2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

An Elementary Proof for the Exactness of (D, G) Scaling

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***Abstract*— The goal of this paper is to provide an elementary proof for the exactness of the (D, G) scaling applied to the uncertainty structure with one repeated real scalar block and one full complex matrix block. The (D, G) scaling has vast application area around control theory, optimization and signal processing. This is because, by applying the (D, G) scaling, we can convert inequality conditions depending on an uncertain parameter to linear matrix inequalities (LMIs) in an exact fashion. However, its exactness proof is tough, and this stems from the fact that the proof requires an involved matrix formula in addition to the standard Lagrange duality theory. To streamline the proof, in the present paper, we clarify that the involved matrix formula is closely related to a norm preserving dilation under structural constraints. By providing an elementary proof for the norm preserving dilation, it follows that basic results such as Schur complement and congruence transformation in conjunction with the Lagrange duality theory are enough to complete a self-contained exactness proof.**

I. INTRODUCTION

The goal of this paper is to provide an elementary proof for the exactness of the (D, G) scaling applied to the uncertainty structure with one repeated real scalar block and one full complex matrix block [11]. The (D, G) scaling was firstly proposed in [6] in the early 90's to compute an upper bound of the structured singular value in μ theory. Subsequently, in the late 90's, the (D, G) scaling was proved to be exact (lossless) when the underlying uncertainty structure consists of one repeated scalar block and one full complex matrix block [11]. As the development of the study on the robustness analysis and synthesis of LTI systems depending on uncertain parameters [2], [13], the (D, G) scaling received renewed interest. This is because, by applying the (D, G) scaling, we can convert inequality conditions depending on an uncertain parameter to linear matrix inequalities (LMIs) in an exact fashion. Currently, the usefulness of the (D, G) scaling is well-recognized, and it has vast application area ranging from control theory to optimization and signal processing.

However, to complete an exactness proof of the (D, G) scaling is undoubtedly a tough problem. This stems from the fact that the proof requires involved matrix formulas [11] in addition to the standard Lagrange duality theory. This is in stark contrast with the standard Kalman-Yakubovich-Popov (KYP) lemma where simple convexity arguments related to the Lagrange duality theory suffices to complete the proof [12]. In the (D, G) scaling, we have to deal with parameter variation over finite interval and this makes the proof more complicated.

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To streamline the proof of the (D, G) scaling, in the present paper, we firstly clarify that the proof can be done straightforwardly if we complete an involved matrix formula that is closely related to a norm preserving dilation [4], [14] under structural constraints. By providing an elementary proof for the norm preserving dilation, it follows that basic results such as Schur complement and congruence transformation in conjunction with the Lagrange duality theory are enough to complete a self-contained exactness proof. We also briefly discuss the proof of the finite frequency KYP lemma [9], [10], which has an strong impact on the recent development of linear control theory. We clarify that its proof readily follows if we simply modify the matrix formula used in the exactness proof of the (D, G) scaling.

We use the following notations in this paper. The symbols $\mathbf{H}_n$, $\mathbf{P}_n$, $\mathbf{Sk}_n$ and $\mathbf{U}_n$ denote respectively the set of $n \times n$ Hermitian, positive-semidefinite Hermitian, skew-symmetric Hermitian and unitary matrices.

For $W \in \mathbf{C}^{n\times m}$, we denote its Moor-Penrose generalized inverse by $W^\dagger$. It is worth mentioning that, for $V \in \mathbf{H}_n$ with its eigenvalue decomposition

$$V = U_1 \Sigma U_1^*, \quad [\ U_1\ U_2\ ] \in \mathbf{U}_n \tag{1}$$

where Σ is non-singular, $V^\dagger$ is given by

$$V^\dagger = U_1 \Sigma^{-1} U_1^*.$$

Moreover, for any $W \in \mathbf{C}^{l\times n}$, the equality $WU_2 = \mathbf{0}$ holds iff $W(\mathbf{1}_n - VV^\dagger) = \mathbf{0}$ holds.

II. (D, G) SCALING AND ITS PROOF

A. (D, G) Scaling

To begin with, let us first recall the (D, G) scaling applied to the uncertainty structure with one repeated real scalar and one full-block complex matrix [11].

Proposition 1: For given $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$, $\Pi \in \mathbf{H}_{n+m}$ and $\mathbf{E} := [-1, 1]$, suppose A has no eigenvalues on $(-\infty, -1] \cup [1, \infty)$ and define $K(\delta) := (I_n - A\delta)^{-1}B$. Then, the following two conditions are equivalent:

(i) The following inequality condition holds:

$$\begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix}^* \Pi \begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix} \prec \mathbf{0} \quad \forall \delta \in \mathbf{E}. \tag{2}$$

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(ii) There exist $D \in \mathbf{P}_n$ and $G \in \mathbf{Sk}_n$ such that

$$\begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} + \Pi \prec \mathbf{0} \qquad (3)$$

It should be noted that the condition (i) is not explicitly dealt with in [11]. Roughly speaking, the problem discussed in [11] is to determine whether the following condition for given $M \in \mathbf{C}^{(n+m)\times(n+m)}$ holds or not:

$$\begin{aligned} &\det\left(I - \mathrm{diag}(\delta I_n, \Delta) M\right) \neq 0 \quad \forall(\delta, \Delta) \in \mathbf{E} \times \boldsymbol{\Delta}, \\ &\boldsymbol{\Delta} := \{\Delta : \ \Delta \in \mathbf{C}^{m\times m}, \|\Delta\| \leq 1\} \end{aligned} \qquad (4)$$

By partitioning M as

$$M = \begin{bmatrix} A & B \\ \mathcal{C} & \mathcal{D} \end{bmatrix}, \ A \in \mathbf{C}^{n\times n}, \ \mathcal{D} \in \mathbf{C}^{m\times m}$$

and assuming that A has no eigenvalues on $(-\infty, -1] \cup [1, \infty)$ as in Proposition 1, it can be readily seen that (4) holds iff

$$\det(I - \Delta(\mathcal{C}\delta K(\delta) + \mathcal{D})) \neq 0 \quad \forall(\delta, \Delta) \in \mathbf{E} \times \boldsymbol{\Delta}$$

or equivalently,

$$\|\mathcal{C}\delta K(\delta) + \mathcal{D}\| < 1 \quad \forall \delta \in \mathbf{E}.$$

It is clear that the above condition holds iff the condition (i) in Proposition 1 holds with

$$\Pi = \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & \mathbf{1}_m \end{bmatrix}^* \begin{bmatrix} \mathbf{1}_n & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_m \end{bmatrix} \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & \mathbf{1}_m \end{bmatrix}$$

For this particularly chosen Π, it was shown in [11] that the conditions (i) and (ii) are equivalent. In this sense, Proposition 1 can be seen as a slight extension of [11].

The (D, G) scaling (in the form of Proposition 1) has vast application area ranging from control theory to optimization and signal processing. In these study areas, it is often the case that we should check the positive definiteness of a univariate polynomial matrix of the form

$$J(\delta) = \sum_{k=0}^{N} \delta^k J_k \ (J_k \in \mathbf{H}_m, \ k = 0, \cdots, N)$$

over $\delta \in \mathbf{E}$. Since this $J(\delta)$ can also be written as

$$\begin{aligned} J(\delta) &= J_0 + C\delta(I_{mN} - A\delta)^{-1}B, \\ A &:= \begin{bmatrix} 0_{m(N-1),m} & I_{m(N-1),m(N-1)} \\ 0_{m,m} & 0_{m,m(N-1)} \end{bmatrix}, \\ B &:= \begin{bmatrix} J_1 \\ \vdots \\ J_N \end{bmatrix}, \quad C := \begin{bmatrix} I_m & 0_{m,m(N-1)} \end{bmatrix} \end{aligned}$$

and since

$$\begin{aligned} & J(\delta) \succ 0 \quad \forall \delta \in \mathbf{E} \\ \Leftrightarrow \quad & \begin{bmatrix} J(\delta) \\ \mathbf{1}_m \end{bmatrix}^* \begin{bmatrix} 0 & -\mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{bmatrix} \begin{bmatrix} J(\delta) \\ \mathbf{1}_m \end{bmatrix} \prec 0 \quad \forall \delta \in \mathbf{E} \\ \Leftrightarrow \quad & (3) \text{ with} \end{aligned}$$

$$\Pi = \begin{bmatrix} C & J_0 \\ 0 & \mathbf{1}_m \end{bmatrix}^* \begin{bmatrix} 0 & -\mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{bmatrix} \begin{bmatrix} C & J_0 \\ 0 & \mathbf{1}_m \end{bmatrix} \qquad (5)$$

holds, the positive definiteness of $J(\delta)$ can be checked by testing the feasibility of the LMI (3) with Π given in (5) via numerical computation.

B. Proof of (D, G) Scaling based on a Matrix Formula

Let us move on to the proof of the (D, G) scaling. It should be noted that the proof for the implication (ii) ⇒ (i) is fairly easy. To see this, we first note that

$$\begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix} = \begin{bmatrix} I \\ \delta I \end{bmatrix} K(\delta).$$

Thus if (3) in (ii) holds, it follows for all $\delta \in \mathbf{E}$ that

$$\begin{aligned} & \begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix}^* \Pi \begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix} \\ \prec & -\begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix}^* \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta K(\delta) \\ \mathbf{1}_m \end{bmatrix} \\ = & -K(\delta)^* D(\mathbf{1}_n - \delta^2 \mathbf{1}_n) K(\delta) \\ \preceq & \ 0. \end{aligned}$$

This clearly shows that the condition (i) holds.

On the other hand, the proofs for (i)⇒(ii) found in the literature, i.e., the proof for the exactness of the (D, G) scaling, is rather involved. To complete the proof, the next matrix formula, proved later in this paper, plays a crucial role.

Lemma 1: For given matrices $\mathcal{F}, \mathcal{G} \in \mathbf{C}^{n\times m}$, the following two conditions are equivalent:

(i) The following two conditions hold:

$$\mathcal{F}\mathcal{F}^* \succeq \mathcal{G}\mathcal{G}^*, \quad \mathcal{F}\mathcal{G}^* = \mathcal{G}\mathcal{F}^*. \qquad (6)$$

(ii) There exists $\Omega \in \mathbf{H}_m$ such that $\mathcal{G} = \mathcal{F}\Omega$ and $\|\Omega\| \leq 1$.

Once we have accepted this matrix formula, however, the exactness proof of the (D, G) scaling can be done straightforwardly in conjunction with the Lagrange duality theory.

Proof of (i)⇒(ii) in Proposition 1: To complete the proof via contradiction, suppose (3) does not hold for any $D \in \mathbf{P}_n$ and $G \in \mathbf{Sk}_n$. Then, from the Lagrange duality theory which is elegantly summarized in [1], [8], [13], we see that there exists $H \in \mathbf{P}_{n+m}$ such that

$$\mathrm{trace}(\Pi H) \geq 0, \quad \mathrm{trace}(H) = 1,$$

$$\Gamma_{11}(H) \succeq \Gamma_{22}(H), \quad \Gamma_{12}(H) = \Gamma_{12}^*(H) \qquad (7)$$

where

$$\begin{bmatrix} \Gamma_{11}(H) & \Gamma_{12}(H) \\ \Gamma_{12}^*(H) & \Gamma_{22}(H) \end{bmatrix} := \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} H \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix}^*. \quad (8)$$

Let us denote the full-rank factorization of H as

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^*, \ H_1 \in \mathbf{C}^{n\times r},\ H_2 \in \mathbf{C}^{m\times r} \quad (9)$$

where $r := \mathrm{rank}(H)$. Then, (7) implies that the two conditions in (6) holds with $\mathcal{F} = AH_1 + BH_2$ and $\mathcal{G} = H_1$. It follows from Lemma 1 that there exists $\Omega \in \mathbf{H}_r$ such that

$$H_1 = (AH_1 + BH_2)\Omega, \quad \|\Omega\| \le 1. \quad (10)$$

If we denote the eigenvalue decomposition of Ω by

$$\Omega = U\Lambda U^*,\ \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_r),\ \lambda_i \in \mathbf{E},\ U \in \mathbf{U}_r,$$

we have

$$H_1 U = (AH_1 + BH_2)U\Lambda. \quad (11)$$

More precisely, if we define

$$\begin{bmatrix} h_1 & \cdots & h_r \\ g_1 & \cdots & g_r \end{bmatrix} := \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} U, \quad (12)$$

we note that (11) can be rewritten, equivalently, as

$$h_i = \lambda_i(\mathbf{1}_n - A\lambda_i)^{-1}Bg_i = \lambda_i K(\lambda_i) g_i \quad (i = 1, \cdots, r).$$

Thus we readily obtain

$$\begin{aligned} H &= \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} UU^* \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \\ &= \sum_{i=1}^{r} \begin{bmatrix} \lambda_i K(\lambda_i) \\ \mathbf{1}_m \end{bmatrix} g_i g_i^* \begin{bmatrix} \lambda_i K(\lambda_i) \\ \mathbf{1}_m \end{bmatrix}^*. \end{aligned}$$

Since $\mathrm{trace}(\Pi H) \ge 0$, the above expression of H implies that there exists at least one index j such that

$$\mathrm{trace}\left(\Pi \begin{bmatrix} \lambda_j K(\lambda_j) \\ \mathbf{1}_m \end{bmatrix} g_j g_j^* \begin{bmatrix} \lambda_j K(\lambda_j) \\ \mathbf{1}_m \end{bmatrix}^*\right) \ge 0$$

or equivalently,

$$g_j^* \begin{bmatrix} \lambda_j K(\lambda_j) \\ \mathbf{1}_m \end{bmatrix}^* \Pi \begin{bmatrix} \lambda_j K(\lambda_j) \\ \mathbf{1}_m \end{bmatrix} g_j \ge 0.$$

We note that $g_j \ne 0$ since $[\ H_1^*\ H_2^*\ ]^*$ is full-column rank. The above inequality clearly shows that (i) does not hold for $\lambda_j \in \mathbf{E}$. This completes the proof. ■

C. Finite Frequency KYP Lemma and Its Proof

The finite frequency KYP lemma was firstly introduced in [9]. This is an extension of the standard KYP lemma and particularly useful for loop shaping in low frequency range. Since this publication, several extensions of KYP lemma in terms of frequency range and system description were successfully done by Iwasaki et al. Recently, those results are unified as a generalized KYP lemma [10].

As clearly mentioned in [9], the finite frequency KYP lemma is closely related to the (D, G) scaling. Indeed, we will show that simple modification of Lemma 1 is enough for its proof. To see this, let us first describe the finite frequency KYP lemma explicitly in the next proposition.

Proposition 2 (Finite Frequency KYP Lemma [9], [10]): For given $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$ and $\Pi \in \mathbf{H}_{n+m}$, suppose A has no eigenvalues on $j\mathbf{E}$. Let us define $L(j\omega) := (j\omega I - A)^{-1}B$. Then, the following two conditions are equivalent:

(i) The following inequality condition holds:

$$\begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix}^* \Pi \begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix} \prec 0 \quad \forall\omega \in \mathbf{E}. \quad (13)$$

(ii) There exist $P \in \mathbf{H}_n$ and $Q \in \mathbf{P}_n$ such that

$$\begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & Q \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} + \Pi \prec 0 \quad (14)$$

Even though the proof of Proposition 2 is thoroughly given in [9], we will revisit the proof to demonstrate the usefulness of Lemma 1. To prove Proposition 2, we need the next lemma which was already derived in [9]. In our context, this lemma can be viewed as an obvious modification of Lemma 1 as explicated in the subsequent proof.

Lemma 2: For given matrices $F, G \in \mathbf{C}^{n\times m}$, the following two conditions are equivalent:

(i) The following two conditions hold:

$$\mathcal{F}\mathcal{F}^* \succeq \mathcal{G}\mathcal{G}^*, \quad \mathcal{F}\mathcal{G}^* = -\mathcal{G}\mathcal{F}^*. \quad (15)$$

(ii) There exists $\Xi \in \mathbf{Sk}_m$ such that $\mathcal{G} = \mathcal{F}\Xi$ and $\|\Xi\| \le 1$.

Proof of Lemma 2: It is obvious that the two conditions in (i) can be rewritten, equivalently as

$$(j\mathcal{F})(j\mathcal{F})^* \succeq \mathcal{G}\mathcal{G}^*, \quad (j\mathcal{F})\mathcal{G}^* = \mathcal{G}(j\mathcal{F})^*.$$

From Lemma 1, the above conditions hold iff there exists $\Omega \in \mathbf{H}_n$ such that $\mathcal{G} = j\mathcal{F}\Omega = \mathcal{F}(j\Omega)$. This is nothing but the condition (ii) with $\Xi = j\Omega \in \mathbf{Sk}_m$. ■

If we admit Lemma 2, we can provide a very concise proof for Proposition 2.

Proof of Proposition 2: (i)⇒(ii): Again, to complete the proof via contradiction, suppose (14) does not hold for any $P \in \mathbf{H}_n$ and $Q \in \mathbf{P}_n$. Then, from the Lagrange duality theory, there exists $H \in \mathbf{P}_{n+m}$ such that

$$\mathrm{trace}(\Pi H) \ge 0, \quad \mathrm{trace}(H) = 1,$$

$$\Gamma_{11}(H) \preceq \Gamma_{22}(H), \quad \Gamma_{12}(H) = -\Gamma_{12}^*(H) \quad (16)$$

where $\Gamma_{11}(H)$ and so on are defined in (8). If we denote the full-rank factorization of H as in (9), the conditions in (16) imply that the two conditions in (15) hold with $\mathcal{F} = H_1$ and $\mathcal{G} = AH_1 + BH_2$. It follows that there exists $\Xi \in \mathbf{Sk}_r$ such that $AH_1 + BH_2 = H_1\Xi$ and $\|\Xi\| \le 1$. Furthermore, if we denote the eigenvalue decomposition of Ξ by

$$\Xi = U\Lambda U^*,\ \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_r),\ \lambda_i \in j\mathbf{E},\ U \in \mathbf{U}_r,$$

we have $(AH_1 + BH_2)U = H_1 U\Lambda$. This implies $h_i = L(\lambda_i)g_i\ (i = 1, \cdots, r)$ where h_i and $g_i\ (i = 1, \cdots, r)$ are defined by (12). Thus we readily obtain

$$H = \sum_{i=1}^{r} \begin{bmatrix} L(\lambda_i) \\ \mathbf{1}_m \end{bmatrix} g_i g_i^* \begin{bmatrix} L(\lambda_i) \\ \mathbf{1}_m \end{bmatrix}^*.$$

Since $\mathrm{trace}(\Pi H) \ge 0$, we can conclude that there exists at least one index j such that

$$g_j^* \begin{bmatrix} L(\lambda_j) \\ \mathbf{1}_m \end{bmatrix}^* \Pi \begin{bmatrix} L(\lambda_j) \\ \mathbf{1}_m \end{bmatrix} g_j \ge 0.$$

This clearly shows that (i) does not hold for $\lambda_j \in j\mathbf{E}$.

(ii)⇒(i): We first note that

$$\begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix} = \begin{bmatrix} j\omega I \\ I \end{bmatrix} L(j\omega).$$

Thus if (14) holds, it follows for all $\omega \in \mathbf{E}$ that

$$\begin{aligned} &\begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix}^* \Pi \begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix} \\ &\prec -\begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix}^* \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & Q \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{1}_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} L(j\omega) \\ \mathbf{1}_m \end{bmatrix} \\ &= -L(j\omega)^* D(\mathbf{1}_n - \omega^2 \mathbf{1}_n) L(j\omega) \\ &\preceq 0. \end{aligned}$$

This completes the proof. ■

To summarize the arguments in this section, we see that the matrix formula in Lemma 1 plays a very important role for the exactness proof of the (D, G) scaling and the finite frequency KYP lemma. We note that a self-contained proof for Lemma 2 is given in [9]. In our opinion, however, the proof is rather complicated since Lemma 2 involves a skew-symmetric Hermitian matrix Ξ and we need a special care for its treatment. In the next section, we provide a concise proof for Lemma 1, by noting that the matrix formula is closely related to a norm preserving dilation under symmetric structure constraints.

III. Elementary Proof for the Matrix Formula

We first note that the next lemma holds.

Lemma 3 (Norm Preserving Dilation): For given $X \in \mathbf{H}_n$ and $Y \in \mathbf{C}^{n\times m}$, suppose $\|[\ X\ Y\ ]\| \le 1$!%Then, there exists $\mathcal{Z} \in \mathbf{H}_m$ such that

$$\left\| \begin{bmatrix} X & Y \\ Y^* & \mathcal{Z} \end{bmatrix} \right\| \le 1. \tag{17}$$

Moreover, one of such $\mathcal{Z}$ is given by

$$\mathcal{Z} = \mathbf{1}_m - Y^*(\mathbf{1}_n - X)^\dagger Y \tag{18}$$

Under this lemma (proved later), the proof of Lemma 1 is straightforward.

Proof of Lemma 1: It is obvious that if (ii) holds then (i) holds. To prove the converse, suppose (i) holds and let us define $U \in \mathbf{U}_m$ such that $FU = [\ F_1\ 0\ ]$ where $F_1 \in \mathbf{C}^{n\times r}$ is full column rank with $r = \mathrm{rank}(F)$. Then, we see from the first inequality in (i) that there exist $X \in \mathbf{C}^{r\times r}$ and $Y \in \mathbf{C}^{r\times(m-r)}$ such that

$$GU = F_1[\ X\ Y\ ], \quad \|[\ X\ Y\ ]\| \le 1.$$

Moreover, since the second equality reduces to

$$F_1 X^* F_1^* = F_1 X F_1^*$$

and since F_1 is full-column rank, we have $X = X^*$. It follows from Lemma 3 that there exists $\mathcal{Z} \in \mathbf{H}_{m-r}$ such that

$$GU = [\ F_1\ 0\ ] \begin{bmatrix} X & Y \\ Y^* & \mathcal{Z} \end{bmatrix}, \quad \left\| \begin{bmatrix} X & Y \\ Y^* & \mathcal{Z} \end{bmatrix} \right\| \le 1.$$

This can be rewritten as

$$G = F\Omega, \quad \|\Omega\| \le 1, \quad \Omega := U \begin{bmatrix} X & Y \\ Y^* & \mathcal{Z} \end{bmatrix} U^*,$$

which clearly shows that (ii) holds. ■

Our task now is to prove Lemma 3. It should be noted that the norm preserving dilation problem in general setting is fully studied in the literature [4], [7], [14]. However, in Lemma 3, we should achieve dilation under symmetricity constraint and the inequalities involved are non-strict. Therefore its proof is not necessarily obvious. Moreover, it would be of benefit if we can provide a concise LMI-based proof by making use of its particular symmetricity structure. This is indeed possible, and to realize this, we introduce the following three lemmas that would be fairly easy to understand.

Lemma 4: (**Schur complements for non-strict inequalities [3]**) For given $J \in \mathbf{H}_n$, $R \in \mathbf{H}_m$ and $W \in \mathbf{C}^{n\times m}$, the following two conditions are equivalent.

(i) $\begin{bmatrix} J & W \\ W^* & R \end{bmatrix} \succeq \mathbf{0}.$

(ii) $R \succeq \mathbf{0}, \quad W(I_m - RR^\dagger) = \mathbf{0}, \quad J - WR^\dagger W^* \succeq \mathbf{0}.$

Lemma 5: For given $P \in \mathbf{H}_n$, $Q, S \in \mathbf{H}_m$, $R \in \mathbf{H}_l$, $V \in \mathbf{C}^{n\times m}$ and $W \in \mathbf{C}^{m\times l}$, the following conditions are equivalent.

(i) The following inequality condition holds:

$$\begin{bmatrix} P & V & \mathbf{0} \\ V^* & Q+S & W \\ \mathbf{0} & W^* & R \end{bmatrix} \succeq \mathbf{0}. \tag{19}$$

(ii) There exists $\mathcal{Z} \in \mathbf{H}_m$ such that

$$\begin{bmatrix} P & V \\ V^* & Q+\mathcal{Z} \end{bmatrix} \succeq \mathbf{0}, \quad \begin{bmatrix} S-\mathcal{Z} & W \\ W^* & R \end{bmatrix} \succeq \mathbf{0}. \quad (20)$$

If the conditions in (ii) are satisfied for some $\mathcal{Z}$, one of such $\mathcal{Z}$ is given by

$$\mathcal{Z} = S - WR^{\dagger}W^*. \quad (21)$$

Proof of Lemma 5: (i)⇒(ii): From Lemma 4, we see that (19) holds iff

$$R \succeq \mathbf{0}, \quad W(I_l - RR^{\dagger}) = \mathbf{0}, \quad (22)$$

$$\begin{bmatrix} P & V \\ V^* & Q+S-WR^{\dagger}W^* \end{bmatrix} \succeq \mathbf{0}.$$

Hence, if we define $\mathcal{Z}$ by (21), the first inequality in (ii) obviously holds. It is also apparent that the second inequality in (ii) holds with $\mathcal{Z}$ in (21) since (22) and

$$S - \mathcal{Z} - WR^{\dagger}W^* = \mathbf{0}$$

do hold. Hence, again from Lemma 4, the second inequality in (ii) readily follows.

(ii)⇒(i): The proof of this direction is exactly the same as [5]. Indeed, it is obvious that (ii) holds iff

$$\begin{bmatrix} P & V & \mathbf{0} & \mathbf{0} \\ V^* & Q+\mathcal{Z} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S-\mathcal{Z} & W \\ \mathbf{0} & \mathbf{0} & W^* & R \end{bmatrix} \succeq \mathbf{0}. \quad (23)$$

Then we have

$$\begin{bmatrix} \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_l \end{bmatrix}^* \begin{bmatrix} P & V & \mathbf{0} & \mathbf{0} \\ V^* & Q+\mathcal{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S-\mathcal{X} & W \\ \mathbf{0} & \mathbf{0} & W^* & R \end{bmatrix} \begin{bmatrix} \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_l \end{bmatrix}$$

$$= \begin{bmatrix} P & V & \mathbf{0} \\ V^* & Q+S & W \\ \mathbf{0} & W^* & R \end{bmatrix} \succeq \mathbf{0}.$$

This completes the proof. ∎

Lemma 6: For given $X \in \mathbf{H}_n$ and $Y \in \mathbf{C}^{n\times m}$, the following three conditions are equivalent:

(i) $\|[\ X\ Y\ ]\| \le 1$.

(ii) $\begin{bmatrix} \mathbf{1}_n & X & Y \\ X & \mathbf{1}_n & \mathbf{0} \\ Y^* & \mathbf{0} & \mathbf{1}_m \end{bmatrix} \succeq \mathbf{0}.$

(iii) $\begin{bmatrix} \mathbf{1}_n + X & Y & \mathbf{0} \\ Y^* & 2\mathbf{1}_m & Y^* \\ \mathbf{0} & Y & \mathbf{1}_n - X \end{bmatrix} \succeq \mathbf{0}. \quad (24)$

Proof of Lemma 6: The equivalence of (i) and (ii) is an elementary fact. To prove the equivalence of (ii) and (iii), let us define a non-singular matrix $T \in \mathbf{R}^{(2n+m)\times(2n+m)}$ by

$$T := \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{1}_n & \mathbf{1}_n & \mathbf{0}_{n,m} \\ \mathbf{0}_{m,n} & \mathbf{0}_{m,n} & 2\mathbf{1}_m \\ \mathbf{1}_n & -\mathbf{1}_n & \mathbf{0}_{n,m} \end{bmatrix}.$$

Then, the equivalence readily follows via congruence transformation with T if we note

$$T \begin{bmatrix} \mathbf{1}_n & X & Y \\ X & \mathbf{1}_n & \mathbf{0} \\ Y^* & \mathbf{0} & \mathbf{1}_m \end{bmatrix} T^*$$

$$= \frac{1}{2} \begin{bmatrix} \mathbf{1}_n + X & \mathbf{1}_n + X & Y \\ 2Y^* & \mathbf{0}_{m,n} & 2I_m \\ \mathbf{1}_n - X & -(\mathbf{1}_n - X) & Y \end{bmatrix} \begin{bmatrix} \mathbf{1}_n & \mathbf{0}_{n,m} & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_{n,m} & -\mathbf{1}_n \\ \mathbf{0}_{m,n} & 2\mathbf{1}_m & \mathbf{0}_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1}_n + X & Y & \mathbf{0} \\ Y^* & 2\mathbf{1}_m & Y^* \\ \mathbf{0} & Y & \mathbf{1}_n - X \end{bmatrix}.$$

∎

Now we are ready to state the proof of Lemma 3.

Proof of Lemma 3: From Lemma 6, we first note that $\|[\ X\ Y\ ]\| \le 1$ holds iff (24) holds. From Lemma 5, we see that (24) holds iff there exists $\mathcal{Z}$ such that

$$\begin{bmatrix} \mathbf{1}_n + X & Y \\ Y^* & \mathbf{1}_m + \mathcal{Z} \end{bmatrix} \succeq \mathbf{0}, \quad \begin{bmatrix} \mathbf{1}_m - \mathcal{Z} & Y^* \\ Y & \mathbf{1}_n - X \end{bmatrix} \succeq \mathbf{0}$$

ore equivalently,

$$\mathbf{1}_{n+m} \succeq \begin{bmatrix} X & Y \\ Y^* & \mathcal{Z} \end{bmatrix} \succeq -\mathbf{1}_{n+m}.$$

Moreover, Lemma 5 clearly shows that one of such $\mathcal{Z}$ is given by $\mathcal{Z} = \mathbf{1}_m - Y^*(\mathbf{1}_n - X)^{\dagger}Y$. This completes the proof. ∎

Thus we have completed the proof of Lemma 3 without any difficulties, by means of such basic results as Schur complement and congruence transformation. As mentioned earlier, the exactness proof of the (D, G) scaling is straightforward if we complete Lemma 3 and hence Lemma 1.

IV. Conclusion

In this paper, we provided an elementary proof for the (D, G) scaling applied to the uncertainty structure with one repeated real scalar block and one full complex matrix block. We first showed that the proof can be done straightforwardly if we accept a certain matrix formula and the Lagrange duality theory. We further showed that the matrix formula is closely related to a norm preserving dilation under structural constraints. By providing an elementary proof for the norm preserving dilation, we clarified that basic results such as Schur complement and congruence transformation in conjunction with the Lagrange duality theory are enough to complete a self-contained exactness proof. Since the (D, G)

scaling is frequently used as a basic tool in up-to-date control theory, we believe that the result in this paper is useful certainly from a pedagogical point of view.

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