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Discrete Time-Varying Attitude Complementary Filter

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Abstract— **This paper presents the development of an attitude complementary filter for an Attitude and Heading Reference System (AHRS). Using strapdown inertial measurements and vector observations, the proposed complementary filter provides attitude estimates in Euler angles representation, while compensating for rate gyro bias. Stability and performance properties of the proposed filter under operating conditions usually found in oceanic applications are derived, and the tuning of the filter parameters in the frequency domain is emphasized. The proposed solution poses small computational requirements, and is suitable for implementation on low-power hardware using low-cost sensors. Experimental results obtained with an implementation of the algorithm running on-board the DELFIMx catamaran are presented and discussed.**

I. Introduction

Complementary filters have been widely used in the past in sensor fusion problems. The frequency domain formulation and simple filter structure allow for straightforward implementation and testing in digital or analog hardware without requiring high performance signal processing hardware, see [1], and references therein. This paper presents the development and experimental evaluation of an Attitude and Heading Reference System (AHRS) using a complementary filter, by exploiting information provided by the vehicle sensor suite over distinct, yet complementary frequency regions. Merging inertial measurements from rate gyros and accelerometers with Earth's magnetic field observations, the filter is required to yield accurate attitude estimates that will be central to stabilize the platform and support the implementation of reliable control strategies. The implementation of the proposed architecture is straightforward and the performance results of the navigation system are demonstrated using experimental data obtained in tests at sea with the DELFIMx catamaran, depicted in Fig. 1.

While a unified error analysis for Inertial Navigation System (INS) has been carried out in the literature [2], several filtering architectures may be used in navigation systems. The Extended Kalman Filter (EKF) is one of the most well known and widely adopted filtering algorithms, see [3], [4], [5] and references therein, however filter divergence due to the linearization of the system and large state initialization error is a frequent stumbling block to the implementation of the filter. The Unscented Kalman Filter (UKF) has been put forth as an alternative to the EKF [3], [6], [7], which numerically approximates the mean and covariance of the state estimate parameterized in Euclidean spaces. Also, there has been an increasing interest in the design of nonlinear attitude observers that are theoretically stable and yield explicit regions of attraction [8], [9].

Fig. 1. The DELFIMx autonomous surface craft

The attitude filter proposed in this work is based on the complementary filtering theory, deeply rooted in the work of Wiener [10]: an unknown signal can be estimated using corrupted measurements from one or more sensors whose information naturally stands in distinct and complementary frequency bands [11], [12]. The minimum mean-square estimation (MMSE criteria) error was first solved by Wiener [10], assuming that the unknown signal had noise-like characteristics, which usually does not fit the signal description. Complementary filtering explores the sensor redundancy to successfully reject measurement disturbances in complementary frequency regions without distorting the signal. The loss of optimality in complementary filters due to ignoring noise stochastic description is slight and can even be beneficial for cases where it is better to consider irregular measures that occur out of the expected variance, as convincingly argued in [11].

The paper is organized as follows. The proposed complementary filter for attitude estimation is presented in Section II, where stability properties and conditions that guarantee performance are also derived. Section III focuses on the implementation of the attitude filter. Experimental results obtained during DELFIMx catamaran sea trials are presented in Section IV to illustrate the performance of the proposed AHRS. Concluding remarks and future work are pointed out in Section V.

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NOMENCLATURE

The notation adopted is fairly standard. The set of $n \times m$ matrices with real entries is denoted as $\mathrm{M}(n, m)$. The Gaussian distribution with mean μ and variance σ^2 is denoted as $\mathcal{N}(\mu, \sigma^2)$. The identity and zero matrices are respectively denoted as $\mathbf{I}$ and $\mathbf{0}$. The dimensions of the vector and matrices are clear from the context. In general, the vectors are elements (or a concatenation of elements) of $\mathbb{R}^3$.

II. ATTITUDE COMPLEMENTARY FILTER

In this section, a complementary filter for attitude estimation is proposed, and its stability and performance properties are derived. The design of the filter in the frequency domain is justified by discussing the complementary characteristics of the inertial and aiding sensors in the frequency domain.

Let $\bar{\boldsymbol{\lambda}} = \begin{bmatrix}\bar{\psi} & \bar{\theta} & \bar{\phi}\end{bmatrix}'$ denote the vector containing the yaw, pitch and roll Euler angles, respectively [13]. The Euler angle kinematics are described by

$$\dot{\bar{\boldsymbol{\lambda}}} = \mathbf{Q}(\bar{\boldsymbol{\lambda}})\bar{\boldsymbol{\omega}},\ \mathbf{Q}(\boldsymbol{\lambda}) = \begin{bmatrix} 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \\ 0 & \cos\phi & -\sin\phi \\ 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \end{bmatrix}, \quad (1)$$

where $\bar{\boldsymbol{\omega}}$ is the body angular velocity expressed in body frame coordinates. The discrete-time equivalent of the system (1) considered here is obtained by the Euler method [14] with the right-hand side subject to sample-and-hold, yielding

$$\bar{\boldsymbol{\lambda}}_{k+1} = \bar{\boldsymbol{\lambda}}_k + T\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k)\bar{\boldsymbol{\omega}}_k, \quad (2)$$

where T is the sampling interval and the index k abbreviates the time instant $t = kT$. In this work, the attitude is estimated by exploiting the angular velocity and attitude measurements provided by strapdown sensors. The angular velocity is measured by a rate gyro affected by noise and random-walk bias [15],

$$\boldsymbol{\omega}_{r\,k} = \bar{\boldsymbol{\omega}}_k + \bar{\mathbf{b}}_{\omega\,k} + \mathbf{w}_{\omega_r\,k}, \quad \bar{\mathbf{b}}_{\omega\,k+1} = \bar{\mathbf{b}}_{\omega\,k} + \mathbf{w}_{b\,k}, \quad (3)$$

where $\mathbf{w}_{\omega_r} \sim \mathcal{N}(\mathbf{0}, \Xi_\omega)$ is zero-mean, Gaussian white noise and $\bar{\mathbf{b}}_\omega$ is the sensor bias driven by the Gaussian white noise $\mathbf{w}_b \sim \mathcal{N}(\mathbf{0}, \Xi_b)$. Rewriting the Euler angles kinematics (2-3) in state space form gives

$$\begin{bmatrix}\bar{\boldsymbol{\lambda}}_{k+1} \\ \bar{\mathbf{b}}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & -T\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k) \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{\lambda}}_k \\ \bar{\mathbf{b}}_k\end{bmatrix} + \begin{bmatrix}T\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k) \\ \mathbf{0}\end{bmatrix}\boldsymbol{\omega}_{r\,k} + \begin{bmatrix}-T\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{w}_{\omega_r\,k} \\ \mathbf{w}_{b\,k}\end{bmatrix}. \quad (4)$$

Consider the following nonlinear feedback system as the proposed attitude filter

$$\begin{bmatrix}\hat{\boldsymbol{\lambda}}_{k+1} \\ \hat{\mathbf{b}}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & -T\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k) \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\lambda}}_k \\ \hat{\mathbf{b}}_k\end{bmatrix} + \begin{bmatrix}T\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k) \\ \mathbf{0}\end{bmatrix}\boldsymbol{\omega}_{r\,k} + \begin{bmatrix}\mathbf{Q}(\bar{\boldsymbol{\lambda}}_k)(K_{1\lambda} - \mathbf{I}) + \mathbf{Q}(\bar{\boldsymbol{\lambda}}_{k-1}) \\ K_{2\lambda}\end{bmatrix}(\mathbf{y}_{\lambda\,k} - \hat{\mathbf{y}}_{\lambda\,k}), \quad (5a)$$

$$\hat{\mathbf{y}}_{\lambda\,k} = \mathbf{Q}^{-1}(\bar{\boldsymbol{\lambda}}_{k-1})\hat{\boldsymbol{\lambda}}_k,\ \ \mathbf{y}_{\lambda\,k} = \mathbf{Q}^{-1}(\bar{\boldsymbol{\lambda}}_{k-1})\bar{\boldsymbol{\lambda}}_k + \mathbf{v}_{\lambda\,k}, \quad (5b)$$

where $\mathbf{y}_{\lambda\,k}$ is the vector of observed Euler angles transformed to the space of angular rate and corrupted by the Gaussian white observation noise $\mathbf{v}_\lambda \sim \mathcal{N}(\mathbf{0}, \Theta_\lambda)$, and

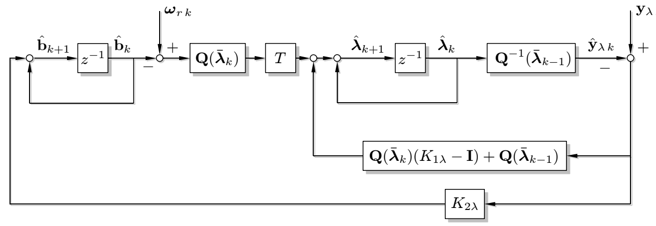


Fig. 2. Attitude complementary filter

$K_{1\lambda}, K_{2\lambda} \in \mathrm{M}(3,3)$ are feedback gain matrices. The block diagram of the proposed attitude filter is depicted in Fig. 2.

The attitude observation $\mathbf{y}_\lambda$ may be determined from vector observations, such as those obtained by magnetometers, pendula, cameras, or star trackers. The problem of determining attitude using vector measurements is known in the literature as the orthogonal Procrustes problem [16] or as Wahba's problem [17] and several solutions have been proposed along time-spread articles [3], [16]. In this work, the pitch and roll angles in $\mathbf{y}_\lambda$ are obtained from Earth's gravitational field, available from two on-board inclinometers (pendula), and the yaw angle in $\mathbf{y}_\lambda$ is computed from the Earth's magnetic field measurements provided by a magnetometer triad.

Consider the following auxiliary linear time invariant system

$$\begin{bmatrix}\mathbf{x}_{\lambda\,k+1} \\ \mathbf{x}_{b\,k+1}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & -T\mathbf{I} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{\lambda\,k} \\ \mathbf{x}_{b\,k}\end{bmatrix} + \begin{bmatrix}-T\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{w}_{\omega_r\,k} \\ \mathbf{w}_{b\,k}\end{bmatrix},$$
$$\mathbf{y}_{x\,k} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{\lambda\,k} \\ \mathbf{x}_{b\,k}\end{bmatrix} + \mathbf{v}_{\lambda\,k}, \quad (6)$$

which will be used in the sequel as the frequency domain design setup for the time-varying attitude filter (5). In the proposed design technique, the feedback gains $K_{1\lambda}$ and $K_{2\lambda}$ in (5) are identified with the steady-state Kalman gains for the system (6), where the covariance matrices Ξ_ω, Ξ_b and Θ_λ act as "tuning knobs" to shape the desired frequency response of the attitude filter.

The time-invariant system (6) adopted for the determination of the feedback gains and associated frequency response is similar to the attitude kinematics (4) for $\mathbf{Q}(\boldsymbol{\lambda}) = \mathbf{Q}(\mathbf{0})$. Although this suggests at first glance that the properties of the proposed filter could be limited to the specific case of $\boldsymbol{\lambda}_k = \mathbf{0}$, the filter is in fact asymptotically stable for any attitude trajectory parametrized by nonsingular Euler angle configurations. The stability properties are derived in the following theorem for the specific case of Z-Y-X Euler angles, however the extension of the results to other Euler angle set conventions [13] is immediate.

Theorem 1: Let $K_{1\lambda}$ and $K_{2\lambda}$ be the steady-state Kalman gains for the system (6) and assume that the pitch angle described by the platform is bounded, $|\theta| \leq \theta_{\max} < \frac{\pi}{2}$. Then the attitude complementary filter (5) is uniformly asymptotically stable (UAS).

Proof: Let $\tilde{\boldsymbol{\lambda}}_k = \bar{\boldsymbol{\lambda}}_k - \hat{\boldsymbol{\lambda}}_k$, $\tilde{\mathbf{b}}_{\omega\,k} = \bar{\mathbf{b}}_{\omega\,k} - \hat{\mathbf{b}}_{\omega\,k}$ denote the estimation errors. The associated estimation error dynamics are given by

$$\begin{bmatrix}\tilde{\boldsymbol\lambda}_{k+1}\\ \tilde{\mathbf b}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_k)(\mathbf I - K_{1\lambda})\mathbf Q^{-1}(\bar{\boldsymbol\lambda}_{k-1}) & -T\mathbf Q(\bar{\boldsymbol\lambda}_k)\\ -K_{2\lambda}\mathbf Q^{-1}(\bar{\boldsymbol\lambda}_{k-1}) & \mathbf I\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol\lambda}_k\\ \tilde{\mathbf b}_k\end{bmatrix} + \begin{bmatrix}-T\mathbf Q(\bar{\boldsymbol\lambda}_k) & \mathbf 0\\ \mathbf 0 & \mathbf I\end{bmatrix}\begin{bmatrix}\mathbf w_{\omega_r k}\\ \mathbf w_{b\,k}\end{bmatrix} + \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_k)(\mathbf I - K_{1\lambda}) - \mathbf Q(\bar{\boldsymbol\lambda}_{k-1})\\ -K_{2\lambda}\end{bmatrix}\mathbf v_{\lambda\,k}. \tag{7}$$

By definition, the filter is said to be UAS if the origin of the system (7) is UAS in the absence of state and measurement noises [18]. However, the state and measurement noises are denoted in the proof for the sake of convenience. The system (6) can be written in the compact state space formulation

$$\mathbf x_{k+1} = \mathbf F\mathbf x_k + \mathbf G\mathbf w_k, \quad \mathbf y_k = \mathbf H\mathbf x_k + \mathbf v_k, \tag{8}$$

where $\mathbf x_k = \begin{bmatrix}\mathbf x'_{\lambda\,k} & \mathbf x'_{b\,k}\end{bmatrix}'$, $\mathbf w_k = \begin{bmatrix}\mathbf w'_{\omega_r k} & \mathbf w'_{b\,k}\end{bmatrix}'$, $\mathbf y_k = \mathbf y_{x\,k}$, $\mathbf v_k = \mathbf v_{\lambda\,k}$, $\mathbf F = \begin{bmatrix}\mathbf I & -T\mathbf I\\ \mathbf 0 & \mathbf I\end{bmatrix}$, $\mathbf G = \begin{bmatrix}-T\mathbf I & \mathbf 0\\ \mathbf 0 & \mathbf I\end{bmatrix}$, and $\mathbf H = \begin{bmatrix}\mathbf I & \mathbf 0\end{bmatrix}$. It is straightforward to show that $[\mathbf F, \mathbf H]$ is detectable and $[\mathbf F, \mathbf G]$ is completely stabilizable, hence the closed-loop system

$$\tilde{\mathbf x}_{k+1} = (\mathbf F - \mathbf K\mathbf H)\tilde{\mathbf x}_k + \mathbf G\mathbf w_k - \mathbf K\mathbf v_k, \tag{9}$$

where $\tilde{\mathbf x}_k = [\tilde{\mathbf x}'_{\lambda\,k}, \tilde{\mathbf x}'_{b\,k}]'$, $\mathbf K = \begin{bmatrix}K'_{1\lambda} & K'_{2\lambda}\end{bmatrix}'$, is UAS [19]. Define the Lyapunov transformation of variables

$$\begin{bmatrix}\tilde{\boldsymbol\lambda}_{x\,k}\\ \tilde{\mathbf b}_{x\,k}\end{bmatrix} = \mathbf T_k\begin{bmatrix}\tilde{\mathbf x}_{\lambda\,k}\\ \tilde{\mathbf x}_{b\,k}\end{bmatrix}, \quad \mathbf T_k = \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_{k-1}) & \mathbf 0\\ \mathbf 0 & \mathbf I\end{bmatrix}, \tag{10}$$

that is well defined [20] because θ is bounded by assumption. Applying the transformation of variables (10) to (9) yields

$$\begin{bmatrix}\tilde{\boldsymbol\lambda}_{x\,k+1}\\ \tilde{\mathbf b}_{x\,k+1}\end{bmatrix} = \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_k)(\mathbf I - K_{1\lambda})\mathbf Q^{-1}(\bar{\boldsymbol\lambda}_{k-1}) & -T\mathbf Q(\bar{\boldsymbol\lambda}_k)\\ -K_{2\lambda}\mathbf Q^{-1}(\bar{\boldsymbol\lambda}_{k-1}) & \mathbf I\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol\lambda}_{x\,k}\\ \tilde{\mathbf b}_{x\,k}\end{bmatrix} + \begin{bmatrix}-T\mathbf Q(\bar{\boldsymbol\lambda}_k)\mathbf w_{\omega_r k}\\ \mathbf w_{b\,k}\end{bmatrix} - \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_k)K_{1\lambda}\\ K_{2\lambda}\end{bmatrix}\mathbf v_{\lambda\,k}. \tag{11}$$

The origin of (9) is UAS and, by the properties of Lyapunov transformations, the origin of (11) is UAS. Hence, the origin of (7) is uniformly asymptotically stable, as desired. ■

The stability results of Theorem 1 can be easily extended for time-varying Kalman gains, however the proposed complementary filter is designed in the frequency domain by means of the time-invariant formulation (6), to obtain a desired transfer function that merges the low-frequency contents of the attitude observations with the high-frequency information from the angular rate readings. Steady-state Kalman filter gains are adopted to yield an asymptotically stable filter that can be easily implemented and tested in low-cost hardware. Interestingly enough, under operating conditions found in some terrestrial and oceanic applications, the gains adopted in the proposed filter (5) are also the Kalman gains for the time-varying system (4).

Theorem 2: Let the state and observation disturbances in the attitude kinematics (4) be characterized by the Gaussian white noises $\mathbf w_{\omega_r} \sim \mathcal N(\mathbf 0, \Xi_\omega)$, $\mathbf w_b \sim \mathcal N(\mathbf 0, \Xi_b)$ and $\mathbf v_\lambda \sim \mathcal N(\mathbf 0, \Theta_\lambda)$, respectively, and assume that the pitch and roll angles are constant. Then the complementary attitude filter (5) is the "steady-state" Kalman filter for the system (4) in the sense that the Kalman feedback gain $K_{\text{opt}\,k}$ converges asymptotically as follows

$$\lim_{k\to\infty}\left\| K_{\text{opt}\,k} - \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_k)(K_{1\lambda} - \mathbf I) + \mathbf Q(\bar{\boldsymbol\lambda}_{k-1})\\ K_{2\lambda}\end{bmatrix}\right\| = 0. \tag{12}$$

Proof: The estimation error covariance matrix of the Kalman filter for the system (6) satisfies

$$\mathbf P_{x\lambda\,k+1|k} = \mathbf F\mathbf P_{x\lambda\,k|k-1}\mathbf F' + \mathbf G\Xi\mathbf G' - \mathbf F\mathbf P_{x\lambda\,k|k-1}\mathbf H'\mathbf S^{-1}_{P\lambda\,k}\mathbf H\mathbf P_{x\lambda\,k|k-1}\mathbf F', \tag{13}$$

where $\mathbf S_{P\lambda\,k} = \mathbf H\mathbf P_{x\lambda\,k|k-1}\mathbf H' + \Theta_\lambda$, $\Xi = \begin{bmatrix}\Xi_\omega & \mathbf 0\\ \mathbf 0 & \Xi_b\end{bmatrix}$, see references [18], [19] for a derivation of (13). Given the transformation of variables (10), the covariance matrix $\boldsymbol\Sigma_{x\lambda\,k+1|k} = E\left(\begin{bmatrix}\tilde{\boldsymbol\lambda}_{x\,k+1}\\ \tilde{\mathbf b}_{x\,k+1}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol\lambda}'_{x\,k+1} & \tilde{\mathbf b}'_{x\,k+1}\end{bmatrix}\right)$ is given by $\boldsymbol\Sigma_{x\lambda\,k+1|k} = \mathbf T_{k+1}\mathbf P_{x\lambda\,k+1|k}\mathbf T'_{k+1}$ and, using (13), satisfies

$$\boldsymbol\Sigma_{x\lambda\,k+1|k} = \mathbf Z_k\boldsymbol\Sigma_{x\lambda\,k|k-1}\mathbf Z'_k + \mathbf T_{k+1}\mathbf G\Xi\mathbf G'\mathbf T'_{k+1} - \mathbf Z_k\boldsymbol\Sigma_{x\lambda\,k|k-1}\mathbf T_k^{-T}\mathbf H'\mathbf S^{-1}_{\Sigma\lambda\,k}\mathbf H\mathbf T_k^{-1}\boldsymbol\Sigma_{x\lambda\,k|k-1}\mathbf Z'_k,$$

where $\mathbf S_{\Sigma\lambda\,k} = \mathbf H\mathbf T_k^{-1}\boldsymbol\Sigma_{x\lambda\,k|k-1}\mathbf T_k^{-T}\mathbf H' + \Theta_\lambda$, and $\mathbf Z_k = \mathbf T_{k+1}\mathbf F\mathbf T_k^{-1}$. With a slight abuse of notation, let $K_{1\lambda\,k}$ and $K_{2\lambda\,k}$ denote the optimal time-varying gains for the system (6) and formulate the attitude filter (5) with time-varying gains

$$\begin{bmatrix}\hat{\boldsymbol\lambda}_{k+1}\\ \hat{\mathbf b}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf I & -T\mathbf Q(\bar{\boldsymbol\lambda}_k)\\ \mathbf 0 & \mathbf I\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol\lambda}_k\\ \hat{\mathbf b}_k\end{bmatrix} + \begin{bmatrix}T\mathbf Q(\bar{\boldsymbol\lambda}_k)\\ \mathbf 0\end{bmatrix}\boldsymbol\omega_{r\,k} + \begin{bmatrix}\mathbf Q(\bar{\boldsymbol\lambda}_k)(K_{1\lambda\,k} - \mathbf I) + \mathbf Q(\bar{\boldsymbol\lambda}_{k-1})\\ K_{2\lambda\,k}\end{bmatrix}(\mathbf y_{\lambda\,k} - \hat{\mathbf y}_{\lambda\,k}). \tag{14}$$

The attitude filter (14) is the Kalman filter for the system (4) if i) $\boldsymbol\Sigma_{x\lambda\,k+1|k}$ is the error covariance of the attitude filter (14) and ii) $\boldsymbol\Sigma_{x\lambda\,k+1|k}$ is the error covariance of the optimal filter for the attitude kinematics (4). If these conditions are verified, the time-varying and the steady-state Kalman filters for the attitude kinematics (4) are respectively given by the attitude complementary filters (14) and (5), for a discussion on the uniqueness of the Kalman gains, the reader is referred to [18], [19].

The condition of constant pitch and roll implies that $\mathbf Q(\bar{\boldsymbol\lambda}_{k+1}) = \mathbf Q(\bar{\boldsymbol\lambda}_k)$, hence the kinematics (7) and (11) are identical (independently of time-varying or steady-state feedback gains), $[\tilde{\boldsymbol\lambda}'_{x\,k} \;\; \tilde{\mathbf b}'_{x\,k}]' = [\tilde{\boldsymbol\lambda}'_k \;\; \tilde{\mathbf b}'_k]'$ and $\boldsymbol\Sigma_{x\lambda\,k+1|k}$ is the error covariance of the attitude filter (14).

The matrix $\boldsymbol\Sigma_{x\lambda\,k+1|k}$ is the covariance error of the Kalman filter for the system

$$\begin{aligned}\mathbf z_{k+1} &= \mathbf Z_k\mathbf z_k + \mathbf T_{k+1}\mathbf G\mathbf w_{z\,k},\\ \mathbf y_{z\,k} &= \mathbf H\mathbf T_k^{-1}\mathbf z_k + \mathbf v_{z\,k},\end{aligned} \tag{15}$$

where $\mathbf z_k \in \mathbb R^6$, $\mathbf w_z \sim \mathcal N(\mathbf 0, \Xi)$, $\mathbf v_z \sim \mathcal N(\mathbf 0, \Theta_\lambda)$. Using $\mathbf Q(\bar{\boldsymbol\lambda}_{k+1}) = \mathbf Q(\bar{\boldsymbol\lambda}_k)$, the matrices of the system (15) are given by

$$\mathbf Z_k = \begin{bmatrix}\mathbf I & -T\mathbf Q(\bar{\boldsymbol\lambda}_k)\\ \mathbf 0 & \mathbf I\end{bmatrix}, \; \mathbf T_{k+1}\mathbf G = \begin{bmatrix}-T\mathbf Q(\bar{\boldsymbol\lambda}_k) & \mathbf 0\\ \mathbf 0 & \mathbf I\end{bmatrix}, \quad \mathbf H\mathbf T_k^{-1} = \begin{bmatrix}\mathbf Q^{-1}(\bar{\boldsymbol\lambda}_{k-1}) & \mathbf 0\end{bmatrix},$$

which are the state space matrices of the attitude kinematics (4) with attitude observation given by (5b). Consequently,

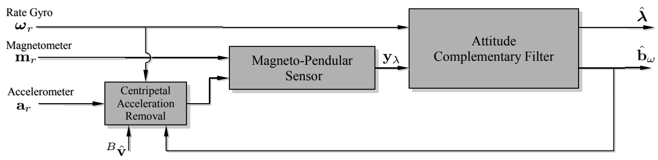


Fig. 3. AHRS architecture

the attitude filter (14) produces the optimal estimation error covariance matrix $\boldsymbol{\Sigma}_{x\lambda\, k+1|k}$ for the system (4) and, by uniqueness, the attitude filter (14) is a Kalman filter. Using $K_{1\lambda\, k} \to K_{1\lambda}$ and $K_{2\lambda\, k} \to K_{2\lambda}$ as $k \to \infty$ yields (12), that completes the proof. ∎

The complementary filter performance results presented in Theorem 2 hold for applications where the pitch and roll angles are constant or, for practical purposes, can be considered approximately constant. For the case of time-varying pitch and roll angles, the performance of the complementary and the optimal filters can be compared offline by computing the estimation error covariances of the filters, as detailed in [19]. Later in this work, the performance of the filter is analyzed using the experimental data obtained on-board the DELFIMx catamaran.

Although performance results are presented in Theorem 2, the design of the feedback gains is performed in the frequency domain due to the characteristics of the attitude aiding sensor at hand. This approach exploits the low-frequency region where the attitude observations are typically more accurate, and the high-frequency region where the integration of the rate gyro yields better attitude estimates.

III. AHRS Implementation: The Magneto-Pendular Sensor

This section presents the overall Attitude and Heading Reference System architecture that builds on the attitude complementary filter derived in the previous section, and discusses the implementation details of the attitude filter using the Magneto-Pendular Sensor (MPS).

A. AHRS architecture

The block diagram of the AHRS is depicted in Fig. 3. The attitude complementary filter, detailed in Section II and illustrated in Fig. 2, merges the angular rate information from the rate gyros with the attitude reconstruction provided by the MPS. The compensation of the centripetal acceleration block improves the attitude estimates in turning maneuvers.

B. Centripetal Acceleration Removal

The computation of pitch and roll angles using directly the accelerometer reading is distorted in the presence of external linear and angular accelerations. The accelerometer measurement model is given by [2]

$$\mathbf{a}_r = \frac{\mathrm{d}\, {}^B\mathbf{v}}{\mathrm{dt}} + \boldsymbol{\omega} \times {}^B\mathbf{v} - {}^B\mathbf{g}, \tag{16}$$

where $\frac{\mathrm{d}\,{}^B\mathbf{v}}{\mathrm{dt}}$ is the linear acceleration, $\boldsymbol{\omega} \times {}^B\mathbf{v}$ is the centripetal acceleration, and ${}^B\mathbf{g}$ is the gravity acceleration vector in body fixed frame coordinates. Typical maneuvers of autonomous vehicles involve mostly short term linear accelerations, which hence are high-frequency and the resulting distortion in pitch and roll can be smoothed out by the complementary low-pass filter. On the other hand, centripetal accelerations occur even in trimming maneuvers, e.g. a helicoidal path, and must be compensated for. As depicted in Fig. 3, the pendular reading estimate $\hat{\mathbf{a}}_p$ is obtained by compensating the centripetal acceleration

$$\hat{\mathbf{a}}_p = \mathbf{a}_r - \hat{\boldsymbol{\omega}} \times^B \hat{\mathbf{v}}, \tag{17}$$

where $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega$ is the angular rate drawn from the rate gyro measurement after bias compensation and ${}^B\hat{\mathbf{v}}$ is the linear velocity estimate provided to the AHRS. The effect of linear acceleration in $\hat{\mathbf{a}}_p$ is compensated in the frequency domain by appropriate design of the complementary filter.

C. Magneto-Pendular Sensor

The attitude observation $\mathbf{y}_{\lambda\, k}$ in Euler angles coordinates is determined using the body and Earth frame representations of two vectors, namely the Earth's magnetic and gravitational fields. Note that $\mathbf{y}_{\lambda\, k}$ can be obtained using other attitude reconstruction algorithms and sensors, for more details see [3] and references therein.

The magnetic field vector is measured in the body frame by the magnetometer

$$\mathbf{m}_r = \mathbf{R}'_X(\phi)\mathbf{R}'_Y(\theta)\mathbf{R}'_Z(\psi)^E\bar{\mathbf{m}} + \mathbf{n}_m, \tag{18}$$

where the magnetic field in Earth frame coordinates, denoted by ${}^E\bar{\mathbf{m}}$, is known, $\mathbf{n}_m$ is the magnetometer measurement noise, and $\mathbf{R}_X(\phi)$, $\mathbf{R}_Y(\theta)$, and $\mathbf{R}_Z(\psi)$ represent the roll, pitch, and yaw elementary rotation matrices, respectively. Denoting the projection of the magnetometer reading on the x-y plane by ${}^P\mathbf{m} = \mathbf{R}_Y(\theta)\mathbf{R}_X(\phi)\mathbf{m}_r$, the yaw angle is obtained by algebraic manipulation of (18), producing

$$\psi = \arctan 2\left({}^E m_y {}^P m_x - {}^E m_x {}^P m_y, {}^E m_x {}^P m_x + {}^E m_y {}^P m_y\right), \tag{19}$$

where the four quadrant $\arctan$, denoted as $\arctan 2$, was adopted. The pitch and roll angles are obtained from the accelerometer, which is regarded as a pendular sensor

$$\hat{\mathbf{a}}_p \approx -{}^B\mathbf{g} = -\mathbf{R}'_X(\phi)\mathbf{R}'_Y(\theta)^E\mathbf{g} = \begin{bmatrix} g\sin\theta \\ -g\cos\theta\sin\phi \\ -g\cos\theta\cos\phi \end{bmatrix}, \tag{20}$$

where ${}^E\mathbf{g} = \begin{bmatrix}0 & 0 & g\end{bmatrix}'$ is the gravity vector in Earth frame coordinates, and g is the local gravitational acceleration. The pitch and roll angles are given by algebraic manipulation of (20), producing

$$\begin{aligned} \phi &= \arctan 2\left(-a_y, -a_z\right), \\ \theta &= \begin{cases} \arctan\left(-\frac{a_x \sin\phi}{a_y}\right), & \sin\phi \neq 0 \\ \arctan\left(-\frac{a_x \cos\phi}{a_z}\right), & \cos\phi \neq 0 \end{cases}. \end{aligned} \tag{21}$$

The yaw, pitch, and roll observations (19,21) define a virtual attitude sensor measurement that is referred to as Magneto-Pendular Sensor (MPS). The MPS observation

TABLE I
COMPLEMENTARY FILTER PARAMETERS

	State Weights	Observation Weight	Filter Gain
Attitude Filter	$\Xi_\omega = 3\mathbf{I}$ $\Xi_b = 10^{-10}\mathbf{I}$	$\Theta_\lambda = 0.8 \times 10^{-2}\mathbf{I}$	$K_{1\,\lambda} = 2.97 \times 10^{-1}\mathbf{I}$ $K_{2\,\lambda} = 9.41 \times 10^{-5}\mathbf{I}$

noise $\mathbf{v}_\lambda$ is a nonlinear function of the inertial sensor noises, and of the acceleration compensation errors, and is mostly high-frequency due to the influence of linear accelerations. Consequently, the observation noise weight matrix Θ_λ is tuned to yield good steady-state high-frequency rejection of the MPS noise.

The theoretical stability property of the attitude filters derived in Section II cannot be directly inferred to the overall AHRS due to the use of $\hat{\boldsymbol{\omega}}$ in the computation of the attitude aiding observation. This fact, which is specific to the adopted attitude aiding sensor, can be easily overcome by resorting to nonpendular attitude aiding devices, e. g. vision based techniques. Alternatively, the direct measurements of the rate gyros can be used after obtaining, off-line, a good bias estimate by performing straight-line trajectories prior to compensating for centripetal acceleration in the filter. The proposed AHRS implementation is focused on low-cost and simplicity and Monte Carlo simulations were adopted to validate in practice the MPS attitude aiding integration in the AHRS architecture.

IV. EXPERIMENTAL RESULTS

The AHRS is validated in this section using a low-power hardware architecture enclosing low-cost sensors and mounted on-board the DELFIMx catamaran. The properties of the complementary filters in the frequency domain are discussed and the resulting performance of the proposed filter is analyzed. The attitude estimation results using the experimental data collected in the catamaran sea tests are presented, and the initial calibration errors of the rate gyros biases are addressed.

The DELFIMx surface craft, depicted in Fig. 1, is a small Catamaran 4.5 m long and 2.45 m wide, with a mass of 300 Kg. Propulsion is ensured by two propellers driven by electrical motors, and the maximum rated speed of the vehicle with respect to the water is 6 knots, the reader is referred to [21] for further details.

The Inertial Measurement Unit (IMU) installed on-board the DELFIMx craft is a strapdown system comprising a triaxial XBOW CXL02LF3 accelerometer and three single axes Silicon Sensing CRS03 rate gyros mounted along three orthogonal axes. The inertial sensors are sampled at 56 Hz. The hardware architecture is also equipped with a Honeywell HMR3300 magnetometer, interfaced by a serial port connection.

A. Filter Parameter Design

The attitude filter derived in Section II is designed to produce a closed-loop frequency response which blends the complementary frequency contents of the inertial and the aiding sensor measurements. In this frequency domain framework, the state and measurement weight matrices are used as tuning parameters and the filter gains are identified with the steady-state Kalman filter gains. The adopted weights and corresponding gains are detailed in Table I.

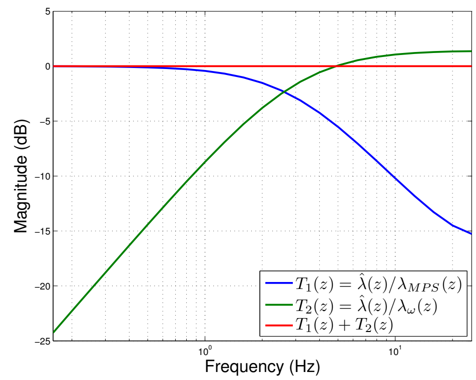
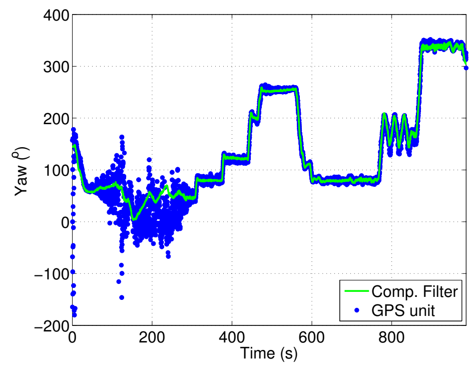
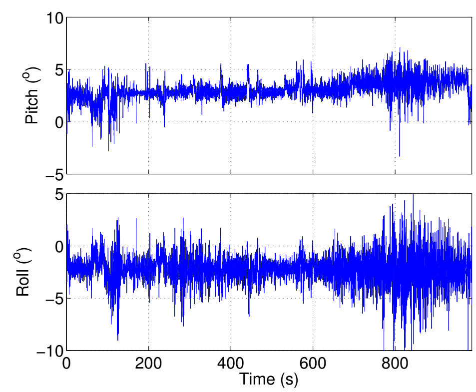


Fig. 4. Complementary filter transfer functions



(a) Yaw



(b) Pitch and Roll

Fig. 5. Attitude estimation results

The complementary frequency response of the closed-loop filters is depicted in Fig. 4 and was obtained by considering $\mathbf{Q}(\boldsymbol{\lambda}) = \mathbf{Q}(\mathbf{0})$, i.e. the frequency response of the time invariant system (6) used in the filter design. As shown in Fig. 4, the low-frequency region of the MPS is blended with the high-frequency contents of the open-loop integration of the rate gyros measurements. The complementary transfer functions are validated in practice with the experimental data

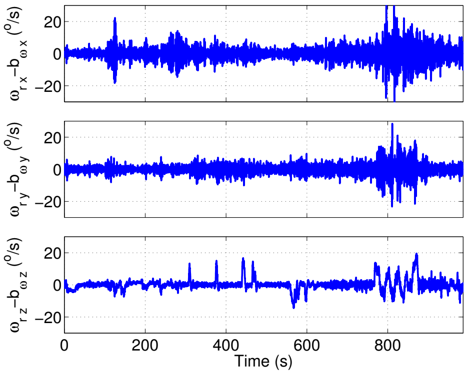


Fig. 6. Angular velocity estimation results

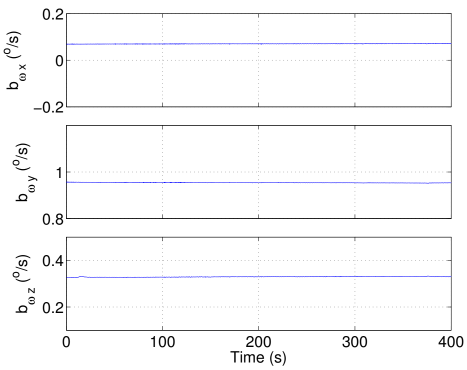


Fig. 7. Rate gyro bias estimates

obtained on-board the DELFIMx catamaran.

B. Experimental Results Analysis

This section presents the attitude filter estimation results obtained with the experimental data collected on-board the DELFIMx catamaran during tests at sea using the hardware architecture detailed previously. The trajectory described by the DELFIMx is mainly characterized by straightline and circular paths to assess the performance of the resultant AHRS in realistic operational scenarios.

The attitude and angular velocity estimation results are presented in Figs. 5-6, where the yaw measurements obtained from the GPS unit installed on-board are also shown for the sake of comparison. The attitude estimation results are as expected, namely the yaw estimate is consistent with the turning maneuvers performed by the platform and with the yaw measurement given by the GPS unit, and the mean of the estimated pitch and roll angles corresponds to the installation angles of the AHRS in the DELFIMx. Interestingly enough, the standard deviations of the roll and pitch are of small amplitude, as shown by the estimation results depicted in Fig. 5(b), which suggests that the performance degradation of the attitude filter due to time-varying pitch and roll is small. The rate gyro bias estimation results are presented in Fig. 7 and show that the attitude complementary filter compensates for slowly time-varying bias, by means of the small design weight Ξ_b in the computation of the feedback gain, see Table I for details.

V. Conclusions

A discrete time-varying complementary filter for attitude estimation was proposed and its stability and performance properties were derived. Using the Euler angles parametrization, the attitude filter compensates for rate gyro bias and is stable for trajectories described by nonsingular configurations. The steady-state filter gains are computed to shape a frequency response that blends the frequency contents of the aiding and the inertial sensors. Implementation aspects were detailed, namely an attitude aiding observation based on magnetic and pendular measurements was derived. The structure of the resulting Attitude and Heading Reference System which can be represented in a simple block diagram, was easily implemented on a low-cost hardware, and was validated using experimental data, in tests at sea with the DELFIMx catamaran.

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