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Finite difference solution of discrete-time regulator equation and its application to digital output regulation problem

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Abstract— **A method for the solution of the regulator equation for the discrete-time nonlinear output regulation problem is presented. This method is based on solution of the regulator equation by finite differences. Moreover, the algebraic condition guaranteeing zero tracking error is replaced by an error functional whose value decreases to zero in an iterative process. The conditions guaranteeing convergence are shown together with some important details for implementation of the scheme. The method is verified on an example.**

I. Introduction

The problem of tracking a desired reference signal and rejecting disturbances is one of major problems of the recent control theory. The output regulation problem also fits to this category as it is a special case when the reference signal as well as the disturbance signals are generated by an autonomous system. First references to the solution of this problem are for example [4]. Analysis of the discrete-time output regulation problem is thoroughly discussed in [1], [2], [11]. A recent survey of results in this field is contained in [7].

Recently, numerous results has been achieved for the output regulation problem of continuous time systems ([6], [5], [12], [3], [9]), including various approximate methods to solve the so-called output regulation equation being the crucial ingredient of all those results. Nevertheless, during the practical digital implementation, the obtained feedback controllers are applied at certain sampled time moments, causing additional unpredictable inaccuracies that are not taken into the account during the controller design.

The natural solution to such a drawback is to solve and then digitally implement the discrete time output regulation problem for suitable discretization of the original continuous-time system. The best option is the discretization based on the input-parameterized flow of the system, tentatively called in the sequel as the **exact sampled discretization**. More specifically, consider the continuous time system $\dot{x} = f(x,u),\ y = h(x)$ then its **exact sampled discretization** with sampling time t is the following discrete time system. $x(t+1) = \phi(x(t), u(t)),\ y(t) = h(x(t)), \phi(x,u) := \Phi_t^u(x)$, where $\Phi_t^u(x)$ is u-parameterized flow of the above system, i.e. $\frac{\mathrm{d}\Phi_t^u}{\mathrm{d}t} = f(\Phi_t^u(x), u), \Phi_0^u(x) = x$. Obviously, the exact sampled discretization is the most suitable one, if solving the appropriate control problem for it and then digitally implement this control with sampling time equal to t, one should obtain the most exact results. Obtaining the sampled flow exactly is not always possible and various approximations can be used.

Nevertheless, quite satisfactory approximation of the exact sampled discretization can be obtained off line by solving appropriate ODE and function ϕ may be considered as given numerically with good precision. Summarizing, considering and solving directly output regulation problem is of eminent importance for the digital implementation of obtained feedback controllers.

This task constitutes the main goal of this paper. It will be based on the numerical approximate solution of the discrete regulator equation being the analogue of the well known regulator equation [7]. While the original RE is PDE combined with algebraic equation, in case of digital output regulation problem (DORP) it is a functional equation, also combined with an algebraic equation.

To explain the essence of the approach to solving discrete time output regulation problem, let us first briefly recall basic facts from continuous time output regulation problem.

The key issue is finding the solution of the regulator equation. Its solution builds up a base for the construction of the regulator. This is a system of equations, one equation being algebraic. This equation expresses the requirement for zero tracking error. The other equations are partial differential in the continuous case or functional equations in the discrete-time case.

The classical method for the solution of the regulator equation is based on decomposition of all involved functions into Taylor series. The solution is then found in the form of a Taylor polynomial as well, simply by comparing the coefficients. This method is thoroughly demonstrated in [7] for both discrete-time and continuous cases. For further works concerning this topic, see the list of references there.

This method provides neither any hint of its convergence region nor an error analysis. This is due to the fact that it is a mere comparison of coefficients of a Taylor series. Other disadvantage is that it is hardly algoritmizable.

Another approach, described in [3] and further developed in [8], was adopted for the continuous-time nonlinear output regulation problems. Instead of satisfying the algebraic condition, an iterative process is defined. During this process, the partial differential equation is solved in each iteration with a fixed control signal. Then, the error made in the algebraic condition is measured through an error functional. This value helps to adjust the control in the next iteration. In other

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words, the requirement of validity of the algebraic condition is replaced by the iterative scheme that converges to the solution of the output regulation problem. A description of this scheme together with conditions of convergence and error estimates is contained in [8]. Some implementation details are also mentioned. The solution is done using the Finite-Element Method. This is advantageous as there is numerical software for solution of such problems widely available. This made algoritmization of this process quite straightforward. It is also worth a remark that the plant was pre-stabilized using a state feedback first which allowed to apply this method also for nonminimum-phase systems for which the Taylor-based approach fails.

A similar approach is adopted here for the case of discrete-time systems. The main difference from the continuous case is that the regulator equation does not contain partial differential equations anymore, rather it contains functional equations. However, as the Finite-Element Method is difficult to employ in the case of functional equations the method of finite differences was used. Other features remained unchanged: the controlled plant is stabilized using a state feedback, then, for this system, the regulator equations are formulated. An error functional is defined which is used in the iterative process as outlined above. Convergence conditions and error estimates are presented together with some implementation details.

The proposed method for the solution of the DORP exhibits, like the corresponding methods for the solution of the continuous ORP, the following features:

- Simple algoritmizability as the use of the finite-difference method (used in this article) is quite straightforward.
- Properties of the solution on a predefined neighborhood of the origin are guaranteed (in contrast to the Taylor series-based method where this neighborhood is not defined).
- Convergence criterium is derived.

The aim of the paper is to thoroughly present the new method, to briefly describe its implementation and to find conditions guaranteeing its convergence. It is organized as follows: next section introduce the DORP in detail, including basic results underlying the key role of the discrete-time regulator equation (DRE). The algorithm to solve DRE and its implementation details are presented in Sections III,IV while the algorithm convergence is analyzed in Section V. Section VI presents the practical case study of the gyroscopical platform to illustrate our approach. Some conclusions are drawn in the final section.

II. Discrete-Time Output Regulation Problem

Let us introduce the nonlinear discrete-time output regulation problem precisely. Consider a sufficiently smooth plant described by the equations

$$\begin{aligned} x(t+1) &= f(x(t)) + g(x(t))u(t) && (1)\\ y(t) &= h(x(t)) && (2) \end{aligned}$$

The meaning of the symbols is as follows: $x(t)$ is an n-dimensional state of the system, $u(t)$ is the one-dimensional input, $y(t)$ represents its scalar output. Besides, let us consider the following autonomous system

$$\begin{aligned} v(t+1) &= Sv(t) && (3)\\ w(t) &= Qv(t). && (4) \end{aligned}$$

Here, $v(t)$ is its μ-dimensional state, $w(t)$ is its scalar output. The output of the system represents the trajectory to be followed or the disturbance to be rejected (this case not being investigated in this paper). It will be assumed that the matrix S has all eigenvalues lying on the unit circle. This autonomous system is called the **exosystem**. Let us remark that the assumption about eigenvalues of the matrix S can be generalized to the case of nonlinear exosystems by introducing the so-called neutral stability.

The system is in the "input-affine" shape in order to simplify the text, however, the results are applicable for more general systems. Similarly, without greater effort the derived results hold for nonlinear exosystems.

The goal of the regulation is to find a feedback compensator such that the resulting closed loop system is internally stable and its output asymptotically tracks the reference trajectory generated by the exosystem.

These requirements mean in particular the following: a function $u = u(x, v)$ (which is the feedback from the states of both the system and the exosystem) is to be found so that:

1) the equilibrium $x = 0$ is asymptotically stable in the case no exogenous signal is present,
2) there exists a neighborhood $U \subset R^{n+\mu}$ of the origin so that for each initial condition $(x(0), v(0)) \in U$ holds:

$$\lim_{t\to+\infty} \|y(t) - w(t)\| \to 0.$$

The output regulation problem can be formulated more generally. A survey of these more general settings can be found in [7].

The crucial result concerning solvability of the DORP is the following one:

The output regulation problem is locally solvable around the origin if the plant (1) has an asymptotically stabilizable linearization at the origin and there exists a solution of the equation:

$$\begin{aligned} \mathbf{x}(Sv) &= f(\mathbf{x}(v)) + g(\mathbf{x}(v))c(v) && (5)\\ h(\mathbf{x}(v)) &= Qv && (6) \end{aligned}$$

with the condition $x(0) = 0$.

The system of equations (5,6) is called the DRE. More precisely:

Lemma 1 ([7]): If the condition 1) holds then the condition 2) holds if and only if there exists a solution of the system (5,6) (denoted by $(x(v), u(v))$. The manifold $(x(v), v)$ is the output-zeroing manifold if $x(v)$ is the solution of (5,6). Then there exists a function $c(v)$ and a matrix L such that the control to be found is

$$u(t) = u(x(t), v(t)) = \\ L(x(t) - \mathbf{x}(v(t))) + c(\mathbf{x}(v(t)), v(t)).$$

The system (5,6) consists of n functional equations and one algebraic equation for the unknown functions $x(v)$ and $c(v)$. It is defined on the space R^{μ}. The usual approach assumes the system has well defined relative degree and has hyperbolic zero dynamics. After some change of coordinates, the regulator equation is reduced to a pure functional equation whose solution coincides with the output zeroing manifold described above. Due to hyperbolicity, such a manifold exists. See [7] and references therein for details.

In this paper, an approach avoiding the need for elimination is presented. The main idea of the described way are the following: first, the plant is stabilized using the state feedback. Then the functional equations are solved using the finite-difference method with a fixed function $c(v)$. After that, the value of a certain penalty functional is evaluated. In the next iteration, the value of the function u is changed so that the value of the cost functional decreases. Notice that, in the case the value of the cost functional is zero, the precise solution of the regulator equation was achieved.

Let us underline that this approach is applicable even in the case of nonminimum-phase systems. This is thanks to the pre-stabilization of the system. Stability is kept throughout the whole process as no elimination of variables is done. Price for these features is a larger amount of variables and the need for an iterative algorithm which, moreover, includes a solution of a functional equation using the finite-difference scheme.

III. Algorithm for solving the regulator equation

Let us describe the procedure outlined in the previous section in detail. Suppose that the plant (1) has an asymptotically stabilizable linear approximation at the origin.

The first step is a stabilization of the plant. Namely, a feedback gain matrix L is sought so that the system

$$x(t+1) = f(x(t)) + g(x(t))(u(t) - Lx(t))$$

is stable. Let us introduce the notation

$$\tilde{f}(x) = f(x) - g(x)Lx. \tag{7}$$

In the second step, the regulator equation for the already stabilized system is formed. The part of RE consisting of the functional equations reads:

$$x(Sv) = \tilde{f}(\mathbf{x}(v)) + g(\mathbf{x}(v))c(v). \tag{8}$$

The functional equation (8) is solved using the finite-difference method. This involves the following steps: a bounded domain $\Omega \subset R^{\mu}$ is chosen so that $0 \in \Omega$. Then the solution of the equation (8) is sought on this domain. This is due to the fact that the finite-difference scheme cannot be applied to unbounded domains. Thus, as will become clear later, this domain should contain all the trajectories of the exosystem that will be used for tracking. Moreover, the domain is supposed to be invariant with respect to the exosystem. This means:

$$x \in \Omega \Rightarrow Sx \in \Omega. \tag{9}$$

This fact guarantees that a trajectory starting in the chosen set Ω stays in this set.

To do this, a rectangular grid is defined in the set Ω. The function u is replaced by its values on this grid only. Denote the number of these points by N. Similarly, the quantity to be sought are the values of the function x on this grid. (If other values of these functions are necessary if tracking is to be implemented, then some interpolation scheme must be used.) This turns the equation (8) into a set of algebraic equations.

The next step is connected with adjusting the function u. To do this, one defines the functional

$$J(u) = \int_{\Omega} (h(\mathbf{x}(v)) - Qv)^2 dv \tag{10}$$

or its discrete counterpart

$$J_D(u) = \sum_{i=1}^{N} (h(\mathbf{x}(v_i)) - Qv_i)^2 \tag{11}$$

where $v_1, \ldots, v_N$ are the nodes of the grid. Both functionals can be used. The functional (10) is natural as it arises from the original formulation, however, use of (11) does not involve any interpolation. Moreover, the function $\mathbf{x}(v)$ in the definition of the above functionals is the finite-difference solution of (8) with function u in the right-hand side.

Let us conclude this section by the remark that the need for invariance of the domain Ω is not restrictive upon the requirement of all eigenvalues of the exosystem being on the unit circle.

Proposition: Let Ω be an arbitrary bounded domain containing the origin. Then the set $\bigcup_{k=1}^{\infty} S\Omega$ is a bounded invariant domain.

Proof: invariance is clear. Boundedness: let r be such that $r > \|v\|$ for all $v \in \Omega$. Then $\|S^k v\| \leq \|S\|^k \|v\| \leq r$.

Let us conclude this section with the announced algorithm.

1) Choose a domain Ω which is forward-invariant with respect to the exosystem.
2) Choose an initial guess of the feedforward c defined on the domain Ω. Moreover, $c(0) = 0$.
3) Solve the equation (8) with the feedforward fixed.
4) Evaluate the functional (11) for the solution of the functional equation computed in the previous step.
5) Decide whether to stop or not. The value of the cost functional evaluated in the previous step can provide a help as it is closely related with the tracking error. This error is thoroughly commented in the fifth section.

IV. Implementation details

Unfortunately, analysis of convergence of the sequence of the solutions of the equation (8) depends on the way how the finite-difference method is implemented. Therefore, some remarks about implementation are summarized here.

The main problem of the finite-difference method in its "pure" form is the large amount of variables. This method was applied to partial differential equations. Some of them, namely the elliptic equations, possess certain "smoothing" properties that cannot be expected here in the case of functional equations. This could mean that some oscillations are met. To avoid this phenomenon, the function $\mathbf{x}(v)$ was approximated by its Taylor polynomial of m-th degree (in the simulations, $m = 2$ was chosen). In the μ-dimensional space, such a polynomial has the form

$$P(v, \bar{\xi}) = \xi_0 + \sum_{i=1}^{\mu} \xi_i v_i + \ldots + \sum_{I}^{\mu} \xi_{i_1,\ldots,i_m} v_1^{i_1} \ldots v_\mu^{i_\mu} \quad (12)$$

where $I = \{(i_1, \ldots, i_\mu) \in N, i_1 + \ldots + i_\mu = m$ and the vector $\bar{\xi}$ contains all the coefficients ξ that appear in the Taylor expansion (12).

Let $\bar{x}^{(i)}$ be the above defined coefficients of the Taylor series of the function $\mathbf{x}_i(v)$, $i = 1, \ldots, n$. Define the vector $\bar{x}$ by $\bar{x}^T = (\bar{x}^{(1)T}, \ldots, \bar{x}^{(n)T})$.

Moreover, one defines the matrix P_1 so that its i-th row contain the values of polynomials $1, v_1, v_2, \ldots, v_\mu, v_1^2, v_1 v_2, \ldots$ up to the order m evaluated at the point $v^i = (v_1^i, \ldots, v_\mu^i)$. This means

$$P_1 = \begin{pmatrix} 1 & v_1^1 & \ldots & v_{\mu-1}^1 (v_\mu^1)^{m-1} & (v_\mu^1)^m \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & v_1^N & \ldots & v_{\mu-1}^N (v_\mu^N)^{m-1} & (v_\mu^N)^m \end{pmatrix}$$

the function $x_j(v), j = 1 \ldots, n$ is approximated by the Taylor polynomial whose coefficients are contained int the column vector $\bar{x}^j$. Thus the approximations of the values of $x_j(v)$ at the nodes of the grid are given as $P_1 \bar{x}^j$. Analogously, the matrix P_2 is defined, but it is composed of the values of the "transformed" mesh, namely of the points $Sv^1, \ldots, Sv^N$. Let also the functions φ, ψ be defined by the relations

$$\tilde{f}(x) = Ax + \varphi(x), g(x) = B + \psi(x)$$

where A is the Jacobi matrix of $\tilde{f}$ ($\tilde{f}$ being defined by (7) and $B = g(0)$.

$$\bar{P}_1 = A \otimes P_1, \bar{P}_2 = I \otimes P_2, \bar{B} = B \otimes P_1.$$

Let $\bar{f}_i(\bar{x})$, resp. $\bar{g}_i(\bar{x})$ be defined as the vector of values $f_i(P(v, \bar{x}), g_i(P(v, \bar{x})$, respectively. Finally, let $\bar{f}(\bar{x})^T = (\bar{f}_1^T(\bar{x}), \ldots, \bar{f}_n^T(\bar{x}))$, $\bar{g}$ being defined analogously. Then (8) can be approximated as

$$\bar{P}_2 \bar{x} = \bar{f}(\bar{x}) + \bar{g}(\bar{x}) c(v) + \varepsilon \quad (13)$$

where ε represents the approximation error. The equation (13) is solved using the least squares so as to minimize the norm of the error ε. Defining the functions $\bar{\varphi}, \bar{\psi}$ in the similar way to the definition of $\bar{f}$ allows to rewrite (13) into the form

$$\bar{P}_2 \bar{x} = \bar{P}_1 \bar{x} + \bar{B} \bar{u} + \bar{\varphi}(\bar{x}) + \bar{\psi}(\bar{x}) \quad (14)$$

which may be more suitable to deal with.

Another issue is minimization of the error functional. This is done through selecting values of Taylor coefficients of the control u, after which the equation (14) is solved. With this solution, the value of the functional (11) is evaluated. This problem is in our framework a rather technical one, yet very complicated. Moreover, some optimization software can be used as it was the case in the example given below, where the SciLab package was used. Therefore, we omit the description of this problem.

V. Convergence analysis

A remark about convexity of the functional (11) is made on this chapter. If the controlled plant is linear, then this functional is convex. This is due to the fact that the mapping

$$(x(v^1), \ldots, x(v^N)) \mapsto \sum_{i=1}^{N} (h(x(v^i)) - Qv^i)^2$$

is convex and the equation (8) is linear. If this is not the case, one can apply the exact linearization procedure to transform the system so that this assumption holds. Moreover, the precise solution of the regulator equation is the minimizing point as (11) attains zero value there.

To ensure convergence of the iterative scheme, one needs uniqueness of the minimizer. This is usually guaranteed by coercivity of the minimized functional. This means $J_D(u) \to +\infty$ is required if $\|u\| \to +\infty$. This is not valid in general as there can be the case that the solution of (8) remains bounded even if the norm of u increases to infinity.

For convenience, we restrict ourselves on the case when the function h is linear, namely there exists a matrix H so that $h(x) = Hx$. This is not restrictive as the original problem can be transformed into this form using a state transformation.

Using the results of the previous section, one can write

$$\bar{H} \bar{x} = \bar{H} (\bar{P}_2 - \bar{P}_1)^{-1} \bar{B} \bar{u}.$$

Thus, one has if $\inf_{\|u\|=1} \|\bar{H}(\bar{P}_2 - \bar{P}_1)^{-1} \bar{B} \bar{u}\| \geq k > 0$. This is realistic as the vector u contains the coefficients of the Taylor polynomial approximating the control while the expression $\bar{H}(\bar{P}_2 - \bar{P}_1)^{-1} \bar{B} \bar{u}$ is a vector of output values at each node of the grid. Number of these nodes is usually very high compared to the coefficients to be chosen. Moreover, if in this case $\|u\| \to +\infty$ then also $\|\bar{H}\bar{x}\| \to +\infty$. This immediately implies coercivity of the functional (11) in the linear case.

In the nonlinear case, one deals with the decomposition introduced in (14). The following additional assumption is necessary in this case:

1) $\sup_{\bar{x} \neq 0} \|\bar{\varphi}(\bar{x})\| < \infty$
2) There exists a constant $k > 0$ such that $\sup_{\bar{x}} \|\bar{H}(\bar{P}_2 - \bar{P}_1)^{-1} (\bar{B} + \bar{\psi}(\bar{x})) u\| \geq k \|\bar{u}\|$ for each u.

Then

$$\bar{H} \bar{x} = \bar{H} (\bar{P}_2 - \bar{P}_1)^{-1} (\bar{B} u + \bar{\varphi}(\bar{x}) + \bar{\psi}(\bar{x})) u.$$

Hence

$$\|\bar{H}\bar{x}\| + \|\bar{\varphi}(\bar{x})\| \geq \|\bar{H}(\bar{P}_2 - \bar{P}_1)^{-1} (\bar{B} - \bar{\psi}(\bar{x})) u\| \geq k \|u\|.$$

Thus, if $\|u\| \to \infty$ then also $\|x\| \to \infty$.

Let us mention another important meaning of the value of the error functional. In practice, its value will rarely be exactly equal to zero. This is due to several facts. First, the minimization of the error functional must be terminated after a finite number of iterations. Further, solution of the regulator equation cannot be exact due to various imprecisions and approximations. However, the value of the error functional is closely related to the tracking error. To be more specific, the following holds:

Theorem There exist a positive constant R and a positive nonincereasing sequence $z(t), t \in N$ such that $\lim_{t\to\infty} = 0$ such that if the value of the discrete error functional (11) is less than $\varepsilon > 0$ then the tracking error

$$\sup_{t\in N} \|y(t) - Qv(t)\| < \varepsilon + Rh + z(t)$$

where $h = \sup_{v\in\Omega} \inf_{j=1,\dots,N} \|v - v_j\|$.

Proof: Let the state of the plant be denoted by x. The tracking starts from the initial condition $x_0 = x(0)$, the initial condition of the exosystem is supposed to be $v_0 = v(0)$. One has to take into account that in general $\mathbf{x}(v_0) \neq x_0$. Moreover, let the exact solution of the regulator equation be denoted by $\bar{\mathbf{x}}(v)$ while the computed (inexact) approximation of this manifold be denoted by $\mathbf{x}(v)$.

Thus

$$\|y(t) - Qv(t)\| = \|h(x(t)) - Qv(t)\| \leq$$
$$\|h(x(t)) - h(\mathbf{x}(v(t)))\| + \|h(\mathbf{x}(v(t))) - h(\bar{\mathbf{x}}(v(t))\|.$$

From the properties of the center manifold follows that $z(t) = \|h(x(t)) - h(\mathbf{x}(v(t)))\| \to 0$ for $t \to \infty$.

Let $v \in \Omega$ and let $v_i \in \{v_1, \dots, v_N\}$ be the closest point from the grid to the point v. Then the term $\|h(\mathbf{x}(v)) - h(\bar{\mathbf{x}}(v))\|$ can be estimated as follows:

$$\begin{gathered}\|h(\mathbf{x}(v)) - h(\bar{\mathbf{x}}(v))\| \leq \\ \|h(\mathbf{x}(v)) - h(\mathbf{x}(v_i))\| + \|h(\mathbf{x}(v_i)) - h(\bar{\mathbf{x}}(v_i))\| + \\ \|h(\bar{\mathbf{x}}(v_i)) - h(\bar{\mathbf{x}}(v))\|\end{gathered} \tag{15}$$

First, observe that

$$\|h(\mathbf{x}(v_i)) - h(\bar{\mathbf{x}}(v_i))\| \leq \varepsilon$$

Further, one has

$$\|h(\bar{\mathbf{x}}(v_i)) - h(\bar{\mathbf{x}}(v))\| =$$
$$\|Q(v_i) - Q(v)\| \leq \|Q\|\|v_i - v\| \leq \|Q\|h.$$

The first term in (15) is estimated analogously using the mean value theorem. This theorem implies existence of points η lying in the segment $[\mathbf{x}(v), \mathbf{x}(v_i)]$ and ν from the segment $[v, v_i]$ such that the following holds:

$$\|h(\mathbf{x}(v)) - h(\mathbf{x}(v_i))\| \leq \|\nabla h(\eta)\|\|\mathbf{x}(v) - \mathbf{x}(v_i)\| \leq$$
$$\|\nabla h(\eta)\|\|\nabla \mathbf{x}(\nu)\|\|v - v_i\|.$$

The function h is smooth. The function $\mathbf{x}$ is defined as a polynomial, thus it is also smooth and its values are bounded on a bounded domain, hence one infers with $C = \sup_{\nu\in\Omega, \eta\in\mathbf{x}(\Omega)} \|\nabla h(\eta)\|\|\nabla \mathbf{x}(\nu)\|$ that

$$\|h(\mathbf{x}(v)) - h(\mathbf{x}(v_i))\| \leq Ch.$$

The proof is completed by setting $R = \|Q\| + C$.

VI. Example

The method described so far was verified using the discretization of the model of the gyroscope. This system was described in [10], later, this model was used to obtain numerous results concerning output regulation in the continuous case. These results can be found in the references, they include both full information as well as the error feedback. Moreover, the previously developed methods for the continuous nonlinear output regulation were successfully applied to the control of the real system, not only on its mathematical model. Thus, the idea of employing this system for the discrete-time output regulation is quite straightforward.

The equations describing this system are as follows:

$$\begin{aligned}&0.00544\ddot{\alpha} + 0.472\dot{\psi}\cos\alpha - 0.000488\dot{\psi}^2 \sin\alpha\cos\alpha \\ &= 2.46\tan\alpha \\ &0.002\ddot{\psi} + 0.000847\cos^2\alpha\ddot{\psi} + 0.00133\sin^2\alpha\ddot{\psi} - \\ &0.472\dot{\alpha}\cos\alpha + 0.000976\dot{\psi}\dot{\alpha}\sin\alpha\cos\alpha = \\ &0.113u - 0.0104\dot{\psi}\end{aligned} \tag{16}$$

For the sake of brevity denote

a_1	a_2	a_3	a_4
0.0054435	0.4717409	−0.0004879	2.4610918
b_1	b_2	b_3	b_4
0.002	0.0008470	0.001335	−0.4717
b_5	b_6	b_7	
0.0009758	0.1126816997	−0.01044	

In the following text the modified notation is used:

$$x_1 = \psi,\ x_2 = \dot{\psi},\ x_3 = \alpha,\ x_4 = \dot{\alpha}.$$

This system can be rewritten for the control purpose in the form (see (16))

$$\dot{x}_1 = x_2 \tag{17}$$
$$\dot{x}_2 = \frac{a_4}{a_1}\tan x_1 - \cos x_1(\frac{a_2}{a_1}x_4 - \frac{a_3}{a_1}x_4^2\sin x_1) \tag{18}$$
$$\dot{x}_3 = x_4 \tag{19}$$
$$\dot{x}_4 = \frac{b_6 u + b_7 x_4 - x_2\cos x_1(b_4 - b_5 x_4 \sin x_1)}{b_1 + b_2\cos^2 x_1 + b_3 \sin^2 x_1} \tag{20}$$

The control was discrete-time with period 0.1s, it was constant in the intervals $[k * 0.1, (k+1) * 0.1)s$. Such a control was fed into the original (continuous) system.

The exosystem was described by the equations (3) with

$$S = \begin{pmatrix} 0.995 & -0.0998 \\ 0.0998 & 0.995 \end{pmatrix},\ Q = (1, 0).$$

The system was stabilized by the feedback gain

$$K = [-1.0254, -0.03407, 0.02096, 0.1838].$$

To compute the controller, the plant was discretized with the sampling period 0.1s. Fig. 1 shows how the trajectory is tracked. Only the points $0, 0.1, 0.2, \dots$ are shown. The asterix represents the output of the controlled system, the cross stands for the reference trajectory. Fig. 2 shows the control that was applied to the system. The values of the cost functional are seen in Fig. 3. The standard built-in optimization procedure `optim` from the package SciLab was used, hence no information about its behavior can be given. Maybe a more sophisticated optimization method would allow to eliminate the peaks.

The center manifold was approximated by its second-order Taylor polynomial in two variables v_1, v_2. This means:

$$\mathbf{x}_i(v_1, v_2) = \xi(i,1)v_1 + \xi(i_2)v_2 + \xi(i,3)v_1^2 \\ +\xi(i,4)v_1v_2 + \xi(i,5)v_2^2.$$

The matrix ξ was obtained by optimization through the algorithm described above. Its resulting value is given in the following matrix.

The initial iteration of the optimization algorithm was chosen as the value of the solution of the linear output regulation problem that was obtained by the linearization of the controlled system. The solution of this problem was also found using the presented algorithm, however, in the linear case, the convergence to the solution is very fast. (Moreover, use of matrix equations describing this solution (see [7]) would also be possible.)

$$\xi = \begin{pmatrix} 0.864 & -0.00054 & -0.0069 & -0.0297 & 0.0028 \\ 1.03 & -0.940 & -0.0316 & 0.00270 & 0.0315 \\ 4.78 & -0.0171 & -0.00528 & -0.166 & 0.00533 \\ 0.201 & 6.220 & 0.0412 & -0.00582 & -0.0419 \end{pmatrix}$$

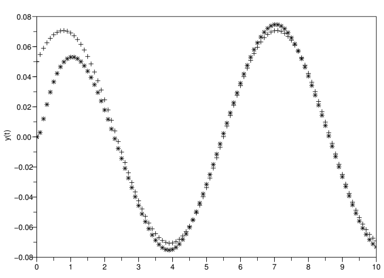

Fig. 1. Tracking

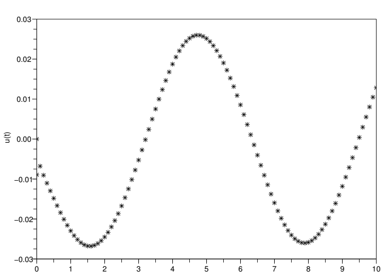

Fig. 2. Control

It is worth a notice that a similar approach for the continuous-time case was successfully applied for the control of this system even in the real time. The results are described in [9].

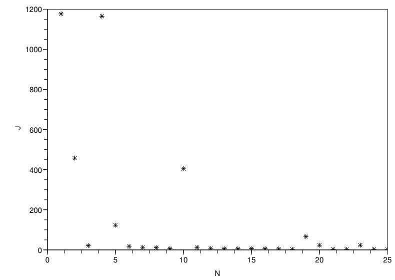

Fig. 3. Functional

VII. Conclusion

An algorithm for numerical solution of the regulator equation (that appears as a part of solution of the discrete-time nonlinear output regulation problem) is presented. The method is based on a sequence of solutions of the functional equation (being a part of the regulator equation) with a fixed control and then calculation of an error functional that measures the error made in the algebraic condition. The control is adjusted in the next iteration so that this value decreases. Convergence analysis as well as the investigation of influence of imprecision in the algebraic condition (that occurs due to the numerical process) on the tracking error are presented.

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