrix} \tilde{A}^T Y_{str} \\ +\tilde{C}_y^T U^T \tilde{B}_u^T Y_{str} \\ +\tilde{C}_y^T W^T Z^T V^T \tilde{B}_u^T \end{pmatrix} & -\mathcal{P} & * & * \\ \begin{pmatrix} \tilde{B}_w^T Y_{str} \\ +\tilde{D}_{yw}^T U^T \tilde{B}_u^T Y_{str} \\ +\tilde{D}_{yw}^T W^T Z^T V^T \tilde{B}_u^T \end{pmatrix} & 0 & -\gamma I_{n_w} & * \\ 0 & \tilde{C}_z & \tilde{D}_{zw} & -\gamma I_{n_z} \end{pmatrix} < 0, \tag{15}$$

$$\tilde{B}_u V R = Y_{str}^T \tilde{B}_u V, \tag{16}$$

where $R = \mathrm{diag}(R_0, R_1, \ldots, R_{N-1})$ and $Z = \mathrm{diag}(Z_0, Z_1, \ldots, Z_{N-1})$.

Proof: The proof is omitted due to the page limit. ■

Remark 1: Note that when $N = 1$, Theorem 1 offers an improved version of sufficient conditions in [17], where an instrumental variable with a fixed block diagonal structure is employed, that is, $Y_{str} = \mathrm{diag}(Y_1, Y_4)$. It is shown that the instrumental variable presented in the present paper provides more relaxed form of structure. Consider the singular value decomposition of $\tilde{B}_u$, that is, $\tilde{B}_u = U_1 \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V_1^T$, where U_1 and V_1 are unitary and Σ_1 is diagonal. The equality constraint (16) is rewritten as

$$\begin{bmatrix} \Sigma_1 V_1^T R \\ 0 \end{bmatrix} = U_1^T Y_{str}^T U_1 \begin{bmatrix} \Sigma_1 V_1^T \\ 0 \end{bmatrix}. \tag{17}$$

Define $\bar{Y}_{str} \triangleq U_1^T Y_{str} U_1$ and $\bar{Y}_{str} = \begin{bmatrix} \bar{Y}_1 & \bar{Y}_2 \\ \bar{Y}_3 & \bar{Y}_4 \end{bmatrix}$. Then, from the above equality, we have $R = V_1 \Sigma_1^{-1} \bar{Y}_1 \Sigma_1 V_1^T$ and $\bar{Y}_2 = 0$. Hence, the explicit structure of the instrumental variable Y_{str} is given by

$$Y_{str} = U_1 \begin{bmatrix} \bar{Y}_1 & 0 \\ \bar{Y}_3 & \bar{Y}_4 \end{bmatrix} U_1^T. \tag{18}$$

Therefore, we can reformulate LMI Problem 1 as follows:

$$\text{Minimize}_{\{\mathcal{P}, \bar{Y}_1, \bar{Y}_3, \bar{Y}_4, Z, \gamma\}}\ \gamma \text{ subject to (15) and (18)} \tag{19}$$

A memoryless output feedback controller is given by $u = V_1 \Sigma_1^{-1} \bar{Y}_1^{-1} \Sigma_1 V_1^T Z y$ by solving the above problem. This clearly improves the results in [17] and produces much less conservative output feedback solutions. Note that the equality constraint (16) in Problem 1 can be directly handled by using the solvers in [18], [19]. Regarding reference papers on memoryless output feedback, see references provided in [17]. ◇

Theorem 2: Assume that D_{yw} is a null matrix and C_y is a full row rank matrix. The system Σ is stabilizable by the FIR controller Σ_c and $\|T_{wz}\|_\infty < \gamma$, if there exists a solution $\{\mathcal{Q}, X_{str}, L, S, \gamma\}$ to the following LMI Problem 2. A suitable H_∞ FIR controller is given by $\mathscr{K}_{str} = SL^{-1}$, i.e., $H_i = S_i L_i^{-1}$, for all $i = 0, 1, \ldots, N-1$.

Problem 2:

$$\text{Minimize}_{\{\mathcal{Q}, X_{str}, W, S, \gamma\}}\ \gamma$$

subject to

$$\begin{pmatrix} \mathcal{Q} - (X_{str} + X_{str}^T) & * & * & * \\ 0 & -\gamma I_{n_w} & * & * \\ \begin{pmatrix} \tilde{A}X_{str}^T + \tilde{B}_u U \tilde{C}_y X_{str}^T \\ +\tilde{B}_u V S W \tilde{C}_y \end{pmatrix} & \tilde{B}_w & -\mathcal{Q} & * \\ \begin{pmatrix} \tilde{C}_z X_{str}^T + \tilde{D}_{zu} U \tilde{C}_y X_{str}^T \\ +\tilde{D}_{zu} V S W \tilde{C}_y \end{pmatrix} & \tilde{D}_{zw} & 0 & -\gamma I_{n_z} \end{pmatrix} < 0, \tag{20}$$

$$L W \tilde{C}_y = W \tilde{C}_y X_{str}^T, \tag{21}$$

where $L = \text{diag}(L_0, L_1, \ldots, L_{N-1})$ and $S = \text{diag}(S_0, S_1, \ldots, S_{N-1})$.

Proof: The proof is omitted due to the page limit. ■

IV. FIR Control Local Search Algorithms

A. Nonconvexity in FIR Control Design

In this section, we shall suggest an efficient iterative LMI method for finding a local optimal solution of the BMI problem (11) with the FIR control structure (12).

Theorem 3: Given $\gamma > 0$, there exists an FIR controller such that $\|T_{wz}\|_\infty < \gamma$ if and only if there exists a solution $\{\mathcal{P}, \mathcal{Q}, \mathscr{K}_{str}\}$ to the following Problem 3.

Problem 3: Find $\mathcal{P}$, $\mathcal{Q}$, and $\mathscr{K}_{str}$ satisfying the LMI (22) and the nonlinear matrix equality

$$\mathcal{Q}\mathcal{P} = I_{n_x + n_\psi}. \tag{23}$$

Proof: Pre- and post-multiplying $\text{diag}(\mathcal{P}^{-1}, I_{n_x}, I_{n_u}, I_{n_y})$ to (11) we have the following inequalities

$$\begin{pmatrix} -\mathcal{P}^{-1} & A_{cl} & B_{cl} & 0 \\ A_{cl}^T & -\mathcal{P} & 0 & C_{cl}^T \\ B_{cl}^T & 0 & -\gamma I_{n_w} & D_{cl}^T \\ 0 & C_{cl} & D_{cl} & -\gamma I_{n_z} \end{pmatrix} < 0. \tag{24}$$

We define a reciprocal matrix $\mathcal{Q}$ such that $\mathcal{Q} := \mathcal{P}^{-1}$. From the definition of the reciprocal matrices, we have (23). Also, by the definition of (10) and (12), $A_{cl} = \tilde{A} + \tilde{B}_u \mathscr{K} \tilde{C}_y$, $B_{cl} = \tilde{B}_w + \tilde{B}_u \mathscr{K} \tilde{D}_{yw}$, $C_{cl} = \tilde{C}_z + \tilde{D}_{zu} \mathscr{K} \tilde{C}_y$, $D_{cl} = \tilde{D}_{zw} + \tilde{D}_{zu} \mathscr{K} \tilde{D}_{yw}$, and $\mathscr{K} = U + V \mathscr{K}_{str} W$, where $\mathscr{K}_{str}$ is a matrix variable defined in (12), hence we have (22). Therefore, it is shown that Problem 3 is equivalent to the original nonconvex problem (11). This completes the proof. ■

B. Concave Minimization Algorithm

Note that, for any matrices $P > 0$ and $Q > 0$, $P, Q \in \mathbf{R}^{n \times n}$, if $Q - P^{-1} \geq 0_n$ is feasible, then $\mathbf{tr}(QP) \geq n$, and $\mathbf{tr}(QP) = n$ if and only if $QP = I_n$ [20]. We apply this heuristic technique to deal with the non-convexity in (23) of Problem 3. From the inequality $Q - P^{-1} \geq 0_n$, we have $\mathcal{Q} - \mathcal{P}^{-1} \geq 0_{n_x + n_\psi}$, which can be converted into the following LMI.

$$\begin{pmatrix} \mathcal{Q} & I_{n_x + n_\psi} \\ I_{n_x + n_\psi} & \mathcal{P} \end{pmatrix} \geq 0. \tag{25}$$

Hence, a solution of Problem 3 can be obtained from the following concave minimization problem.

Problem 4: Minimize$_{\{\mathcal{P}, \mathcal{Q}, \mathscr{K}_{str}\}}$ $\mathbf{tr}(\mathcal{Q}\mathcal{P})$ subject to (22), (25). ◇

We may see that if the optimal solution of Problem 4 satisfies $\mathbf{tr}(\mathcal{Q}\mathcal{P}) = n_x + n_\psi$, then Problem 3 is solved; otherwise Problem 3 is infeasible. Hence the H_∞ FIR control problem is now reduced to the problem of finding a global solution of Problem 4. This is, however, still a difficult issue since the objective function of Problem 4 is nonconvex. Note that there exists no global algorithm that always guarantees a global minimum of such a concave minimization problem in feasible time. Instead, to search a local optimal minimum of Problem 4, the popular conditional gradient algorithm, also called the Frank and Wolfe algorithm [21], can be applied to Problem 4, such as in [5], [20]. Note that Problem 4 is a new type of concave minimization problem for FIR control design. A computational algorithm for obtaining a local solution of the concave minimization problem, Problem 4, is given in the following. To this end, define a convex set by the set of LMIs as

$$\mathscr{C}_{(\mathcal{P}, \mathcal{Q}, \mathscr{K}_{str})} \triangleq \{(\mathcal{P}, \mathcal{Q}, \mathscr{K}_{str}) : (22), (25), \mathcal{P} > 0, \mathcal{Q} > 0\}. \tag{26}$$

Algorithm 1: Let $\gamma > 0$ be given.

1. Find an initial feasible solution $(\mathcal{P}^0, \mathcal{Q}^0) \in \mathscr{C}_{(\mathcal{P}, \mathcal{Q}, \mathscr{K}_{str})}$. Set $k = 0$.
2. Set $S^k = \mathcal{P}^k$, $T^k = \mathcal{Q}^k$. Linearize the concave objective function of Problem 4 at a given point (S^k, T^k) and define a linear function as
$$f_k(\mathcal{Q}, \mathcal{P}) \triangleq \mathbf{tr}(S^k \mathcal{Q} + T^k \mathcal{P}). \tag{27}$$
3. Find $(\mathcal{P}^{k+1}, \mathcal{Q}^{k+1}, \mathscr{K}_{str}^{k+1})$ solving the following LMI problem.
$$\text{Minimize}_{\{\mathcal{P}, \mathcal{Q}, \mathscr{K}_{str}\}}\ f_k(\mathcal{Q}, \mathcal{P}) \text{ subject to } (22), (25).$$
4. If $f_{k+1}(\mathcal{Q}^{k+1}, \mathcal{P}^{k+1}) - 2(n_x + n_\psi) < \epsilon_1$, where ϵ_1 is a pre-determined tolerance, is satisfied, then exit. Otherwise set $k = k + 1$ and return to Step 2. ◇

The following theorem shows that f_k is decreasing and bounded below, and thus it converges. It is also shown that if f_k tends to 0, then Problem 3 is feasible, which implies that an H_∞ finite memory controller is found for a given $\gamma > 0$.

Theorem 4: The algorithm has the following properties.

(i) $2(n_x + n_\psi) \leq f_{k+1} \leq f_k$.

(ii) $\lim_{k \to \infty} f_k = 2(n_x + n_\psi)$ if and only if $\mathcal{Q}\mathcal{P} = I_{n_x + n_\psi}$ at the optimum.

Proof: Refer to [20], [21]. ■

Remark 2: In Step 4 of Algorithm 1, the following stopping criterion can also be used: $\varepsilon := \mathbf{tr}(\mathcal{Q}\mathcal{P}) - (n_x + n_\psi) < \epsilon$, where ϵ is a pre-determined tolerance. LMI Problem 1, 2 and Algorithm 1 presented in this paper are easily implemented using semidefinite programming algorithms or existing LMI packages [2], [18], [19]. ◇

$$\begin{pmatrix} -\mathcal{Q} & * & * & * \\ \tilde{A}^T + \tilde{C}_y^T U^T \tilde{B}_u^T + \tilde{C}_y^T W^T \mathscr{K}_{str}^T V^T \tilde{B}_u^T & -\mathcal{P} & * & * \\ \tilde{B}_w^T + \tilde{D}_{yw}^T U^T \tilde{B}_u^T + \tilde{D}_{yw}^T W^T \mathscr{K}_{str}^T V^T \tilde{B}_u^T & 0 & -\gamma I_{n_w} & * \\ 0 & \begin{pmatrix} \tilde{C}_z + \tilde{D}_{zu} U \tilde{C}_y \\ + \tilde{D}_{zu} V \mathscr{K}_{str} W \tilde{C}_y \end{pmatrix} & \begin{pmatrix} \tilde{D}_{zw} + \tilde{D}_{zu} U \tilde{D}_{yw} \\ + \tilde{D}_{zu} V \mathscr{K}_{str} W \tilde{D}_{yw} \end{pmatrix} & -\gamma I_{n_z} \end{pmatrix} < 0 \qquad (22)$$

V. Illustrative Examples

Consider an unstable plant given by

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & -0.3 & 0.6 \\ 0 & 0 & 1 \\ 0.29 & -0.8+\delta_k & 1 \end{pmatrix} x(k) \\ &+ 2w_1(k) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} u(k) \\ y(k) &= x_1(k) + x_2(k) + 2w_2(k) \end{aligned} \qquad (28)$$

and a performance output $z = \begin{pmatrix} x_1 \\ u \end{pmatrix}$, where δ_k is an uncertain model parameter and considered as $\delta_k = 0$ in the controller design, which will be discussed later on. The open-loop poles of the above plant are $\{1.1153, 0.4424 \pm 0.6660i\}$. In this example, an FIR controller with H_∞ performance shall be addressed for stabilization of the linear discrete-time system (28). We are interested in H_∞ performance from w to z. For reference, it is noted that the optimal γ obtained by a full-order output feedback H_∞ controller of the IIR type is 4.0169 for the system (28). The IIR full-order controller is solved by convex optimization [22]. To implement the proposed methods, we used MATLAB and LMI solvers in [18], [19] on Intel(R) Core(TM)2 Duo CPU T5200 1.6GHz and Windows Vista with 1GB RAM.

For fair comparison purpose with the existing control schemes, an achievable minimum value of γ needs to be found. So Algorithm 1 will be mainly used to obtain a local optimal solution in this example. Now let $\gamma = 4.1$, $N = 3$, and the stopping criterion $\epsilon = 10^{-8}$. Using Algorithm 1, we obtain a controller of the finite impulse response structure

$$u_k = \begin{bmatrix} -0.5313 \\ -0.0594 \end{bmatrix} y_k + \begin{bmatrix} 0.4448 \\ 0.0068 \end{bmatrix} y_{k-1} + \begin{bmatrix} -0.3007 \\ -0.0020 \end{bmatrix} y_{k-2}, \qquad (29)$$

which satisfies the actual H_∞ performance $\|T_{wz}\|_\infty = 4.0998$ with the accuracy tolerance $\varepsilon = 5.1178 \times 10^{-9}$. It took around 100 seconds to reach the stopping criterion. However, it is affordable and the computation is run off-line. It is shown that the H_∞ performance of the FIR controller (29) is comparable to that of the IIR full-order controller. As N is large, the FIR type of control will approximate the IIR type of control well.

It is noticed that, due to the finite memory structure, the FIR control will be robust in most cases against temporary modeling uncertainties and thus has a better recovery or tracking ability compared with IIR type of control [6], [7], [9]. In contrast, IIR type of control with the infinite memory with respect to the measurements takes a long time to recover or track to the normal state after temporary modeling uncertainties disappears. To illustrate this point, the IIR full-order controller and the proposed FIR controller are compared when a system has actually temporary modeling uncertainty. Now the uncertain model parameter δ_k in (28) is considered as $\delta_k = 1.2$ for the time period $350 \le k \le 450$; otherwise $\delta_k = 0$. Fig. 1 compares the robustness of two controllers given temporary modeling uncertainty δ_k. This figure shows that the suggested FIR controller is more robust than the IIR type of controller when applied to systems with model parameter uncertainties. In addition, FIR controllers can be preferred in the real implementation to *full-order* or *reduced-order dynamic control* of the IIR type because of the structurally much simpler advantage.

One may try to obtain an FIR controller by directly solving (11) with a BMI solver, i.e., PENBMI developed by Kočvara and Stingl [13]. We tried to find an FIR controller using PENBMI. A random search technique was used to find several local solutions, as in [5]. Actually, PENBMI works quite well for simple memoryless output feedback control problems. However, for the FIR control problem that has the memory structure in (8), it could not get a comparable result to the proposed algorithm. That is, PENBMI failed to optimize H_∞ performance subject to the FIR control structure. It can be said that the proposed algorithm is numerically efficient for the implementation of FIR control.

Now consider H_∞ memoryless output feedback control. Let $\gamma = 7.58$ for memoryless feedback. Using Algorithm 1, we obtain a local optimal memoryless output feedback controller

$$u = \begin{bmatrix} -0.4477 \\ -0.4562 \end{bmatrix} y \qquad (30)$$

with H_∞ performance $\|T_{wz}\|_\infty = 7.5787$ and the accuracy $\varepsilon = 6.6929 \cdot 10^{-12}$ as shown in Table I. Now let $\gamma = 5.3$ and consider H_∞ PID output feedback control. Using Algorithm 1, we obtain

$$u_k = \begin{bmatrix} -0.2999 \\ -0.0828 \end{bmatrix} y_k + \begin{bmatrix} -0.0202 \\ -0.0185 \end{bmatrix} \sum_{i=0}^{k} y_i + \begin{bmatrix} -0.1725 \\ -0.3012 \end{bmatrix} \Delta y_k, \qquad (31)$$

with H_∞ performance $\|T_{wz}\|_\infty = 5.3$ and $\varepsilon = 0.8752 \times 10^{-10}$ as shown in Table I. As expected, it is clearly demonstrated that the proposed FIR controller (29) outperforms the memoryless output feedback controller (30). Also the proposed FIR controller (29) outperforms the output feedback PID controller (31) because the FIR control includes additional measurement terms, which provide additional information to predict the behavior of the dynamic system.

VI. Conclusions

A new type of output feedback control with the finite impulse response (FIR) structure is proposed for H_∞ output feedback stabilization of linear systems, which is called FIR control in this paper. For construction of FIR control, an

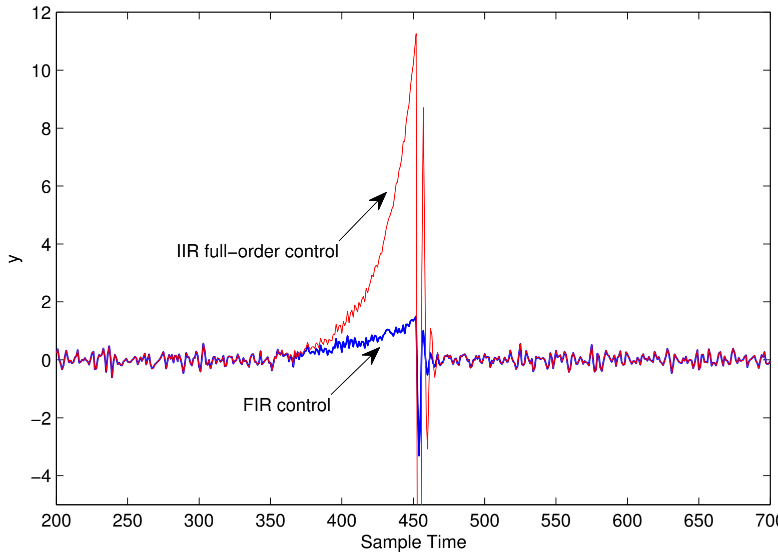

Fig. 1. Comparison of FIR and IIR controls for temporary modeling uncertainties.

TABLE I
MEMORYLESS OUTPUT FEEDBACK, PID CONTROL V.S. FIR CONTROL SCHEME

Control scheme	γ	$\|T_{wz}\|_\infty$	ε
Memoryless output feedback	7.58	7.5787	6.6929×10^{-12}
Output feedback PID control	5.3	5.3	0.8752×10^{-10}
FIR control ($N = 3$)	4.1	4.0998	5.1178×10^{-09}

efficient design approach is exploited and optimization techniques are developed based on semidefinite programming. Sufficient solvability conditions are suggested. A necessary and sufficient condition for the existence of H_∞ FIR control is presented through a set of a linear matrix inequality and a reciprocal matrices equality constraint. Explicit computational local search algorithms have been proposed to solve the matrix inequalities characterizing a solution of H_∞ FIR control. The advantages of the proposed FIR control are illustrated through numerical examples as well as the numerical efficiency of the proposed algorithm. This paper can be applied not only to different types of performance criteria, such as LQG and mixed H_2/H_∞, but also many types of systems, such as uncertain systems, time-delay systems, Markovian systems, etc.

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