2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Compensating a String PDE in the Actuation or Sensing Path of an Unstable ODE

Miroslav Krstic

Abstract— **How to control an unstable linear system with a long pure delay in the actuator path? This question was resolved using 'predictor' or 'finite spectrum assignment' designs in the 1970s. Here we address a more challenging question: How to control an unstable linear system with a wave PDE in the actuation path? Physically one can think of this problem as having to stabilize a system to whose input one has access through a string. The challenges of overcoming string/wave dynamics in the actuation path include their infinite dimension, finite propagation speed of the control signal, and the fact that all of their (infinitely many) eigenvalues are on the imaginary axis. In this paper we provide an explicit feedback law that compensates the wave PDE dynamics at the input of an LTI ODE and stabilizes the overall system. In addition, we prove robustness of the feedback to the error in a priori knowledge of the propagation speed in the wave PDE. Finally, we consider a dual problem where the wave PDE is in the sensing path and design an exponentially convergent observer.**

I. Introduction

The 'Smith predictor' and its extensions developed since the 1970s [1], [3], [4], [5], [6], [7], [8], [13], [9], [14], [15], [16], [17], [18], [19], [21], [22], [25], [26], [27], [28], [29], [30], [31] are important tools in several application areas. They allow to compensate a pure delay of arbitrary length in either the actuation or sensing path of a linear system, even when the system is unstable. Several results in adaptive control for unknown ODE parameters have been published [2], [20]. Extensions to nonlinear systems are also beginning to emerge [10].

In [11] we presented a first attempt of compensating infinite-dimensional actuator dynamics of more complex type than pure delay. We presented a design for diffusion-dominated PDE dynamics (such as the heat equation). While these dynamics do not have a finite speed of propagation, they are 'low-pass' and 'phase-lag' to the extreme, as they have infinitely many (stable) poles and no zeros.

In this paper we tackle a problem from a different class of PDE dynamics in the actuation or sensing path—the wave/string equation. The wave equation is challenging due to the fact that all of its (infinitely many) eigenvalues are on the imaginary axis, and due to the fact that it has a finite (limited) speed of propagation (large control doesn't help).

The problem studied here is more challenging than in [11] due to another difficulty—the PDE system is second order in time, which means that the state is 'doubly infinite dimensional' (distributed displacement and distributed velocity). This is not so much or a problem dimensionally, as it is a problem in constructing the state transformations for compensating the PDE dynamics. One has to deal with the coupling of two infinite-dimensional states.

As in [13] for delay-ODE cascades, and in [11] for heat-PDE-ODE cascades, we design feedback laws that are given by explicit formulae. We start in Section II with an actuator compensation design with full state feedback. In Section III we approach the question of robustness of our infinite-dimensional feedback law with respect to small uncertainty in the wave propagation speed and provide an affirmative answer. Finally, in Section IV we develop a dual of our actuator dynamics compensator and design an infinite-dimensional observer which compensates the wave PDE dynamics of the sensor.

II. Stabilization With Full-State Feedback

We consider the cascade of a wave (string) equation and an LTI finite-dimensional system given by

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{1}$$
$$u_{tt}(x,t) = u_{xx}(x,t) \tag{2}$$
$$u_x(0,t) = 0 \tag{3}$$
$$u_x(D,t) = U(t)\,, \tag{4}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, and $u(x,t)$ is the state of the PDE dynamics of the actuator governed by a wave equation. The cascade system is depicted in Figure 1.

The length of the PDE domain, D, is arbitrary. Thus, we take the wave propagation speed to be unity without loss of generality. We assume that the pair (A,B) is stabilizable and take K to be a known vector such that $A+BK$ is Hurwitz.

We recall from [13] that, if (2), (3) are replaced by the delay/transport equation,

$$u_t(x,t) = u_x(x,t)\,, \tag{5}$$

then the predictor-based control law

$$U(t) = K\left[e^{AD}X(t) + \int_0^D e^{A(D-y)}Bu(y,t)dy\right] \tag{6}$$

achieves perfect compensation of the actuator delay and achieves exponential stability at $u \equiv 0, X = 0$. When the pure delay actuator dynamics are replaced by the wave equation dynamics, a much more involved feedback law is needed.

We seek an invertible transformation $(X,u,u_t) \mapsto (X,v,v_t)$ that converts (1)–(3) into

$$\dot{X}(t) = (A+BK)X(t) + Bv(0,t) \tag{7}$$
$$v_{tt}(x,t) = v_{xx}(x,t) \tag{8}$$
$$v_x(0,t) = 0 \tag{9}$$

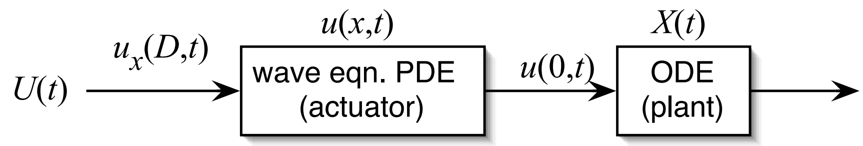

Fig. 1. The cascade of the wave equation PDE dynamics of the actuator with the ODE dynamics of the plant.

and then another transformation $(X,v,v_t) \mapsto (X,w,w_t)$ that converts (7)–(9) into

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{10}$$
$$w_{tt}(x,t) = w_{xx}(x,t) \tag{11}$$
$$w_x(0,t) = c_0 w(0,t)\,, \qquad c_0 > 0. \tag{12}$$

We also seek a feedback law that achieves

$$w_x(D,t) = -c_1 w_t(D,t)\,, \qquad c_1 > 0\,. \tag{13}$$

The system (10)–(13) is exponentially stable, as we shall see. With the invertibility of the composite transformation $(X,u,u_t) \mapsto (X,w,w_t)$, we will achieve exponential stability of the closed-loop system in the original variables (X,u,u_t).

We postulate the transformation $(X,u,u_t) \mapsto (X,v,v_t)$ in the form

$$\begin{aligned} v(x,t) =& u(x,t) - \int_0^x k(x,y)u(y,t)dx - \int_0^x l(x,y)u_t(y,t)dy \\ &- \gamma(x)X(t)\,, \end{aligned} \tag{14}$$

where the kernel functions $k(x,y)$, $l(x,y)$, and $\gamma(x)$ are to be found. By matching the systems (1)–(3) and (7)–(9), a lengthy but straightforward calculation leads to the following conditions on the kernels:

$$\gamma''(x) = \gamma(x)A^2 \tag{15}$$
$$\gamma(0) = K \tag{16}$$
$$\gamma'(0) = 0 \tag{17}$$

$$l_{xx}(x,y) = l_{yy}(x,y) \tag{18}$$
$$l(x,x) = 0 \tag{19}$$
$$l_y(x,0) = -\gamma(x)B \tag{20}$$

$$k_{xx}(x,y) = k_{yy}(x,y) \tag{21}$$
$$k(x,x) = 0 \tag{22}$$
$$k_y(x,0) = -\gamma(x)AB\,. \tag{23}$$

These differential equations can be solved explicitly. The solutions are

$$\gamma(x) = KM(x) \tag{24}$$
$$l(x,y) = m(x-y) \tag{25}$$
$$k(x,y) = \mu(x-y) \tag{26}$$
$$M(x) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{27}$$
$$m(s) = \int_0^s \gamma(\xi)Bd\xi \tag{28}$$
$$\mu(s) = \int_0^s \gamma(\xi)ABd\xi\,. \tag{29}$$

Thus the transformation $(X,u,u_t) \mapsto (X,v,v_t)$ is defined as

$$\begin{aligned} v(x,t) =& u(x,t) - \int_0^x \mu(x-y)u(y,t)dx \\ &- \int_0^x m(x-y)u_t(y,t)dy - \gamma(x)X(t) \end{aligned} \tag{30}$$
$$\begin{aligned} v_t(x,t) =& u_t(x,t) - KBu(x,t) - \int_0^x \mu(x-y)u_t(y,t)dx \\ &- \int_0^x m''(x-y)u(y,t)dy - \gamma(x)AX(t)\,. \end{aligned} \tag{31}$$

With similar derivations, one can show that the inverse of the transformation $(X,u,u_t) \mapsto (X,v,v_t)$ is defined as

$$\begin{aligned} u(x,t) =& v(x,t) - \int_0^x \sigma(x-y)v(y,t)dx \\ &- \int_0^x n(x-y)v_t(y,t)dy - \rho(x)X(t) \end{aligned} \tag{32}$$
$$\begin{aligned} u_t(x,t) =& v_t(x,t) + KBv(x,t) - \int_0^x \sigma(x-y)v_t(y,t)dx \\ &- \int_0^x n''(x-y)v(y,t)dy - \rho(x)AX(t)\,, \end{aligned} \tag{33}$$

where

$$\rho(x) = -KN(x) \tag{34}$$
$$N(x) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & (A+BK)^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{35}$$
$$n(s) = \int_0^s \rho(\xi)Bd\xi \tag{36}$$
$$\sigma(s) = \int_0^s \rho(\xi)ABd\xi\,. \tag{37}$$

The transformation $(X,v,v_t) \mapsto (X,w,w_t)$ is simpler and given by

$$w(x,t) = v(x,t) + c_0 \int_0^x v(y,t)dy \tag{38}$$
$$w_t(x,t) = v_t(x,t) + c_0 \int_0^x v_t(y,t)dy\,, \tag{39}$$

whereas its inverse is

$$v(x,t) = w(x,t) - c_0 \int_0^x \mathrm{e}^{-c_0(x-y)} w(y,t)dy \tag{40}$$
$$v_t(x,t) = w_t(x,t) - c_0 \int_0^x \mathrm{e}^{-c_0(x-y)} w_t(y,t)dy\,. \tag{41}$$

The composite transformation $(X,u,u_t)\mapsto(X,w,w_t)$ is

$$\begin{aligned}w(x,t)=&u(x,t)\\&+\int_0^x\left(c_0-\mu(x-y)-c_0\int_0^{x-y}\mu(\xi)d\xi\right)u(y,t)dy\\&-\int_0^x\left(m(x-y)+c_0\int_0^{x-y}m(\xi)d\xi\right)u_t(y,t)dy\\&-\left(\gamma(x)+c_0\int_0^x\gamma(\xi)d\xi\right)X(t)\end{aligned}\tag{42}$$

$$\begin{aligned}w_t(x,t)=&u_t(x,t)-KBu(x,t)\\&-\int_0^x\left(c_0m'(x-y)+m''(x-y)\right)u(y,t)dy\\&+\int_0^x\left(c_0-\mu(x-y)-c_0\int_0^{x-y}\mu(\xi)d\xi\right)u_t(y,t)dy\\&-\left(\gamma(x)+c_0\int_0^x\gamma(\xi)d\xi\right)AX(t)\end{aligned}\tag{43}$$

and its inverse is

$$\begin{aligned}u(x,t)=&w(x,t)\\&-\int_0^x\Big(c_0\mathrm{e}^{-c_0(x-y)}+\sigma(x-y)\\&-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\Big)w(y,t)dy\\&-\int_0^x\left(n(x-y)-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\right)\\&\times w_t(y,t)dy\\&-\rho(x)X(t)\end{aligned}\tag{44}$$

$$\begin{aligned}u_t(x,t)=&w_t(x,t)+KBw(x,t)\\&-\int_0^x\Big(n''(x-y)-c_0n'(x-y)+c_0^2n(x-y)\\&+c_0^3\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\Big)w(y,t)dy\\&-\int_0^x\Big(c_0\mathrm{e}^{-c_0(x-y)}+\sigma(x-y)\\&-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\Big)w_t(y,t)dy\\&-\rho(x)AX(t)\,.\end{aligned}\tag{45}$$

Next, we design a controller that satisfies the boundary condition (13). First, from (42) we get

$$\begin{aligned}w_x(x,t)=&u_x(x,t)+c_0u(x,t)\\&-\int_0^x\left(\mu'(x-y)+c_0\mu(x-y)\right)u(y,t)dy\\&-\int_0^x\left(m'(x-y)+c_0m(x-y)\right)u_t(y,t)dy\\&-\left(\gamma'(x)+c_0\gamma(x)\right)X(t)\,.\end{aligned}\tag{46}$$

Then, the control law is

$$\begin{aligned}U(t)=&(-c_0+c_1KB)u(D,t)-c_1u_t(D,t)\\&+\int_0^Dp(D-y)u(y,t)dy+\int_0^Dq(D-y)u_t(y,t)dy\\&+\pi(D)X(t)\end{aligned}\tag{47}$$

where

$$p(s)=\mu'(s)+c_0\mu(s)+c_1\left(m''(s)+c_0m'(s)\right)\tag{48}$$

$$q(s)=m'(s)+c_0m(s)+c_1\left(\mu(s)+c_0\int_0^s\mu(\xi)d\xi-c_0\right)\tag{49}$$

$$\pi(x)=\gamma'(x)+\gamma(x)(c_0I+c_1A)+c_1c_0\int_0^x\gamma(\xi)d\xi A\,.\tag{50}$$

Next we state a new controller that compensates the wave PDE actuator dynamics and prove exponential stability of the resulting closed-loop system.

Theorem 1: (Stabilization) Consider a closed-loop system consisting of the plant (1)–(4) and the control law (47). For any initial condition such that $u(\cdot,0)\in H^1$ and $u_t(\cdot,0)\in L^2$, the closed-loop system has a unique solution $\big(X(t),u(\cdot,t),u_t(\cdot,t)\big)\in C([0,\infty),\mathbb{R}^n\times H^1(0,D)\times L^2(0,D))$ and is exponentially stable in the sense of the norm

$$\left(|X(t)|^2+u(0,t)^2+\int_0^Du_x(x,t)^2dx+\int_0^Du_t(x,t)^2dx\right)^{1/2}.\tag{51}$$

Moreover, if the initial condition $\big(u(\cdot,0),u_t(\cdot,0)\big)$ is compatible with the control law (47) and belongs to $H^2(0,D)\times H^1(0,D)$, then $\big(X(t),u(\cdot,t),u_t(\cdot,t)\big)\in C^1([0,\infty),\mathbb{R}^n\times H^1(0,D)\times L^2(0,D))$ is the classical solution of the closed-loop system.

Proof: We will use the system norms

$$\Omega(t)=u(0,t)^2+\|u_x(t)\|^2+\|u_t(t)\|^2+|X(t)|^2\tag{52}$$

$$\Xi(t)=w(0,t)^2+\|w_x(t)\|^2+\|w_t(t)\|^2+|X(t)|^2\,,\tag{53}$$

where $\|u(t)\|^2$ is a compact notation for $\int_0^Du(x,t)^2dx$. In addition, we employ a Lyapunov function

$$V(t)=X(t)^TPX(t)+aE(t)\,,\tag{54}$$

where the matrix $P=P^T>0$ is the solution to the Lyapunov equation $P(A+BK)+(A+BK)^TP=-Q$ for some $Q=Q^T>0$, the parameter $a>0$ is to be chosen later, and the function $E(t)$ is defined by

$$\begin{aligned}E(t)=&\frac{1}{2}\left(c_0w(0,t)^2+\|w_x(t)\|^2+\|w_t(t)\|^2\right)\\&+\delta\int_0^D(1+x)w_x(y,t)w_t(y,t)dy\,,\end{aligned}\tag{55}$$

By using (43), (46), (45), and

$$\begin{aligned}u_x(x,t)=&w_x(x,t)-c_0w(x,t)\\&-\int_0^x\Big(-c_0^2\mathrm{e}^{-c_0(x-y)}+\sigma'(x-y)-c_0\sigma(x-y)\\&+c_0^2\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\Big)w(y,t)dy\\&-\int_0^x\Big(n'(x-y)-c_0n(x-y)\\&+c_0^2\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\Big)w_t(y,t)dy\\&-\rho'(x)X(t)\end{aligned}\tag{56}$$

$$u(0,t)=w(0,t)+KX(t)\,,\tag{57}$$

and by using Poincare's inequality, for sufficiently small δ it is possible to show that there exist positive constants $\theta_1, \theta_2, \theta_3, \theta_4$ such that

$$\theta_1 \Xi \leq \Omega \leq \theta_2 \Xi \tag{58}$$

$$\theta_3 \Xi \leq V \leq \theta_4 \Xi . \tag{59}$$

Furthermore, it is readily shown that

$$\begin{aligned} \dot{E}(t) = & -\left(c_1 - \delta \frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2 \\ & -\frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2\right) \\ & -\frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right) . \end{aligned} \tag{60}$$

Then, by choosing

$$a \geq \frac{8|PB|^2}{\delta c_0^2 \lambda_{\min}(Q)}, \tag{61}$$

we get

$$\dot{V} \leq -\eta V \tag{62}$$

for some sufficiently small positive η. From (58), (59), (62), it follows that

$$\Omega(t) \leq \frac{\theta_2 \theta_4}{\theta_1 \theta_3} \Omega(0) \mathrm{e}^{-\eta t} . \tag{63}$$

The rest of the argument is almost identical to [12]. ■

III. Robustness to Uncertainty in the Wave Propagation Speed

We now study robustness of the feedback law (47) to a small perturbation of the propagation speed in the actuator dynamics, i.e., we study stability robustness of the closed-loop system

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{64}$$

$$u_{tt}(x,t) = (1+\varepsilon) u_{xx}(x,t) \tag{65}$$

$$u_x(0,t) = 0 \tag{66}$$

$$\begin{aligned} u_x(D,t) = & (-c_0 + c_1 KB) u(D,t) - c_1 u_t(D,t) \\ & + \int_0^D p(D-y) u(y,t) dy + \int_0^D q(D-y) u_t(y,t) dy \\ & + \pi(D) X(t) \end{aligned} \tag{67}$$

to the perturbation parameter ε, which we allow to be either positive or negative but small.

With a very long calculation, we arrive at the representation of the system (64)–(67) in the w-variable:

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{68}$$

$$w_{tt}(x,t) = (1+\varepsilon) w_{xx}(x,t) + \varepsilon \Pi(x,t) \tag{69}$$

$$w_x(0,t) = c_0 w(0,t) \tag{70}$$

$$w_x(D,t) = -c_1 w_t(D,t) , \tag{71}$$

where

$$\begin{aligned} \Pi(x,t) = & \left(\gamma(x) + c_0 \int_0^x \gamma(\xi) d\xi\right) \\ & \times ((A+BK)X(t) + Bw(0,t)) \\ & + KBw_t(x,t) + (KB)^2 w(x,t) \\ & + \int_0^x g(x-y) w(y,t) dy + \int_0^x h(x-y) w_t(y,t) dy \\ & - \left(KB\rho(x) + \int_0^x \omega(x-\xi)\rho(\xi) d\xi\right) AX(t) \end{aligned} \tag{72}$$

and where

$$\omega(x) = m''(x) + c_0 m'(x) \tag{73}$$

$$g(x) = KB\phi(x) + KB\omega(x) + \int_0^x \omega(x-y)\phi(y) dy \tag{74}$$

$$h(x) = KB\psi(x) + \omega(x) + \int_0^x \omega(x-y)\psi(y) dy \tag{75}$$

$$\phi(x) = -n''(x) + c_0 n'(x) - c_0^2 n(x) + c_0^3 \int_0^x \mathrm{e}^{-c_0(x-y)} n(y) dy \tag{76}$$

$$\psi(x) = -c_0 \mathrm{e}^{-c_0 x} - \sigma(x) + c_0 \int_0^x \mathrm{e}^{-c_0(x-y)} \sigma(y) dy . \tag{77}$$

The state perturbation $\Pi(x,t)$ is very complicated in appearance but $\int_0^D \Pi(x,t)^2 dx$ can be bounded in terms of $\Xi(t)$ as defined in (53), and hence also in terms of $V(t)$ as defined in (54). Consequently, the same kind of Lyapunov analysis can be conducted as in the proof of Theorem 1, dominating the effect of $\Pi(x,t)$ for small ε, to prove the following robustness result.

Theorem 2: (Robustness to Small Error in Wave Propagation Speed) Consider the closed-loop system (64)–(67). There exists a sufficiently small $\varepsilon^* > 0$ such that for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ the closed-loop system is exponentially stable in the same sense as in Theorem 1.

IV. Observer Design

Consider the LTI ODE system in cascade with a wave PDE in the sensing path (as depicted in Figure 2),

$$Y(t) = u(0,t)) \tag{78}$$

$$u_{tt}(x,t) = u_{xx}(x,t) \tag{79}$$

$$u_x(0,t) = 0 \tag{80}$$

$$u(D,t) = CX(t) \tag{81}$$

$$\dot{X}(t) = AX(t) + BU(t) . \tag{82}$$

We recall from [13] that, if (79), (80) are replaced by the delay/transport equation, $u_t(x,t) = u_x(x,t)$, then the predictor-based observer

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + C\mathrm{e}^{Ax} L(Y(t) - \hat{u}(0,t)) \tag{83}$$

$$\hat{u}(D,t) = C\hat{X}(t) \tag{84}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \mathrm{e}^{AD} L(Y(t) - \hat{u}(0,t)) \tag{85}$$

achieves perfect compensation of the observer delay and achieves exponential stability at $u - \hat{u} \equiv 0, X - \hat{X} = 0$.

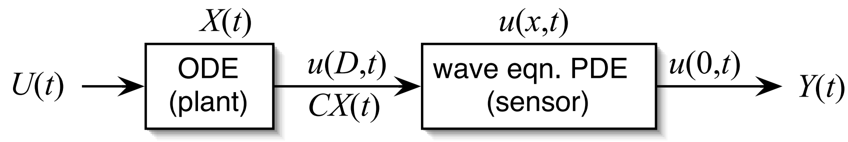

Fig. 2. The cascade of the ODE dynamics of the plant with the wave equation PDE dynamics of the sensor.

We are seeking an observer of the form

$$\begin{aligned}\hat{u}_t(x,t) &= \hat{u}_{xx}(x,t)+\alpha(x)\left(Y(t)-\hat{u}(0,t)\right)\\ &\quad+\beta(x)\left(\dot{Y}(t)-\hat{u}_t(0,t)\right) \qquad (86)\\ \hat{u}_x(0,t) &= -a\left(Y(t)-\hat{u}(0,t)\right)-b\left(\dot{Y}(t)-\hat{u}_t(0,t)\right) \qquad (87)\\ \hat{u}(D,t) &= C\hat{X}(t) \qquad (88)\\ \dot{\hat{X}}(t) &= A\hat{X}(t)+BU(t)+\Lambda\left(Y(t)-\hat{u}(0,t)\right), \qquad (89)\end{aligned}$$

where the functions $\alpha(x),\beta(x)$, the scalars a,b, and the vector Λ are to be determined, to achieve exponential stability of the observer error system

$$\begin{aligned}\tilde{u}_t(x,t) &= \tilde{u}_{xx}(x,t)-\alpha(x)\tilde{u}(0,t)-\beta(x)\tilde{u}_t(0,t) \qquad (90)\\ \tilde{u}_x(0,t) &= a\tilde{u}(0,t)+b\tilde{u}_t(0,t) \qquad (91)\\ \tilde{u}(D,t) &= C\tilde{X}(t) \qquad (92)\\ \dot{\tilde{X}}(t) &= A\tilde{X}(t)-\Lambda\tilde{u}(0,t), \qquad (93)\end{aligned}$$

where

$$\begin{aligned}\tilde{u}(x,t) &= u(x,t)-\hat{u}(x,t) \qquad (94)\\ \tilde{X}(t) &= X(t)-\hat{X}(t). \qquad (95)\end{aligned}$$

We consider the transformation

$$\tilde{w}(x)=\tilde{u}(x)-\Gamma(x)\tilde{X} \qquad (96)$$

and try to find $\Gamma(x)$, along with $\alpha(x),\beta(x),a,b,\Lambda$, that convert (90)–(93) into the exponentially stable system

$$\begin{aligned}\tilde{w}_{tt}(x,t) &= \tilde{w}_{xx}(x,t) \qquad (97)\\ \tilde{w}_x(0,t) &= c_0\tilde{w}_t(0,t) \qquad (98)\\ \tilde{w}(D,t) &= 0 \qquad (99)\\ \dot{\tilde{X}}(t) &= \left(A-\Lambda\Gamma(0)\right)\tilde{X}-\Lambda\tilde{w}(0,t), \qquad (100)\end{aligned}$$

where $c_0>0$ and $A-\Lambda\Gamma(0)$ is a Hurwitz matrix.

By matching the systems (90)–(93) and (97)–(100), we obtain the conditions

$$\begin{aligned}\Gamma''(x) &= \Gamma(x)A^2 \qquad (101)\\ \Gamma'(0) &= c_0\Gamma(0)A \qquad (102)\\ \Gamma(D) &= C, \qquad (103)\end{aligned}$$

as well as

$$\begin{aligned}\alpha(x) &= \Gamma(x)A\Lambda \qquad (104)\\ \beta(x) &= \Gamma(x)\Lambda \qquad (105)\\ a &= c_0\Gamma(0)\Lambda \qquad (106)\\ b &= c_0. \qquad (107)\end{aligned}$$

Solving the linear ODE two-point-boundary-value problem (101)–(103), we obtain

$$\Gamma(x)=\Gamma(0)G(x), \qquad (108)$$

where

$$\begin{aligned}\Gamma(0) &= CG(D)^{-1} \qquad (109)\\ G(x) &= \begin{bmatrix} I & c_0A\end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2\\ I & 0\end{bmatrix}x}\begin{bmatrix} I\\ 0\end{bmatrix}. \qquad (110)\end{aligned}$$

Thus, we have determined all the quantities needed to implement the observer (86)–(89) except Λ, which needs to be chosen so that the matrix $A-\Lambda\Gamma(0)$ is Hurwitz. We pick Λ as

$$\Lambda=G(D)L, \qquad (111)$$

where L is chosen so that the matrix $A-LC$ is Hurwitz. Since A and $G(D)$ commute, using $G(D)$ as a similarity transformation for the matrix $A-\Lambda\Gamma(0)=A-G(D)LCG(D)^{-1}$, we get that the matrices $A-LC$ and $A-\Lambda\Gamma(0)$ have the same eigenvalues, so the latter matrix is Hurwitz.

So the system (97)–(100) is a cascade of a wave equation (97)–(99), which is exponentially stable due to the 'damping' boundary condition (98), and of the exponentially stable ODE (100). So, the entire observer error system is exponentially stable.

Theorem 3: (Observer Design and Convergence) Assume that the matrix $G(D)$ is non-singular. The observer (86)–(89), with gains defined through (104)–(111), guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, i.e., more precisely, that the observer error system is exponentially stable in the sense of the norm

$$\left(|X(t)-\hat{X}(t)|^2 + \int_0^D \left(u_x(x,t)-\hat{u}_x(x,t)\right)^2dx + \int_0^D \left(u_t(x,t)-\hat{u}_t(x,t)\right)^2dx\right)^{1/2}. \qquad (112)$$

Proof: Very similar to the proof of Theorem 1, with a Lyapunov function

$$V(t)=\tilde{X}(t)^TM(D)^{-T}PM(D)^{-1}\tilde{X}(t)+aE(t), \qquad (113)$$

where $P=P^T>0$ is the solution to the Lyapunov equation $P(A-LC)+(A-LC)^TP=-Q$ for some $Q=Q^T>0$, and with

$$\begin{aligned}E(t) &= \frac{1}{2}\left(\|\tilde{w}_x(t)\|^2+\|\tilde{w}_t(t)\|^2\right)\\ &\quad+\delta\int_0^D(-1-D+x)\tilde{w}_x(y,t)\tilde{w}_t(y,t)dy, \qquad (114)\end{aligned}$$

The system norms are simpler,

$$\Omega(t) = \|\tilde{u}_x(t)\|^2 + \|\tilde{u}_t(t)\|^2 + |\tilde{X}(t)|^2 \quad (115)$$

$$\Xi(t) = \|\tilde{w}_x(t)\|^2 + \|\tilde{w}_t(t)\|^2 + |\tilde{X}(t)|^2, \quad (116)$$

and the system transformations are much simpler,

$$\tilde{w}_x(x,t) = \tilde{u}_x(x,t) - \Gamma'(x)\tilde{X}(t) \quad (117)$$

$$\begin{aligned} \tilde{w}_t(x,t) &= \tilde{u}_t(x,t) - \Gamma(x)A\tilde{X}(t) + \Gamma(x)\Lambda\tilde{u}(0,t) \\ \tilde{u}_t(x,t) &= \tilde{w}_t(x,t) + \Gamma(x)(A - \Lambda\Gamma(0))\tilde{X}(t) \\ &\quad -\Gamma(x)\Lambda\tilde{w}(0,t). \end{aligned} \quad (118)$$

One obtains the inequalities (58), (59) with the help of Agmon's inequality, or, with the help of Poincare's inequality and the alternative representation of the state transformation,

$$\begin{aligned} \tilde{w}_t(x,t) &= \tilde{u}_t(x,t) + \Gamma(x)\Lambda\tilde{u}(x,t) - \Gamma(x)\Lambda\int_0^D \tilde{u}_x(y,t)dy \\ &\quad -\Gamma(x)A\tilde{X}(t) \\ \tilde{u}_t(x,t) &= \tilde{w}_t(x,t) - \Gamma(x)\Lambda\tilde{w}(x,t) + \Gamma(x)\Lambda\int_0^d \tilde{w}_x(y,t)dy \\ &\quad +\Gamma(x)(A - \Lambda\Gamma(0))\tilde{X}(t). \end{aligned} \quad (119)$$

Then, one obtains (63), which completes the proof. ■

REFERENCES

[1] Z. Artstein, "Linear systems with delayed controls: a reduction," *IEEE Trans. on Automatic Control*, 27, pp. 869–879, 1982.

[2] S. Evesque, A. M. Annaswamy, S. Niculescu, and A. P. Dowling, "Adaptive Control of A Class of Time-delay Systems," *ASME Transactions on Dynamics, Systems, Measurement, and Control*, vol. 125, pp. 186–193, 2003.

[3] Y. A. Fiagbedzi and A. E. Pearson, "Feedback stabilization of linear autonomous time lag systems," *IEEE Trans. Automatic Control*, 31, pp. 847–855, 1986.

[4] K. Gu and S.-I. Niculescu, "Survey on recent results in the stability and control of time-delay systems," *Transactions of ASME*, 125, pp. 158–165, 2003.

[5] M. Jankovic, "Forwarding, backstepping, and finite spectrum assignment for time delay systems," *2006 American Control Conference*.

[6] M. Jankovic, "Control of cascade systems with time delay—the integral cross-term approach," *2006 IEEE Conference on Decision and Control*.

[7] M. Jankovic, "Recursive predictor design for linear systems with time delay," *2008 American Control Conference*.

[8] J. Klamka, "Observer for linear feedback control of systems with distributed delays in controls and outputs," *Syst Contr Lett*, vol. 1, pp. 326-331, 1982.

[9] M. Krstic, "Lyapunov tools for predictor feedbacks for delay systems: Inverse optimality and robustness to delay mismatch," *Automatica*, in press.

[10] M. Krstic, "On compensating long actuator delays in nonlinear control," *IEEE Transactions on Automatic Control*, in press.

[11] M. Krstic, "Predictor-like feedback for actuator and sensor dynamics governed by diffusion PDEs," submitted to *2009 American Control Conference*.

[12] M. Krstic, B.-Z. Guo, A. Balogh, and A. Smyshlyaev, "Output-feedback stabilization of an unstable wave equation," *Automatica*, vol. 44 , pp. 63–74, 2008.

[13] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *Systems & Control Letters*, vol. 57, pp. 750–758, 2008.

[14] W. H. Kwon and A. E. Pearson, "Feedback stabilization of linear systems with delayed control," *IEEE Trans. on Automatic Control*, 25, pp. 266–269, 1980.

[15] A. Z. Manitius and A. W. Olbrot, "Finite spectrum assignment for systems with delays," *IEEE Trans. on Automatic Control*, 24, pp. 541–553, 1979.

[16] W. Michiels and S.-I. Niculescu, "On the delay sensitivity of Smith predictors," *International Journal of Systems Science*, vol. 34, pp. 543-551, 2003.

[17] W. Michiels and S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eignevalue-Based Approach*, SIAM, 2007.

[18] L. Mirkin, "On the approximation of distributed-delay control laws," *Systems & Control Letters*, vol. 51, pp. 331-342, 2004.

[19] S. Mondie and W. Michiels, "Finite spectrum assignment of unstable time-delay systems with a safe implementation," *IEEE Trans. on Automatic Control*, 48, pp. 2207–2212, 2003.

[20] S.-I. Niculescu and A. M. Annaswamy, "An Adaptive Smith-Controller for Time-delay Systems with Relative Degree $n^* \geq 2$," *Systems & Control Letters*, vol. 49, pp. 347–358, 2003.

[21] A. W. Olbrot, "Stabilizability, detectability, and spectrum assignment for linear autonomous systems with general time delays," *IEEE Trans. on Automatic Control*, 23, pp. 887–890, 1978.

[22] J.-P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667-1694, 2003.

[23] A. Smyshlyaev and M. Krstic, "Closed form boundary state feedbacks for a class of 1D partial integro-differential equations," *IEEE Trans. on Automatic Control*, Vol. 49, No. 12, pp. 2185-2202, 2004.

[24] A. Smyshlyaev and M. Krstic, "Backstepping observers for a class of parabolic PDEs," *Systems and Control Letters*, vol.54, pp. 613-625, 2005.

[25] G. Tadmor, "The standard H_∞ problem in systems with a single input delay," *IEEE Transactions on Automatic Control*, vol. 45, pp. 382–397, 2000.

[26] K. Watanabe, "Finite spectrum assignment and observer for multivariable systems with commensurate delays," *IEEE Transactions on Automatic Control*, vol. 31, pp. 543-550, 1996.

[27] K. Watanabe and M. Ito, "An observer for linear feedback control laws of multivariable systems with multiple delays in controls and outputs," *Systems and Control Letters*, 1, pp. 54–59, 1981.

[28] L. Zaccarian and D. Nesic, "A cascade interpretation of the Smith predictor and the arising enhanced scheme," *2006 American Control Conference*.

[29] Q.-C. Zhong, "On distributed delay in linear control laws—Part I: Discrete-delay implementation," *IEEE Transactions on Automatic Control*, vol. 49, pp. 2074–2080, 2006.

[30] Q.-C. Zhong, *Robust Control of Time-delay Systems*, Springer, 2006.

[31] Q.-C. Zhong and L. Mirkin, "Control of integral processes with dead time—Part 2: Quantitative analysis," *IEE Proc. Control Theory & Appl.* , vol. 149, pp. 291-296, 2002