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Beam-and-Ball System under Limited Control: Stabilization with Large Basin of Attraction

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Abstract— **This article is devoted to the stabilization of the well known straight beam-and-ball system and an original circular beam-and-ball system. The limits on the voltage, fed to the motor, are taken into account explicitly. The feedback control for each system ensures the basin of attraction as large as possible. Simulation results are presented.**

I. INTRODUCTION

The under actuated mechanical systems represent a challenge for the control. An active field of research exists, due to the applications of these systems such as aircrafts, spacecrafts, flexible and legged robots.

This paper deals with the stabilization of two planar under actuated systems. They are of two degrees of freedom but have one actuator only. The first system is the straight beam-and-ball system. The stabilization and the tracking problem using a state or an output feedback have been considered by several researchers ([1], [2], [3], [4] or [5]). In [1], tracking for this system was considered using approximate input-output linearization. Semiglobal stabilization of the straight beam-and-ball system using the state feedback was addressed by [2]. In [3], this system is stabilized using output feedback. The problem of global stabilization of the straight beam-and-ball system was considered in [4]. Semiglobal stabilization of this system, using fixed-point state feedback was addressed by [5].

The second system is an original circular beam-and-ball system. For each system, a control law, based on the Jordan form of the linearized model, is designed. The saturation of the actuator is taken into account explicitly. This kind of control has been previously tested to stabilize a biped with point feet [6], a one-link pendulum with flywheel [7]. The main difference between the system with the straight beam and the system with the circular beam is that the linear model of the second system has two eigenvalues in the right-half complex plane. For the linear model of the system with straight beam, the controllability domain Q and the basin of attraction B can coincide under a linear control law with restriction, [7], [8]. This property is not satisfied for the linear model of the system with circular beam. But the basin B can be made arbitrary close to the domain Q [9].

The paper is organized as follows: Section II is devoted to the straight beam-and-ball system. The second system is studied in Section III. Simulation results in Sections II and III for the complete two nonlinear systems are shown to illustrate the efficiency of the proposed control laws. Finally, Section IV contains our conclusion and perspectives.

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II. STRAIGHT BEAM-AND-BALL SYSTEM

This system consists of a straight beam and a ball on it, Figure 1. The ball is rolling on the beam without slide. C_1 is center of mass of the beam with its holder OA. O is the suspension point. The point C_2 and value r are center and radius of the ball. C_2 is also the center of mass of the ball.

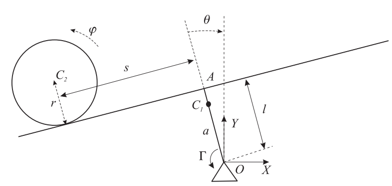

Fig. 1. Diagram of the straight beam-and-ball system.

A. Equations of motion

Let m_1 and m_2 denote the masses of the beam with its holder OA and of the ball, respectively. Let us introduce ρ_1 and ρ_2 the radii of inertia such that $I_1 = m_1\rho_1^2$ and $I_2 = m_2\rho_2^2$ are the inertia moments respectively of the beam with its holder OA around the suspension point O and of the ball around its center C_2; let $OC_1 = a$ and $OA = l$.

Two generalized coordinates, the angles θ and φ characterize the behavior of this system. Position of the ball on the beam is defined also by the distance $s = r\varphi$. Let Γ be the torque, which is directly proportional to the electrical current in the armature winding. By neglecting the armature inductance, this torque can be written in the form [10]:

$$\Gamma = c_u u - c_v\dot{\theta} \tag{1}$$

where u is the voltage, supplied to the motor. The positive constants c_u and c_v for a given motor can be calculated by using its characteristics [10]. Product $c_v\dot{\theta}$ is the torque of the back electromotive force. Let,

$$|u| \leq u_0, \;\; u_0 = const \tag{2}$$

The expressions for the kinetic energy K and the potential energy Π are the following (g is the gravity acceleration):

$$2K = m_1\rho_1^2\dot{\theta}^2 + m_2[r^2\varphi^2 + (r+l)^2]\dot{\theta}^2 + \\ +2m_2r(l+r)\dot{\varphi}\dot{\theta} + m_2(r^2+\rho_2^2)\dot{\varphi}^2$$

$$\Pi = m_1gacos\theta + m_2g[-r\varphi sin\theta + (l+r)cos\theta]$$

The equations of the mechanism motion can be derived, using Lagrange's method:

$$\begin{array}{c}\left[m_1\rho_1^2+m_2(r+l)^2+m_2r^2\varphi^2\right]\ddot{\theta}+\\ +m_2r(r+l)\ddot{\varphi}+2m_2r^2\varphi\dot{\varphi}\dot{\theta}-\\ -g[m_1a+m_2(r+l)]sin\theta-\\ -m_2gr\varphi cos\theta=c_uu-c_v\dot{\theta}-f\dot{\theta}\end{array} \tag{3}$$

$$r(r+l)\ddot{\theta}+(r^2+\rho_2^2)\ddot{\varphi}-r^2\varphi\dot{\theta}^2-grsin\theta=0$$

The term $f\dot{\theta}$ ($f=const>0$) in the first equation describes the torque of the viscous friction force in the joint O.

If $u=0$, system (3) has one unstable equilibrium state:

$$\theta=0,\quad \varphi=0\quad (s=0),\quad \dot{\theta}=0,\quad \dot{\varphi}=0\quad (\dot{s}=0) \tag{4}$$

B. Linearized Model

Corresponding to the nonlinear equations (3) the linear model of the motion near the equilibrium state (4) is:

$$\begin{array}{c}\left[m_1\rho_1^2+m_2(r+l)^2\right]\ddot{\theta}+m_2r(r+l)\ddot{\varphi}-\\ -g[m_1a+m_2(r+l)]\theta-m_2gr\varphi=\\ =c_uu-c_v\dot{\theta}-f\dot{\theta}\end{array} \tag{5}$$

$$r(r+l)\ddot{\theta}+(r^2+\rho_2^2)\ddot{\varphi}-gr\theta=0$$

C. Kalman controllability

The determinant of the controllability matrix [11] for the linear model (5) is not null, if and only if:

$$r^2g^2\left[(2r^2+\rho_2^2)\rho_2^2+r^4\right]\neq 0 \tag{6}$$

Thus, inequality (6) is valid, if $r\neq 0$. If $r=0$, then the ball becomes a material point and we do not consider this case.

D. Spectrum of Linear System

The state form of system (5) using the state vector $x=(\theta,\ \varphi,\ \dot{\theta},\ \dot{\varphi})^T$, is:

$$\begin{array}{c}\dot{x}=Ax+bu=\\ =\left[\begin{array}{cc}0_{2\times 2} & I_{2\times 2}\\ D^{-1}E & D^{-1}\begin{pmatrix}-c_v-f & 0\\ 0 & 0\end{pmatrix}\end{array}\right]x+\\ +\left[\begin{array}{c}0_{2\times 1}\\ D^{-1}\begin{pmatrix}c_u\\ 0\end{pmatrix}\end{array}\right]u\end{array} \tag{7}$$

The notations $I_{2\times 2}$, and $0_{2\times 2}$, $0_{2\times 1}$ define an identity matrix of size (2×2) and zero matrices (2×2), (2×1). The expressions of matrices D and E are

$$\begin{array}{c}D=\begin{pmatrix}m_1\rho_1^2+m_2(r+l)^2 & m_2r(r+l)\\ r(r+l) & r^2+\rho_2^2\end{pmatrix}\\ E=g\begin{pmatrix}m_1a+m_2(r+l) & m_2r\\ r & 0\end{pmatrix}\end{array} \tag{8}$$

Introducing a nondegenerate linear transformation $x=Sy$ with a constant matrix S, it is possible to get the well-known Jordan form of the matrix equation (7)

$$\dot{y}=\Lambda y+du \tag{9}$$

where

$$\begin{array}{c}\Lambda=S^{-1}AS=\begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4\end{pmatrix},\\ d=S^{-1}b=[d_i]^T\quad (i=1,...,4).\end{array} \tag{10}$$

Here, $\lambda_1,...,\lambda_4$ are the eigenvalues of the matrix A. They are the roots of the characteristic equation of system (5):

$$a_0\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4=0 \tag{11}$$

with $a_0=detD>0,\quad a_1=(c_v+f)(r^2+\rho_2^2)>0,$
$a_2=m_2(r+l)(r^2-\rho_2^2)-m_1a(r^2+\rho_2^2),\quad a_3=0,$
$a_4=detE=-m_2g^2r^2<0.$

According to the theorem of Routh-Hurwitz ([12], [5]), equation (11) has one root in the right-half complex plane and three roots in the left-half complex plane.

E. Problem Statement

Let vector $x=0$ be the desired equilibrium state of system (7). Let us design an admissible feedback control $|u(x)|\leq u_0$ to ensure the asymptotic stability of this desired state. Let W be the set of piecewise continuous functions of time $u(t)$, satisfying inequality (2). Let Q be the set of the initial states $x(0)$, from which system (7) can reach the origin $x=0$ with the control $u(t)\in W$. Set Q is called controllability domain. If the matrix A has eigenvalues with positive real parts and $u(t)\in W$, then the controllability domain Q for system (7) is an open subset of the phase space X ([8], [7]).

For any admissible feedback control $u=u(x)$ the corresponding basin of attraction belongs to the controllability domain: $B\subset Q$. Here, as usual, B is the set of initial states $x(0)$, from which system (7), with feedback $u=u(x)$ asymptotically tends to the origin point $x=0$ as $t\rightarrow\infty$.

F. Feedback Control for the Straight beam-and-ball system

A control law is proposed here to stabilize this system with basin of attraction $B=Q$.

1) Control design: Let $\lambda_1>0$, $Re\lambda_i<0$ $(i=2,3,4)$ and let us consider the first scalar differential equation of system (9) corresponding to eigenvalue λ_1,

$$\dot{y}_1=\lambda_1y_1+d_1u \tag{12}$$

System (7) is Kalman controllable, therefore scalar $d_1\neq 0$. The controllability domain Q of the equation (12) and consequently of system (9) is described by the following inequality ([8], [7])

$$|y_1|<|d_1|\,u_0/\lambda_1 \tag{13}$$

The instability of the coordinate y_1 can be "suppressed" by a linear feedback control,

$$u=\gamma y_1\quad with\quad \lambda_1+d_1\gamma<0 \tag{14}$$

For system (7) under the feedback control (14), only the pole λ_1 is replaced by a negative pole $\lambda_1+d_1\gamma$. The poles λ_2, λ_3, λ_4 do not change.

If constraint (2) is taken into account, the linear feedback control (14) becomes with saturation,

$$u = u(y_1) = \begin{cases} u_0, & if \quad \gamma y_1 \geq u_0 \\ \gamma y_1, & if \quad |\gamma y_1| \leq u_0 \\ -u_0, & if \quad \gamma y_1 \leq -u_0 \end{cases} \quad (15)$$

It is possible to see that if $|y_1| < |d_1|u_0/\lambda_1$, then the right part of equation (12) with the nonlinear control (15) is negative when $y_1 > 0$ and positive when $y_1 < 0$. Consequently, if $|y_1(0)| < |d_1|u_0/\lambda_1$, then the solution $y_1(t)$ of system (12), (15) tends to 0 as $t \to \infty$. But if $y_1(t) \to 0$, then, according to expression (15), $u(t) \to 0$. Therefore, the solutions $y_i(t)$ $(i = 2, 3, 4)$ of the second, third and fourth equations of system (9) with any initial conditions $y_i(0)$ $(i = 2, 3, 4)$ converge to zero as $t \to \infty$, because $Re\lambda_i < 0$ for $i = 2, 3, 4$. Thus, under the nonlinear control (15), the basin of attraction B coincides with the controllability domain Q ([8], [7]): $B = Q$. So, the basin of attraction B for system (7), (15) is as large as possible and it is described by inequality (13).

The variable y_1 depends on the original variables from the vector x, according to the transformation $y = S^{-1}x$. Due to this, formula (15) defines the control feedback, which depends on the original variables. All coefficients of the designed control can be defined. Only the constant γ is an arbitrary multiplier, but it has to satisfy inequality (14).

According to theorems from [13], the equilibrium (4) of the nonlinear system (3), (15) is exponentially stable.

2) Numerical results: Let

$$\begin{gathered} m_1 = 1.0\,kg,\, m_2 = 0.2\,kg,\, g = 9.81\,m/s^2, \\ r = 0.05\,m,\, l = 0.2\,m,\, a = 0.15\,m, \\ \rho_1 = 0.2179\,m,\, \rho_2 = 0.1414\,m, \\ c_u = 0.007\,N{\cdot}m/V,\, c_v = 0.0001\,N{\cdot}m/s \\ f = 0.4\;N{\cdot}m{\cdot}s,\, u_0 = 19\,V. \end{gathered} \quad (16)$$

In open-loop, the poles of the linear system (7) (the roots of equation (11)) with parameters (16) are:

$$\lambda_1 = 3.40,\ \lambda_2 = -10.02,\ \lambda_3, \lambda_4 = -0.10 \pm 1.03i. \quad (17)$$

Now we can use inequality (13) to evaluate the basin of attraction B for system (7), (15). If $\theta(0) = \dot{\theta}(0) = \dot{\varphi}(0) = 0$, the upper bound of the initial angles φ, which can be handled for the linear model (7) is $\varphi(0) \cong 77.679°$. The corresponding initial distance $s(0)$ is equal to $0.0678\ m$. This value for the distance s is close to the value

$$s(0) = c_u u_0/(m_2 g) \quad (18)$$

With $\theta = 0$ product sm_2g is the torque about joint O of the gravity force of the ball (see the nonlinear equations (3) and linear (5)), the product $c_u u_0$ is the maximal as possible torque developed by the motor in static. Thus, the point

$$\theta = \dot{\theta} = \dot{s} = 0, \quad s = c_u u_0/(m_2 g) \quad (19)$$

is the equilibrium state (unstable) for our system (nonlinear (3) and linear (5)). It is easily to see that the equilibrium point (19) is located on the boundary of the controllability region (13). Simulation shows that, if

$$\theta(0) = \dot{\theta}(0) = \dot{s}(0) = 0, \quad s(0) \geq c_u u_0/(m_2 g), \quad (20)$$

then it is not possible to bring the nonlinear system (3) under control (15) to the equilibrium (4); but it is possible to do that, if $s(0) < \frac{c_u u_0}{m_2 g}$. Furthermore, we think there is no an admissible control $|u(x)| \leq u_0$ to bring system (3) to the equilibrium (4) from the initial states (20). This opinion is based on the numerical studies and physical feeling.

Figure 2 shows a numerical test with $\varphi(0) = 77.65°$ for the nonlinear system (3), (15) with $\gamma = -122$.

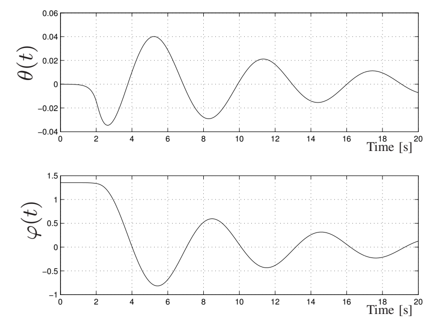

Fig. 2. Stabilization of the Straight beam-and-ball system, $\theta \to 0$ and $\varphi \to 0$ (in radians).

If $\varphi(0) = \dot{\varphi}(0) = \dot{\theta}(0) = 0$, then, using inequality (13), the upper bound of the initial tilts of the beam, which can be handled, for the linear model (7) is $\theta(0) = 3.61°$. The computations show that the upper bound of the initial tilts for the nonlinear system (3), (15) is $\theta(0) = 3.64°$. So, this value is little more important than for system (7), (15).

In the papers [1]- [5], the problem of maximization of the basin of attraction is not discussed.

III. CIRCULAR BEAM-AND-BALL SYSTEM

This system consists of a circular beam with the center C and the radius R and a ball on it.

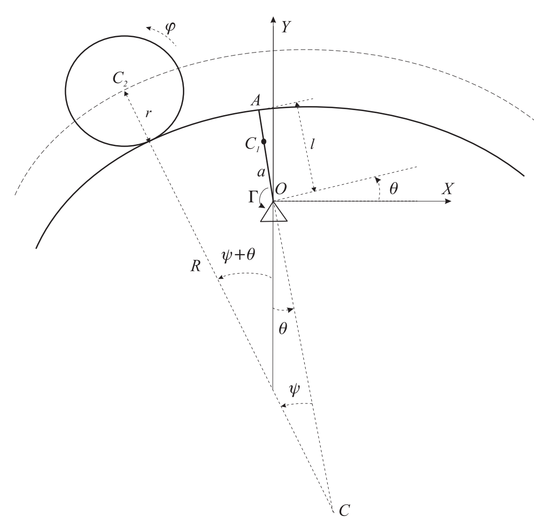

Fig. 3. Diagram of the circular beam-and-ball system.

A. Equations of motion

Here the same notations are used as for the straight beam-and-ball system. The relation between the angles φ and ψ is: $r\varphi = R\psi$. Let us assume that the constants c_u, c_v and u_0 are the same as in Section II.

The kinetic K and the potential Π energies are:

$$\begin{gathered} 2K = \left\{m_1\rho_1^2 + m_2\left[(R+r)^2 + (l-R)^2 + \right.\right. \\ \left.\left. + \ 2(R+r)(l-R)cos\tfrac{r\varphi}{R}\right]\right\}\dot{\theta}^2 + \\ +m_2\left[(R+r)^2 + \tfrac{(\rho_2 R)^2}{r^2}\right]\tfrac{(r\dot{\varphi})^2}{R^2} + \\ +2m_2\left[(R+r)^2 + (R+r)(l-R)cos\tfrac{r\varphi}{R}\right]\dot{\theta}\tfrac{r\dot{\varphi}}{R} \end{gathered}$$

$$\begin{gathered} \Pi = [m_1 a + m_2(l-R)]gcos\theta + \\ +m_2 g(R+r)cos(\tfrac{r}{R}\varphi + \theta) \end{gathered}$$

The mechanism's motion is governed by equations:

$$\begin{gathered} \left[m_1\rho_1^2 + m_2(r^2 + l^2 + 2rlcos\tfrac{r\varphi}{R}) + \right. \\ \left. +2m_2 R(R+r-l)(1-cos\tfrac{r\varphi}{R})\right]\ddot{\theta} + \\ +m_2 r(1+\tfrac{r}{R})\left[R+r+(l-R)cos\tfrac{r\varphi}{R}\right]\ddot{\varphi} + \\ +m_2 r(1+\tfrac{r}{R})(R-l)(2\dot{\theta} + \tfrac{r\dot{\varphi}}{R})\dot{\varphi}sin\tfrac{r\varphi}{R} - \\ -g[m_1 a + m_2(l-R)]sin\theta - \\ -m_2 g(R+r)sin(\theta + \tfrac{r\varphi}{R}) = c_u u - c_v\dot{\theta} - f\dot{\theta} \end{gathered} \tag{21}$$

$$\begin{gathered} r(1+\tfrac{r}{R})\left[R+r+(l-R)cos\tfrac{r\varphi}{R}\right]\ddot{\theta} + \\ + \left[\rho_2^2 + r^2(1+\tfrac{r}{R})^2\right]\ddot{\varphi} + (1+\tfrac{r}{R}) \\ (l-R)\dot{\theta}^2 sin\tfrac{r\varphi}{R} - gr(1+\tfrac{r}{R})sin(\theta + \tfrac{r\varphi}{R}) = 0 \end{gathered}$$

If $u = 0$, this system has one unstable equilibrium state (4).

B. Linearized Model

Linearize the equations (21) near the equilibrium (4):

$$\begin{gathered} \left[m_1\rho_1^2 + m_2(r+l)^2\right]\ddot{\theta} + m_2 r\left(1+\tfrac{r}{R}\right)(r+l)\ddot{\varphi} - \\ -g[m_1 a + m_2(r+l)]\theta - \\ -m_2 g(r+R)\tfrac{r\varphi}{R} = c_u u - c_v\dot{\theta} - f\dot{\theta} \\ \\ r\left(1+\tfrac{r}{R}\right)(r+l)\ddot{\theta} + \left[\rho_2^2 + r^2\left(1+\tfrac{r}{R}\right)^2\right]\ddot{\varphi} - \\ -gr\left(1+\tfrac{r}{R}\right)\left(\theta + \tfrac{r\varphi}{R}\right) = 0 \end{gathered} \tag{22}$$

C. Kalman controllability

The determinant of the controllability matrix for the model (22) is not null, if and only if:

$$Rr^2(R-l) + R^2\rho_2^2 + r^3(R-l) \neq 0 \tag{23}$$

If $r = 0$, then the ball becomes a material point and $\rho_2 = 0$. However, we do not consider a material point on the beam. Now let $r \neq 0$, but the mass of the ball is concentrated in its center ($\rho_2 = 0$) and the suspension point O coincides with the curvature center C of the circular beam ($R = l$). In this case, inequality (23) is not satisfied and the linear system is not controllable. Consider in this case the original nonlinear system (21). Introduce the angle $\alpha = \theta + \frac{r\varphi}{R}$. Instead of the system (21) we come to the separated equations:

$$\begin{gathered} m_1\rho_1^2\ddot{\theta} - m_1 gasin\theta = c_u u - c_v\dot{\theta} - f\dot{\theta} \\ (R+r)\ddot{\alpha} - gsin\alpha = 0 \end{gathered} \tag{24}$$

The control u has no action on the angle α and system (24) is not controllable.

Inequality (23) is equivalent to the following one:

$$l - R \neq \frac{R^2\rho_2^2}{(R+r)r^2} \tag{25}$$

D. Spectrum of Linear System

The state form of system (22) can be presented in the same matrix form (7) as for the straight beam, but with the following submatrices D and E:

$$D = \begin{pmatrix} m_1\rho_1^2 + m_2(r+l)^2 & m_2 r(1+\frac{r}{R})(r+l) \\ r(1+\frac{r}{R})(r+l) & \rho_2^2 + r^2(1+\frac{r}{R})^2 \end{pmatrix}$$

$$E = g\begin{pmatrix} m_1 a + m_2(r+l) & m_2 r(1+\frac{r}{R}) \\ r(1+\frac{r}{R}) & r(1+\frac{r}{R})\frac{r}{R} \end{pmatrix}$$

Using a linear transformation $x = Sy$ with a constant matrix S, we get the Jordan form similar to (9), (10).

The characteristic equation of system (22) has form (11) with

$a_0 = detD > 0, \quad a_1 = (c_v + f)\left[\rho_2^2 + \frac{r^2}{R^2}(R+r)^2\right] > 0,$

$a_2 =$
$- m_1 g\left[(\rho_2^2 + r^2)aR^2 + (2ar + \rho_1^2)Rr^2 + (ar + \rho_1^2)r^3\right] -$
$- m_2 g\left[(r+l)R^2(r^2 - \rho_2^2) + (r^2 - l^2)r^2 R - (r+l)lr^3\right],$

$a_3 = -(c_v + f)g\frac{r^2}{R^2}(R+r) < 0,$

$a_4 = detE = g^2\frac{r^2}{R^2}(R+r)[m_1 a + m_2(l-R)].$

We assume that

$$m_1 a + m_2(l-R) > 0 \tag{26}$$

Inequality (26) is satisfied, if R is sufficiently small. But we have not to forget condition (23) (or (25)) of controllability.

Under condition (26), the coefficient a_4 is positive. According to the theorem of Routh-Hurwitz [12], the characteristic equation (11) has two roots in the right-half complex plane and two roots in the left-half complex plane.

E. Problem Statement

For the circular beam-and-ball system, we consider the same problem, as in Section II.

F. Feedback Control for the Circular beam-and-ball system

Under condition (26), the linear model of the system has two eigenvalues λ_1, λ_2 in the right-half complex plane and two eigenvalues λ_3, λ_4 in the left-half complex plane.

1) Control design: Let λ_1 and λ_2 be the real positive eigenvalues, and let us consider the first two scalar differential equations of system (9), (10), corresponding to them:

$$\dot{y}_1 = \lambda_1 y_1 + d_1 u, \quad \dot{y}_2 = \lambda_2 y_2 + d_2 u \tag{27}$$

Under condition (25), system (7) is Kalman controllable. Therefore, subsystem (27) is controllable too [11] and $d_1 \neq 0$, $d_2 \neq 0$. The controllability domain Q of the equations

(27), and consequently of system (9) is an open bounded set with the following boundaries [14] ($0 \le \tau < \infty$)

$$\begin{aligned} y_1(\tau) &= \pm \frac{d_1 u_0}{\lambda_1} \left(2e^{-\lambda_1 \tau} - 1\right), \\ y_2(\tau) &= \pm \frac{d_2 u_0}{\lambda_2} \left(2e^{-\lambda_2 \tau} - 1\right) \end{aligned} \qquad (28)$$

In the case of two complex poles with positive real parts, instead of (28), we get other formulas [8].

The boundary of the controllability region Q of the system (27) has two corner points:

$$y_1 = \pm d_1 u_0/\lambda_1, \quad y_2 = \pm d_2 u_0/\lambda_2; \qquad (29)$$

They are the equilibrium points of system (27) under the constant controls:

$$u = \pm u_0 \qquad (30)$$

We can "suppress" the instability of the state $y_1 = 0$, $y_2 = 0$ by a linear feedback control,

$$u_l = k_1 y_1 + k_2 y_2 \qquad (31)$$

with $k_1 = const$ and $k_2 = const$. It is shown in paper [9] that using a linear feedback (31) with saturation ($\gamma = const$):

$$u = \begin{cases} u_0, & if \quad \gamma u_l \ge u_0 \\ \gamma u_l, & if \quad |\gamma u_l| \le u_0 \\ -u_0, & if \quad \gamma u_l \le -u_0 \end{cases} \qquad (32)$$

the basin of attraction B can be made arbitrary close to the controllability domain Q.

The straight line crossing two points (29) is the following:

$$k_1 y_1 + k_2 y_2 = 0$$

with

$$k_1 = -d_2/\lambda_2, \quad k_2 = d_1/\lambda_1 \qquad (33)$$

If

$$sign\gamma = sign\left[d_1 d_2 \left(\lambda_1 - \lambda_2\right)\right] \qquad (34)$$

and $|\gamma| \to \infty$, then the basin of attraction B of system (27) under the nonlinear control (32) with coefficients (33) tends to the controllability region Q. Consequently, using the coefficients (33), B can be made arbitrary close to Q. If $|\gamma| \to \infty$, control (32) tends to the bang-bang control.

The solutions $y_1(t)$ and $y_2(t)$ of system (27), (32) tend to 0 as $t \to \infty$ for the initial values $y_1(0)$, $y_2(0)$, belonging to the basin of attraction of system (27), (32). Similarly to the proof in the Section II, we can show that under control (32) with coefficients (33), the basin of attraction of system (9), (32) is described by the same relations, which describe the basin of attraction of system (27), (32).

2) Numerical results: Let

$$\begin{gathered} m_1 = 1.0\,kg, \; m_2 = 0.2\,kg, \\ r = 0.05\,m, \; l = 0.2\,m, \; a = 0.15\,m, \\ \rho_1 = 0.2646\,m, \; \rho_2 = 0.1414\,m, \; f = 0.4 \; N{\cdot}m{\cdot}s. \end{gathered} \qquad (35)$$

The eigenvalues of system (22) depend on R. In Figures 4 and 5, the evolution of the eigenvalues following R is shown. There are two positive real eigenvalues and two negative real eigenvalues, if $R < 0.95$. The value $R = 0.95$ can be found from (26). For $0.95 < R < 0.962$ there are three negative real eigenvalues and one positive real eigenvalue. Two eigenvalues become complex conjugate with negative real part, if $R > 0.962$. And it is naturally because for the straight beam ($R = \infty$) there are two complex conjugate eigenvalues (see data (17)).

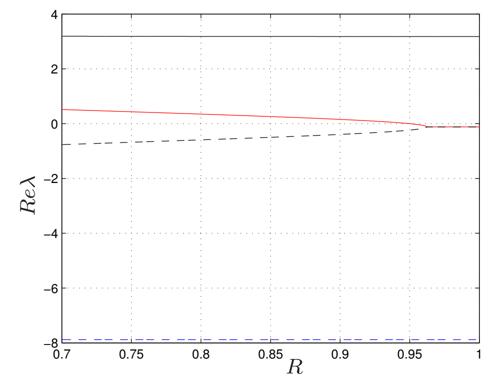

Fig. 4. Real parts of the eigenvalues versus radius R.

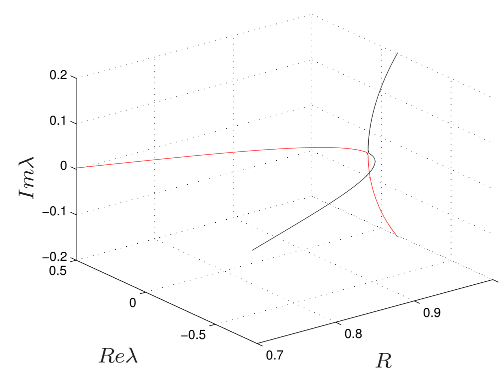

Fig. 5. Evolution of the eigenvalues, becoming complex, if R increases.

In open-loop, considering the parameters (35) and $R = 0.8\ m$ the poles of the linear system (7) are:

$$\lambda_1 = 3.19,\ \lambda_2 = 0.35,\ \lambda_3 = -7.89,\ \lambda_4 = -0.59 \qquad (36)$$

Using formulas (28), the controllability domain Q for system (27) is designed in Figure 6. It is bounded by dashed line. Its boundary contains the corner points (29). Using the linear model (22), we can define the following equilibrium points under controls (30) with the original variables:

$$\begin{gathered} \theta = \mp \frac{c_u u_0}{g[m_1 a + m_2(l-R)]}, \qquad \dot{\theta} = 0, \\ \varphi = -\tfrac{R}{r}\theta \quad (s = -R\theta), \qquad \dot{\varphi} = 0 \quad (\dot{s} = 0). \end{gathered} \qquad (37)$$

Points (37) are located on the boundary of the controllability region.

For the nonlinear model (21), instead of (37), we get:

$$\theta = \mp arcsin\frac{c_u u_0}{g[m_1 a + m_2(l-R)]}, \qquad \dot{\theta} = 0,$$
$$\varphi = -\tfrac{R}{r}\theta \quad (s = -R\theta), \qquad \dot{\varphi} = 0 \quad (\dot{s} = 0). \tag{38}$$

In equilibriums (38), the ball is located on the highest point of the circular beam and consequently in their point of contact the tangent to the beam is horizontal.

The basin of attraction B for system (27) under the control (32) is shown in the same Figure 6. Its boundary is drawn by solid line. This boundary is the periodical motion (cycle) of system (27), (32). This cycle is computed, using the backward motion of system (27), (32) from a state close to the origin $y_1 = y_2 = 0$. The basin B depends on the coefficient γ. We show in Figure 6 the basin of attraction B, with $\gamma = -4000$. If the coefficient γ is smaller in modulus, then the basin of attraction is smaller too.

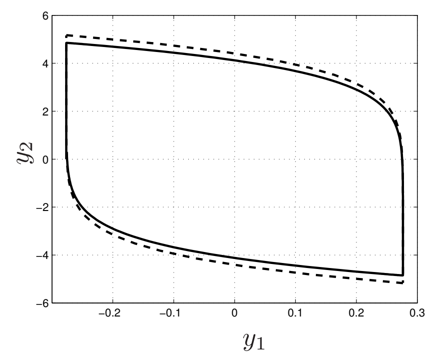

Fig. 6. Controllability domain Q (dashed line) for system (27) and basin of attraction B (solid line) with $\gamma = -4000$.

The control law (32) is applied to the nonlinear model (21). Figure 7 shows the graphs of the angular variables θ and φ. These graphs are designed for the initial angle $\varphi(0) = 64.5°$ and $\theta(0) = \dot{\theta}(0) = \dot{\varphi}(0) = 0$. This value $\varphi(0) = 64.5°$ is close to the upper bound of the initial angles $\varphi(0)$, which are possible to stabilize the equilibrium state (4). The corresponding initial distance $s(0)$ is equal to $0.061\ m$. No oscillations appear during the transient process, because matrix A does not have complex poles.

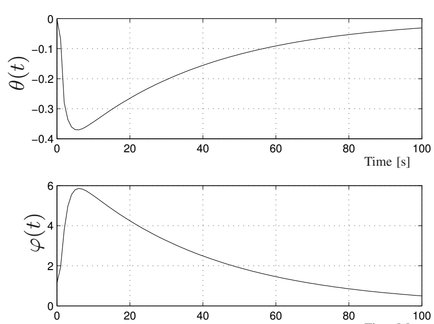

Fig. 7. Stabilization of the Circular beam-and-ball system, $\theta \to 0$ and $\varphi \to 0$ (in radians).

IV. Conclusion

The stabilization problem of unstable equilibriums was considered for the straight beam-and-ball system and an original circular beam-and-ball system. The model linearized near the unstable equilibrium of the first system has one unstable mode. The difficulty is greater to stabilize the circular beam-and-ball system, because its linear model has two unstable modes. Considering the restriction on the voltage of the motor, the objective is to get a large basin of attraction. For each system we use the Jordan form of the linear model to extract the unstable part. The designed feedback control contains the unstable Jordan variables only. All parameters of this control are defined up to a constant multiplier. Simulation results are close for linear and nonlinear systems. They illustrate the efficiency of the designed control laws. Testbed devices can be now imagined to test the designed control laws experimentally. The original circular beam-and-ball system will be interesting for education, demonstrations, and to investigate new nonlinear control laws.

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