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P·SPR·D Control and P·I·SPR·D Control for Affine Nonlinear Systems

—Stabilization Theory Based on Passivity—

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***Abstract*— This paper is concerned with set-point sevo problems by P·SPR·D control and P·I·SPR·D control of affine nonlinear system which is of multi input and multi output. P·SPR·D control consists of proportional (P) action + strict positive real (SPR) action + derivative (D) action. Such control can asymptotically stabilize the affine nonlinear system being passive. Stability analysis of P·SPR·D control and P·I·SPR·D control is made, based on the passivity theory and LaSalle's invariance principle. The effectiveness of the proposed method is demonstrated by the simulation results for an elastic joint robot arm and TORA model.**

I. Introduction

This paper investigates a PID-like control scheme for affine nonlinear systems. In regard to stabilizing control of affine nonlinear systems, there exist many studies as passivity theory $[3, 4, 8, 9, 10]$, exact linearization $[5]$, back stepping method $[6, 10]$, passivity based design of cascaded system $[13]$, etc. But PID control has not been used so much except for the Lagrangian systems like robot manipulators[1]. PID control for dissipative systems is discussed in Ref. [2] considerably.

We study stability analysis of P·SPR·D control imitating PID control for the affine nonlinear systems, based on the passivity theory and LaSalle's invariance principle[7]. (SPR is a short for strict positive real.) When the P·SPR·D controller is applied to a plant possessing the Kalman-Yakubovich-Popov (K.Y.P) property[3,4,10], we can prove that the closed-loop system becomes asymptotically stable by the P·SPR·D control, applying the passivity theory and LaSalle's theorem. Based on the same approach, P·I·SPR·D control is also proposed to apply for more general cases.

Stabilizability by PID control is augmented by adding the SPR to PD or PID in parallel. This is guessed from the passivity theorem of interconnected systems. Note that the SPR element in the structured controller is effective for improvement of transient response, while the I element works for steady state performance.

Section 2 describes the P·SPR·D control generally. Section 3 investigates regulation problem for the affine nonlinear system by the P·SPR·D control. Sections 4 and 5 are devoted for a set-point servo problem by the P·SPR·D control and P·I·SPR·D one, respectively. Section 6 investigates P·SPR·D control of an elastic joint robot arm and TORA medel. The simulation results are shown to demonstrate the effectiveness of the P·SPR·D control.

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II. P·SPR·D Control of Affine Nonlinear System

Let us consider an affine nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{2}$$

where $\boldsymbol{x} \in R^n, \boldsymbol{u} \in R^r, \boldsymbol{y} \in R^m$ are the state vector, the control input and the measurable output, respectively. We assume that system (1),(2) is stabilizable.

When the desired value is $\boldsymbol{y} = \boldsymbol{y}^*$, PID control for a set-point servo problem is given as

$$\boldsymbol{u} = K_P(\boldsymbol{y}^* - \boldsymbol{y}) + K_I \int_0^t (\boldsymbol{y}^* - \boldsymbol{y})dt - K_D\dot{\boldsymbol{y}} + \boldsymbol{m}_0 \tag{3}$$

where $K_P \in R^{r\times m}, K_I \in R^{r\times m}, K_D \in R^{r\times m}$ are gain matrices corresponding to proportional, integral and derivative actions, respectively. $\boldsymbol{m}_0$ is the so-called manual reset quantity.

Introducing here a new state equation (an integlator)

$$\dot{\boldsymbol{\xi}} = \boldsymbol{y}^* - \boldsymbol{y} \tag{4}$$

the PID control (3) can be represented as

$$\boldsymbol{u} = K_P(\boldsymbol{y}^* - \boldsymbol{y}) + K_I\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}} + \boldsymbol{m}_0 \tag{5}$$

Now at the equilibrium that output $\boldsymbol{y}$ is kept $\boldsymbol{y}^*$, the following relation must be satisfied.

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_e) + G(\boldsymbol{x}_e)\overline{\boldsymbol{u}}$$

$$\boldsymbol{y}^* = \boldsymbol{h}(\boldsymbol{x}_e)$$

Since there exist $(n+m)$ equations and $(n+r)$ variables, when $r \geq m$, we can set $(r-m)$ state variables $\boldsymbol{x}_{eN}$ as an arbitrary value $\boldsymbol{x}^*_{eN}$. But the remained state variables $\boldsymbol{x}_{eB}$ and $\overline{\boldsymbol{u}}$ are dependently detemined. Put such an equilibrium as $\boldsymbol{x}^* = \begin{bmatrix} \boldsymbol{x}^*_{eN} \\ \boldsymbol{x}_{eB}(\boldsymbol{x}^*_{eN}, \boldsymbol{y}^*) \end{bmatrix}, \quad \boldsymbol{u}^* = \overline{\boldsymbol{u}}(\boldsymbol{x}^*_{eN}, \boldsymbol{y}^*)$.

Now consider the cascaded system of subsystem Σ_p and subsystem Σ_s :

$$\Sigma_p : \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u} \tag{6}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{7}$$

$$\Sigma_s : \dot{\boldsymbol{\xi}} = D\boldsymbol{\xi} + \boldsymbol{q}(\boldsymbol{y}, \boldsymbol{x}), \quad D < 0 \tag{8}$$

Here D is a negative definite matrix and Σ_s is strict positive real (SPR). Further, $\boldsymbol{q}(\boldsymbol{y}, \boldsymbol{x})$ is properly set depending on the problem. And consider a feedback controller

$$\boldsymbol{u} = K_P(\boldsymbol{y}^* - \boldsymbol{y}) + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}} + \boldsymbol{m}_0 \tag{9}$$

where $K_S \in R^{r\times m}$ is a gain matrix for the SPR action. We call the control law (9) with (8) the P·SPR·D control law.

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Generally speaking, stability of the closed-loop system (6)∼(9) cannot be analyzed except applying Lyapunov's theorem for an individual plant. But we can prove asymptotical stability for the regulation problem and/or the set-point servo problem of passive affine nonlinear systems, applying LaSalle's invariance principle, as mentioned below.

III. P·SPR·D Control for Regulation Problem

In this section we study on the regulation problem of affine nonlinear system. Consider the cascaded system of subsystem Σ_p and subsystem Σ_s :

$$\Sigma_p : \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u} \tag{10}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{11}$$

$$\Sigma_s : \dot{\boldsymbol{\xi}} = D\boldsymbol{\xi} - \boldsymbol{y}, \quad D < 0 \tag{12}$$

Here D is asssumed to be a negative definite diagonal matrix.

Since we discuss on a control scheme based on the passivity, we assume that system (10),(11) is passive with respect to m-dimensional input $\boldsymbol{u} \in R^m$ and output $\boldsymbol{y} \in R^m$. Then, taking a semi-positive definite storage function as $W(\boldsymbol{x}) \geq 0$, $W(\boldsymbol{0}) = 0$, the so-called K-Y-P property holds[3,4,10].

$$W_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) \leq 0 \tag{13}$$

$$W_{\boldsymbol{x}}(\boldsymbol{x})G(\boldsymbol{x}) = \boldsymbol{y}^T \tag{14}$$

[Definition 1] Nonlinear system (10),(11) is *zero state detectable* , if $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$ when $\boldsymbol{u}(t) = \boldsymbol{0},\ \boldsymbol{y}(t) = \boldsymbol{0}\ \ \forall t \geq 0$. Furthermore, it is $\boldsymbol{x}^*$-*state detectable*, if $\boldsymbol{x}(t) \to \boldsymbol{x}^*$ as $t \to \infty$ when $\boldsymbol{u}(t) = \boldsymbol{u}^*,\ \boldsymbol{y}(t) = \boldsymbol{y}^*\ \ \forall t \geq 0$.

First the following is well known[5,10].

⌈ Assume that system (10),(11) with $r = m$ is passive and zero state detectable. Then the output feedback control $\boldsymbol{u} = -K_P\boldsymbol{y}$ aymptotically stabilizes an equilibrium point $\boldsymbol{x}_e = \boldsymbol{0}$, where $K_P \in R^{m\times m}$ is a positive definite matrix. ⌋

However, we investigate below the P·SPR·D control by which performance improvement is expected.

[**Theorem 1**] Suppose that the cascaded system (10)∼(12) of subsystems Σ_p and Σ_s satisfies :

Assumption (a) Subsystem Σ_p is passive.

Assumption (b) Subsystem Σ_s is asymptotically stable as $\boldsymbol{y} = \boldsymbol{0}$.

Then, if the system Σ_p is zero state detectable with respect to the output $\boldsymbol{y}$, the P·SPR·D control

$$\boldsymbol{u} = -K_P\boldsymbol{y} + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}} \tag{15}$$

asymptotically stabilizes the closed-loop system of cascaded system of Σ_p and Σ_s at the equilibrium point $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$, provided that K_P and K_S are positive definite matrices and K_D is semi-positive definite one and $K_SD < 0$.

(**Proof**) For the overall system consider a Lyapunov function candidate (semi-positive definite function)

$$V(\boldsymbol{x}, \boldsymbol{\xi}) = W(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{\xi}^T K_S\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{y}^T K_D\boldsymbol{y} \geq 0 \tag{16}$$

Take a time derivative of $V(\boldsymbol{x}, \boldsymbol{\xi})$ along (10) and (12) and use (13),(14),(15) to get

$$\begin{aligned}
&\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) \\
&= W_{\boldsymbol{x}}(\boldsymbol{x})\dot{\boldsymbol{x}} + \boldsymbol{\xi}^T K_S\dot{\boldsymbol{\xi}} + \boldsymbol{y}^T K_D\dot{\boldsymbol{y}} \\
&= W_{\boldsymbol{x}}(\boldsymbol{x})\{\boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}\} + \boldsymbol{\xi}^T K_S(D\boldsymbol{\xi} - \boldsymbol{y}) + \boldsymbol{y}^T K_D\dot{\boldsymbol{y}} \\
&= W_{\boldsymbol{x}}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) + W_{\boldsymbol{x}}(\boldsymbol{x})G(\boldsymbol{x})(-K_P\boldsymbol{y} + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}}) \\
&\quad + \boldsymbol{\xi}^T K_S\,(D\boldsymbol{\xi} - \boldsymbol{y}) + \boldsymbol{y}^T K_D\dot{\boldsymbol{y}} \\
&\leq \boldsymbol{y}^T(-K_P\boldsymbol{y} + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}}) \\
&\quad + \boldsymbol{\xi}^T K_S D\boldsymbol{\xi} - \boldsymbol{\xi}^T K_S\boldsymbol{y} + \boldsymbol{y}^T K_D\dot{\boldsymbol{y}} \\
&= -\boldsymbol{y}^T K_P\boldsymbol{y} + \boldsymbol{\xi}^T K_S D\boldsymbol{\xi} \leq 0
\end{aligned} \tag{17}$$

Accordingly, Lyapunov's stability theorem cannot be applied, as $V(\boldsymbol{x}, \boldsymbol{\xi})$ is semi-positive definite and $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi})$ is semi-negative definite. So we apply LaSalle's invariance principle[7] to prove that the overall system is asymptotically stable at the equilibrium $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$.

Now let $\Omega_c = \{(\boldsymbol{x}, \boldsymbol{\xi}) | V(\boldsymbol{x}, \boldsymbol{\xi}) \leq c\}$ and suppose Ω_c is bounded and $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$ in Ω_c (c is a positive number such that $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$). Here define Ω_E as a set of all points of Ω_c satisfying $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ and put

$$\Omega_E = \left\{(\boldsymbol{x}, \boldsymbol{\xi}) \mid \dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0,\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega_c\right\}$$

Since $K_P > 0$, $K_SD < 0$ from the condition of the theorem, $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ holds from (17) only when $\boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{y} = \boldsymbol{0}$, that is,

$$\Omega_E = \{(\boldsymbol{x}, \boldsymbol{\xi}) \mid \boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{y} = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega_c\}$$

But when $\boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{y} = \boldsymbol{0}$, one has $\boldsymbol{u} = \boldsymbol{0}$ from (15). Thus it holds in Ω_E that

$$\Omega_E = \{(\boldsymbol{x}, \boldsymbol{\xi}) \mid \boldsymbol{\xi} = \boldsymbol{0},\ \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{y} = \boldsymbol{0},\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \Omega_c\} \tag{18}$$

Subsystem Σ_p is zero state detectable from the condition of the theorem. Therefore, by the definition of zero state detectability, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$ implies that $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$ in Ω_E. Consequently, $(\boldsymbol{x}, \boldsymbol{\xi})$ satisfying $\dot{V}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ consists of only a point $(\boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{0}, \boldsymbol{0})$. Namely, letting Ω_M be the largest invariance set in Ω_E, Ω_M consists of only the equilibrium point $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$. Thus, by LaSalle's invariance principle[7], all trajectories in Ω_c converge to Ω_M as $t \to \infty$, that is, converge to the equilibrium $(\boldsymbol{x}_e, \boldsymbol{\xi}_e) = (\boldsymbol{0}, \boldsymbol{0})$. Q.E.D

By the way, static state feedback control law may be obtained by the passivity based design[10,13] of the cascaded system also. Generally speaking, however, the control law using a storage function is complex. An advantage of the P·SPR·D control is of output feedback with simple structure.

IV. P·SPR·D Control for Set-point Servo Problem

We study a set-point servo problem of affine nonlinear system.

Most passive systems consist of Lagrangian systems like mechanical systems. In that case hypothetical output $\boldsymbol{y}$ is generally taken as velocity of generalized coordinates. In many cases, however, actual controlled output and/or measurable output are position that is an integral of velocity $\boldsymbol{y}$.

We suppose $\boldsymbol{y} \in R^m$ and $\boldsymbol{u} \in R^m$, since we consider the set-point servo problem of passive systems. Such a set-point servo problem is formulated with the SPR element as follows.

$$\Sigma_p : \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{19}$$
$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{20}$$
$$\dot{\boldsymbol{z}} = \boldsymbol{y}, \quad \boldsymbol{z}(0) = \boldsymbol{0} \tag{21}$$
$$\Sigma_s : \dot{\boldsymbol{\xi}} = D\boldsymbol{\xi} + (\boldsymbol{z}^* - \boldsymbol{z}) - \boldsymbol{y}, \quad \boldsymbol{\xi}(0) = \boldsymbol{0}, \ D < 0 \tag{22}$$
$$\boldsymbol{u} = K_P(\boldsymbol{z}^* - \boldsymbol{z}) + K_S\boldsymbol{\xi} - K_D\boldsymbol{y} + \boldsymbol{m}_0 \tag{23}$$

where $\boldsymbol{z}^*$ is the desired value of position $\boldsymbol{z}$ which is an integral of velocity $\boldsymbol{y}$. In case of the set-point servo problem the desired value of velocity $\boldsymbol{y}^*$ is zero. $\boldsymbol{m}_0$ is the so-called manual reset quantity. We suppose all $K_P \in R^{m\times m}, \ K_S \in R^{m\times m}, \ K_D \in R^{m\times m}$ be positive definite matrices.

In particular, let us call a control scheme (21)$\sim$(23) the P·SPR·D control in regard to position.

[Theorem 2] Assume that system (19),(20) be passive and $\boldsymbol{x}^*$-state detectable. Then the closed-loop system (19)$\sim$(23) of affine nonlinear system with P·SPR·D control is asymptotically stable at the equilibrium $(\boldsymbol{x}_e, \boldsymbol{z}_e, \boldsymbol{\xi}_e) = (\boldsymbol{x}^*, \boldsymbol{z}^*, \boldsymbol{0})$, provided that $\boldsymbol{m}_0 = \boldsymbol{u}^*$ and positive definite matrices $K_P, \ K_S, \ K_D$ and negative definite diagonal D are appropriately chosen. Here $\boldsymbol{x}^*$ and $\boldsymbol{u}^*$ are the equilibrium state and the control input corresponding to the desired controlled output $\boldsymbol{z}^*$.

(Proof) At the equilibrium at which the controlled output $\boldsymbol{z}$ is kept $\boldsymbol{z}^*$, the following relation must be satisfied.

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{f}(\boldsymbol{x}_e) + G(\boldsymbol{x}_e)\overline{\boldsymbol{u}} \\ \boldsymbol{z}^* &= \widetilde{\boldsymbol{h}}(\boldsymbol{x}_e) \end{aligned}$$

where $\boldsymbol{z} = \widetilde{\boldsymbol{h}}(\boldsymbol{x})$ denotes the actual controlled output. Thus, in order that $\boldsymbol{z}$ becomes $\boldsymbol{z}^*$, $\boldsymbol{x}_e$ and $\overline{\boldsymbol{u}}$ must be $\boldsymbol{x}^*$ and $\boldsymbol{u}^*$ corresponding to $\boldsymbol{z}^*$, as mentioned in Section 2.

Meanwhile, since an equilibrium of system (19)$\sim$(22) must satisfy

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{f}(\boldsymbol{x}_e) + G(\boldsymbol{x}_e)\{K_P(\boldsymbol{z}^* - \boldsymbol{z}_e) + K_S\boldsymbol{\xi}_e - K_D\boldsymbol{y}_e + \boldsymbol{u}^*\} \\ \boldsymbol{0} &= \boldsymbol{y}_e \\ \boldsymbol{0} &= D\boldsymbol{\xi}_e + (\boldsymbol{z}^* - \boldsymbol{z}_e) - \boldsymbol{y}_e, \end{aligned}$$

it follows that $(\boldsymbol{x}_e = \boldsymbol{x}^*, \ \boldsymbol{z}_e = \boldsymbol{z}^*, \ \boldsymbol{\xi}_e = \boldsymbol{0})$ is an equilibrium point with $\boldsymbol{y}_e = \boldsymbol{h}(\boldsymbol{x}_e) = \boldsymbol{0}$.

Now let us consider a Lyapunov function candidate

$$\begin{aligned} V(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) &= W(\boldsymbol{x}) - \boldsymbol{u}^{*T}\boldsymbol{z} \\ &+ \frac{1}{2}\begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}^T \begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K} \end{bmatrix} \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix} \end{aligned} \tag{24}$$

where $W(\boldsymbol{x})$ is a storage function and $K_P - \overline{K} > 0, \ K_S - \overline{K} > 0$ and $\begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K} \end{bmatrix}$ is a positive definite matrix. The first term in the right-hand side of (24) is a semi-positive definite function. Since the second term plus the third one is a quadrtic function of $\begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}$ whose quadratic term is with the positive definite matrix, it has the minimum. Hence, $V(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi})$ is a function bounded below.

Next calculate its time derivative along (19),(21),(22) with the use of K-Y-P property (13),(14) to get

$$\begin{aligned} &\dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \\ &= W_{\boldsymbol{x}}(\boldsymbol{x})\{\boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u}\} - \boldsymbol{u}^{*T}\boldsymbol{y} \\ &+ \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}^T \begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K} \end{bmatrix} \begin{bmatrix} -\dot{\boldsymbol{z}} \\ \dot{\boldsymbol{\xi}} \end{bmatrix} \\ &\le \boldsymbol{y}^T\boldsymbol{u} - \boldsymbol{u}^{*T}\boldsymbol{y} + \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}^T \begin{bmatrix} K_P - \overline{K} & \overline{K} \\ \overline{K}^T & K_S - \overline{K} \end{bmatrix} \\ &\quad \times \begin{bmatrix} -\boldsymbol{y} \\ D\boldsymbol{\xi} + (\boldsymbol{z}^* - \boldsymbol{z}) - \boldsymbol{y} \end{bmatrix} \\ &= \boldsymbol{y}^T(K_P(\boldsymbol{z}^* - \boldsymbol{z}) + K_S\boldsymbol{\xi} - K_D\boldsymbol{y} + \boldsymbol{u}^*) - \boldsymbol{u}^{*T}\boldsymbol{y} \\ &+ \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}^T \times \\ &\begin{bmatrix} -(K_P - \overline{K})\boldsymbol{y} + \overline{K}D\boldsymbol{\xi} + \overline{K}(\boldsymbol{z}^* - \boldsymbol{z}) - \overline{K}\boldsymbol{y} \\ -\overline{K}^T\boldsymbol{y} + (K_S - \overline{K})D\boldsymbol{\xi} + (K_S - \overline{K})(\boldsymbol{z}^* - \boldsymbol{z}) - (K_S - \overline{K})\boldsymbol{y} \end{bmatrix} \\ &= \boldsymbol{y}^T\{K_P(\boldsymbol{z}^* - \boldsymbol{z}) + K_S\boldsymbol{\xi} - K_D\boldsymbol{y} + \boldsymbol{u}^*\} - \boldsymbol{u}^{*T}\boldsymbol{y} \\ &+ \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}^T \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S - \overline{K} & (K_S - \overline{K})D \end{bmatrix} \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix} \\ &- (\boldsymbol{z}^* - \boldsymbol{z})^T K_P\boldsymbol{y} - \boldsymbol{\xi}^T K_S\boldsymbol{y} \\ &= -\boldsymbol{y}^T K_D\boldsymbol{y} + \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix}^T \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S - \overline{K} & (K_S - \overline{K})D \end{bmatrix} \\ &\quad \times \begin{bmatrix} (\boldsymbol{z}^* - \boldsymbol{z}) \\ \boldsymbol{\xi} \end{bmatrix} \end{aligned} \tag{25}$$

Here we try to make

$$\begin{bmatrix} \overline{K} & \overline{K}D \\ K_S - \overline{K} & (K_S - \overline{K})D \end{bmatrix}$$

be negative definite. For that purpose, set $\overline{K} < 0, \ K_S - \overline{K} = (\overline{K}D)^T$ and $D < -I$ such that we have $K_S = (I + D)\overline{K} > 0$. Then the above matrix becomes

$$\begin{bmatrix} \overline{K} & \overline{K}D \\ (\overline{K}D)^T & D\overline{K}D \end{bmatrix}$$

Since the (1,1) element and the (2,2) element are $\overline{K} < 0, \ D\overline{K}D < 0$, respectively, we can choose $\overline{K} < 0$ and $D < 0$ such that the above matrix becomes negative definite.

Consequently, $\dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi})$ becomes semi-negative definite, and it follows that the P·SPR·D control is stable in the sense of Lyapunov, but it is unknown if asymptotically stable. So we apply LaSalle's invariance principle.

Let $\Omega_c = \{(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \mid V(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \le c\}$ and suppose Ω_c is bounded and $\dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \le 0$ in Ω_c (c is a positive number such that $\dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \le 0$). Here define Ω_E as a set of all points of Ω_c satisfying $\dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) = 0$ and put

$$\Omega_E = \{(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \mid \dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) = 0, \ (\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) \in \Omega_c\} \tag{26}$$

From (25) $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi})$ satisfying $\dot{V}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi}) = 0$ is given as $\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{z}^* - \boldsymbol{z} = \boldsymbol{0}, \ \boldsymbol{\xi} = \boldsymbol{0}$. But at that time we have $\boldsymbol{u} = \boldsymbol{u}^*$

from (23) and by $\boldsymbol{x}^*$-state detectability, we obtain

$$\Omega_E = \{(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})|\boldsymbol{x}=\boldsymbol{x}^*,\boldsymbol{z}=\boldsymbol{z}^*,\boldsymbol{\xi}=\boldsymbol{0},\ (\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\in\Omega_c\} \tag{27}$$

Accordingly, we know from (19),(21),(22) that $(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})$ in Ω_E consists of only the equilibrium point $(\boldsymbol{x}_e,\boldsymbol{z}_e,\boldsymbol{\xi}_e) = (\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{0})$ for $\boldsymbol{u} = \boldsymbol{u}^*$. Thus the largest invariance set Ω_M in Ω_E consists of the equilibrium point $(\boldsymbol{x}_e,\boldsymbol{z}_e,\boldsymbol{\xi}_e) = (\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{0})$. Therefore, by LaSalle's invariance principle all trajectories in Ω_c converge to Ω_M as $t \to \infty$. Thus, $(\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{0})$ is aymptotically stable and $\boldsymbol{z}$ converges to $\boldsymbol{z}^*$. Q.E.D

[Corollary 1] If system (19),(20) is the Euler-Lagrange system, Theorem 2 holds without the assumption of $\boldsymbol{x}^*$-state detectability.

[Remark 1] A theorem corresponding to the above Corollary 1 has been proved in Refs. [11,12] in cases of an inverted pendulum and robot manipulators not being zero state detectable.

[Remark 2] When $\boldsymbol{m}_0 = \boldsymbol{u}^*$ cannot be calculated or is not available, one may adopt P·SPR·D+I control, where $K_I\int_0^t(\boldsymbol{z}^*-\boldsymbol{z})dt$ is substituted for $\boldsymbol{m}_0$.

V. P·I·SPR·D Control for Set-point Servo Problem

When output $\boldsymbol{y}$ is not always velocity of generalized coordinates, we can consider the following P·I·SPR·D control.

Such a set-point servo problem is formulated with the SPR element as follows.

$$\Sigma_p : \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + G(\boldsymbol{x})\boldsymbol{u},\quad \boldsymbol{x}(0)=\boldsymbol{x}_0 \tag{28}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{29}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{y}^* - \boldsymbol{y},\quad \boldsymbol{z}(0)=\boldsymbol{0} \tag{30}$$

$$\Sigma_s :\ \dot{\boldsymbol{\xi}} = D\boldsymbol{\xi} + \boldsymbol{z} + (\boldsymbol{y}^*-\boldsymbol{y}),\quad \boldsymbol{\xi}(0)=\boldsymbol{0},\ D<0 \tag{31}$$

$$\boldsymbol{u} = K_P(\boldsymbol{y}^*-\boldsymbol{y}) + K_I\boldsymbol{z} + K_S\boldsymbol{\xi} - K_D\dot{\boldsymbol{y}} \tag{32}$$

where $\boldsymbol{y}^*$ is the desired value of output $\boldsymbol{y}$. We suppose all $K_P\in R^{m\times m},\ K_I\in R^{m\times m},\ K_S\in R^{m\times m},\ K_D\in R^{m\times m}$ be positive definite matrices.

In particular, let us call a control scheme (30)∼(32) the P·I·SPR·D control in regard to position.

[Theorem 3] Assume that system (28),(29) be passive and $\boldsymbol{x}^*$-state detectable. Then the closed-loop system (28)∼(32) of affine nonlinear system with P·I·SPR·D control is asymptotically stable at the equilibrium $(\boldsymbol{x}_e,\boldsymbol{z}_e,\boldsymbol{\xi}_e)=(\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{\xi}^*)$, provided that positive definite matrices $K_P,\ K_I,\ K_S,\ K_D$ and negative definite diagonal D are appropriately chosen. Here $\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{\xi}^*$ are the desired equilibrium state corresponding to the desired output $\boldsymbol{y}^*$ and control $\boldsymbol{u}^*$.

(Proof) At the equilibrium of system (28),(29), the following relation must be satisfied.

$$\begin{aligned}\boldsymbol{0} &= \boldsymbol{f}(\boldsymbol{x}_e) + G(\boldsymbol{x}_e)\overline{\boldsymbol{u}}\\ \boldsymbol{y}^* &= \boldsymbol{h}(\boldsymbol{x}_e)\end{aligned}$$

Thus, in order that $\boldsymbol{y}$ becomes $\boldsymbol{y}^*$, $\boldsymbol{x}_e$ and $\overline{\boldsymbol{u}}$ must be $\boldsymbol{x}^*$ and $\boldsymbol{u}^*$, as mentioned in Section 2.

Meanwhile, since an equilibrium of system (28)∼(31) must satisfy

$$\begin{aligned}\boldsymbol{0} &= \boldsymbol{f}(\boldsymbol{x}_e) + G(\boldsymbol{x}_e)\{K_P(\boldsymbol{y}^*-\boldsymbol{y}_e) + K_I\boldsymbol{z}_e + K_S\boldsymbol{\xi}_e\\ &\quad - K_D\dot{\boldsymbol{y}}_e\}\\ \boldsymbol{0} &= \boldsymbol{y}^* - \boldsymbol{y}_e\\ \boldsymbol{0} &= D\boldsymbol{\xi}_e + \boldsymbol{z}_e + (\boldsymbol{y}^*-\boldsymbol{y}_e)\\ \boldsymbol{y}_e &= \boldsymbol{h}(\boldsymbol{x}_e)\end{aligned}$$

it follows that $\boldsymbol{x}_e = \boldsymbol{x}^*,\ \boldsymbol{y}_e = \boldsymbol{y}^*,\ \boldsymbol{z}_e = \boldsymbol{z}^*,\ \boldsymbol{\xi}_e = \boldsymbol{\xi}^*$, where $\boldsymbol{z}^*$ and $\boldsymbol{\xi}^*$ must satisfy $K_I\boldsymbol{z}^* + K_S\boldsymbol{\xi}^* = \boldsymbol{u}^*,\ \boldsymbol{0} = D\boldsymbol{\xi}^* + \boldsymbol{z}^*$. Hence we have $\boldsymbol{z}^* = -D(-K_ID+K_S)^{-1}\boldsymbol{u}^*$ and $\boldsymbol{\xi}^* = (-K_ID+K_S)^{-1}\boldsymbol{u}^*$.

Now let us consider a Lyapunov function candidate

$$\begin{aligned}&V(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi}) = W(\boldsymbol{x}) - \boldsymbol{y}^{*T}\int_0^t \boldsymbol{u}dt\\ &+\frac{1}{2}(\boldsymbol{y}^*-\boldsymbol{y})^TK_D(\boldsymbol{y}^*-\boldsymbol{y})\\ &+\frac{1}{2}\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}^T\begin{bmatrix}K_I-\overline{K} & \overline{K}\\ \overline{K}^T & K_S-\overline{K}\end{bmatrix}\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}\\ &-\begin{bmatrix}\boldsymbol{z}^*\\ \boldsymbol{\xi}^*\end{bmatrix}^T\begin{bmatrix}\overline{K} & \overline{K}D\\ K_S-\overline{K} & (K_S-\overline{K})D\end{bmatrix}\int_0^t\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}dt\\ &+(\boldsymbol{z}^{*T}K_I+\boldsymbol{\xi}^{*T}K_S)\int_0^t(\boldsymbol{y}^*-\boldsymbol{y})dt\end{aligned} \tag{33}$$

where $W(\boldsymbol{x})$ is a storage function and $K_I-\overline{K}>0,\ K_S-\overline{K}>0$ and $\begin{bmatrix}K_I-\overline{K} & \overline{K}\\ \overline{K}^T & K_S-\overline{K}\end{bmatrix}$ is a positive definite matrix. Thus, we can show that $V(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})$ is a function bounded below, as well as in Theorem 2.

Next calculate its time derivative along (28),(30)∼(32) with the use of K-Y-P property (13),(14) to get

$$\begin{aligned}&\dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\\ &= W_{\boldsymbol{x}}(\boldsymbol{x})\{\boldsymbol{f}(\boldsymbol{x})+G(\boldsymbol{x})\boldsymbol{u}\} - \boldsymbol{y}^{*T}\boldsymbol{u} - (\boldsymbol{y}^*-\boldsymbol{y})^TK_D\dot{\boldsymbol{y}}\\ &+\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}^T\begin{bmatrix}K_I-\overline{K} & \overline{K}\\ \overline{K}^T & K_S-\overline{K}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{z}}\\ \dot{\boldsymbol{\xi}}\end{bmatrix}\\ &-\begin{bmatrix}\boldsymbol{z}^*\\ \boldsymbol{\xi}^*\end{bmatrix}^T\begin{bmatrix}\overline{K} & \overline{K}D\\ K_S-\overline{K} & (K_S-\overline{K})D\end{bmatrix}\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}\\ &+(\boldsymbol{z}^{*T}K_I+\boldsymbol{\xi}^{*T}K_S)(\boldsymbol{y}^*-\boldsymbol{y})\\ &\le \boldsymbol{y}^T\boldsymbol{u} - \boldsymbol{y}^{*T}\boldsymbol{u} - (\boldsymbol{y}^*-\boldsymbol{y})^TK_D\dot{\boldsymbol{y}}\\ &+\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{\xi}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}^T\begin{bmatrix}K_I-\overline{K} & \overline{K}\\ \overline{K}^T & K_S-\overline{K}\end{bmatrix}\begin{bmatrix}\boldsymbol{y}^*-\boldsymbol{y}\\ D\boldsymbol{\xi}+\boldsymbol{z}+(\boldsymbol{y}^*-\boldsymbol{y})\end{bmatrix}\\ &-\begin{bmatrix}\boldsymbol{z}^*\\ \boldsymbol{\xi}^*\end{bmatrix}^T\begin{bmatrix}\overline{K} & \overline{K}D\\ K_S-\overline{K} & (K_S-\overline{K})D\end{bmatrix}\begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}\\ &+(\boldsymbol{z}^{*T}K_I+\boldsymbol{\xi}^{*T}K_S)(\boldsymbol{y}^*-\boldsymbol{y})\\ &= -(\boldsymbol{y}^*-\boldsymbol{y})^T\{K_P(\boldsymbol{y}^*-\boldsymbol{y})+K_I\boldsymbol{z}+K_S\boldsymbol{\xi}-K_D\dot{\boldsymbol{y}}\}\\ &-(\boldsymbol{y}^*-\boldsymbol{y})^TK_D\dot{\boldsymbol{y}} + \begin{bmatrix}(\boldsymbol{z}-\boldsymbol{z}^*)\\(\boldsymbol{\xi}-\boldsymbol{\xi}^*)\end{bmatrix}^T\times\end{aligned}$$

$$
\begin{aligned}
&\begin{bmatrix} (K_I-\overline{K})(\boldsymbol{y}^*-\boldsymbol{y})+\overline{K}D\boldsymbol{\xi}+\overline{K}\boldsymbol{z}+\overline{K}(\boldsymbol{y}^*-\boldsymbol{y}) \\ \overline{K}^T(\boldsymbol{y}^*-\boldsymbol{y})+(K_S-\overline{K})D\boldsymbol{\xi}+(K_S-\overline{K})\boldsymbol{z} \\ \qquad\qquad +(K_S-\overline{K})(\boldsymbol{y}^*-\boldsymbol{y}) \end{bmatrix} \\
&-\begin{bmatrix} \boldsymbol{z}^* \\ \boldsymbol{\xi}^* \end{bmatrix}^T \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S-\overline{K} & (K_S-\overline{K})D \end{bmatrix} \begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix} \\
&+(\boldsymbol{z}^{*T}K_I+\boldsymbol{\xi}^{*T}K_S)(\boldsymbol{y}^*-\boldsymbol{y}) \\
&=-(\boldsymbol{y}^*-\boldsymbol{y})^T\{K_P(\boldsymbol{y}^*-\boldsymbol{y})+K_I\boldsymbol{z}+K_S\boldsymbol{\xi}-K_D\dot{\boldsymbol{y}}\} \\
&-(\boldsymbol{y}^*-\boldsymbol{y})^TK_D\dot{\boldsymbol{y}} \\
&+\begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix}^T \left\{ \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S-\overline{K} & (K_S-\overline{K})D \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\xi} \end{bmatrix} \right. \\
&\left. +\begin{bmatrix} K_I(\boldsymbol{y}^*-\boldsymbol{y}) \\ K_S(\boldsymbol{y}^*-\boldsymbol{y}) \end{bmatrix} \right\} \\
&-\begin{bmatrix} \boldsymbol{z}^* \\ \boldsymbol{\xi}^* \end{bmatrix}^T \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S-\overline{K} & (K_S-\overline{K})D \end{bmatrix} \begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix} \\
&+(\boldsymbol{z}^{*T}K_I+\boldsymbol{\xi}^{*T}K_S)(\boldsymbol{y}^*-\boldsymbol{y}) \\
&=-(\boldsymbol{y}^*-\boldsymbol{y})^T\{K_P(\boldsymbol{y}^*-\boldsymbol{y})+K_I\boldsymbol{z}+K_S\boldsymbol{\xi}-K_D\dot{\boldsymbol{y}}\} \\
&-(\boldsymbol{y}^*-\boldsymbol{y})^TK_D\dot{\boldsymbol{y}} \\
&+\begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix}^T \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S-\overline{K} & (K_S-\overline{K})D \end{bmatrix} \begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix} \\
&+\begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix}^T \begin{bmatrix} K_I(\boldsymbol{y}^*-\boldsymbol{y}) \\ K_S(\boldsymbol{y}^*-\boldsymbol{y}) \end{bmatrix} \\
&+(\boldsymbol{z}^{*T}K_I+\boldsymbol{\xi}^{*T}K_S)(\boldsymbol{y}^*-\boldsymbol{y}) \\
&=-(\boldsymbol{y}^*-\boldsymbol{y})^TK_P(\boldsymbol{y}^*-\boldsymbol{y}) \\
&+\begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix}^T \begin{bmatrix} \overline{K} & \overline{K}D \\ K_S-\overline{K} & (K_S-\overline{K})D \end{bmatrix} \begin{bmatrix} (\boldsymbol{z}-\boldsymbol{z}^*) \\ (\boldsymbol{\xi}-\boldsymbol{\xi}^*) \end{bmatrix} \qquad (34)
\end{aligned}
$$

Here we try to make

$$\begin{bmatrix} \overline{K} & \overline{K}D \\ K_S-\overline{K} & (K_S-\overline{K})D \end{bmatrix}$$

be negative definite. For that purpose, set $\overline{K}<0,\ K_S-\overline{K}=(\overline{K}D)^T$ and $D<-I$ such that we have $K_S=(I+D)\overline{K}>0$. Then the above matrix becomes

$$\begin{bmatrix} \overline{K} & \overline{K}D \\ (\overline{K}D)^T & D\overline{K}D \end{bmatrix}$$

Since the (1,1) element and the (2,2) element are $\overline{K}<0,\ D\overline{K}D<0$, respectively, we can choose $\overline{K}<0$ and $D<0$ such that the above matrix becomes negative definite.

Consequently, $\dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})$ becomes semi-negative definite, and it follows that the P·I·SPR·D control is stable in the sense of Lyapunov, but it is unknown if asymptotically stable.

Let $\Omega_c=\{(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\mid V(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\le c\}$ and suppose Ω_c is bounded and $\dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\le 0$ in Ω_c (c is a positive number such that $\dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\le 0$). Here define Ω_E as a set of all points of Ω_c satisfying $\dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})=0$ and put

$$\Omega_E=\{(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\mid \dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})=0,\ (\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\in\Omega_c\} \qquad (35)$$

From (34) $(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})$ satisfying $\dot{V}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})=0$ is given as $\boldsymbol{y}=\boldsymbol{y}^*,\ \boldsymbol{z}=\boldsymbol{z}^*,\ \boldsymbol{\xi}=\boldsymbol{\xi}^*$. But at that time we have $\boldsymbol{u}=K_Z\boldsymbol{z}^*+K_S\boldsymbol{\xi}^*=\boldsymbol{u}^*$ from (32). Then by $\boldsymbol{x}^*$-state detectability of system (28),(29), we get

$$\Omega_E=\{(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})|\boldsymbol{x}=\boldsymbol{x}^*,\boldsymbol{z}=\boldsymbol{z}^*,\boldsymbol{\xi}=\boldsymbol{\xi}^*,\ (\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})\in\Omega_c\} \qquad (36)$$

Accordingly, we know from (28),(30),(31) that $(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi})$ in Ω_E consists of only the equilibrium point $(\boldsymbol{x}_e,\boldsymbol{z}_e,\boldsymbol{\xi}_e)=(\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{\xi}^*)$ for $\boldsymbol{u}=\boldsymbol{u}^*$. Thus the largest invariance set Ω_M in Ω_E consists of the equilibrium point $(\boldsymbol{x}_e,\boldsymbol{z}_e,\boldsymbol{\xi}_e)=(\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{\xi}^*)$. Therefore, by LaSalle's invariance principle all trajectories in Ω_c converge to Ω_M as $t\to\infty$. Thus, $(\boldsymbol{x}^*,\boldsymbol{z}^*,\boldsymbol{\xi}^*)$ is aymptotically stable. Q.E.D

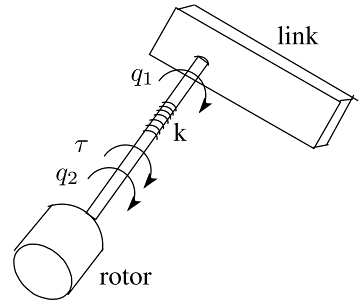

Fig.1 Elastic Joint Robot Arm

VI. SIMULATION

A. *Elastic Joint Robot Arm*

A robot manipulator (1-link manipulator) with an elastic joint, depicted in Fig.1, can be modeled as[2,8].

$$J_1\ddot{q}_1+mgl\sin q_1=k(q_2-q_1) \qquad (37)$$

$$J_2\ddot{q}_2=k(q_2-q_1)+\tau \qquad (38)$$

where q_1 and q_2 are the angles of the link and the rotor schaft and τ is the control torque given by the rotor. J_1, J_2 are the moment of inertia, m, l are the mass and the length of the link, and k is the elasticity coefficient of a spring.

System (19), letting $\boldsymbol{x}=(q_1,\dot{q}_1,q_2,\dot{q}_2)^T$, is expressed by a state equation as follows.

$$\dot{x}_1=x_2 \qquad (39a)$$

$$\dot{x}_2=-\frac{mgl}{J_1}\sin x_1+\frac{k}{J_1}(x_3-x_1) \qquad (39b)$$

$$\dot{x}_3=x_4 \qquad (39c)$$

$$\dot{x}_4=\frac{k}{J_2}(x_1-x_3)+\frac{1}{J_2}\tau \qquad (39d)$$

This system can be shown to be passive with respect to input τ and output $x_4=\dot{q}_2$ [2] (i.e., dissipative with respect to supply rate $\tau\dot{q}_2$.) Hence it is passive with respect to input $\boldsymbol{u}=\begin{bmatrix}0\\ \tau\end{bmatrix}$ and output $\boldsymbol{y}=\begin{bmatrix}x_2\\ x_4\end{bmatrix}$.

Now if the desired equilibrium is $x_{1e}=x_1^*,\ x_{3e}=x_3^*$, it holds from (39b) that $x_{3e}=x_3^*=\frac{mgl}{k}\sin x_1^*+x_1^*$. Namely, x_3^* is a function of x_1^*. And the corresponding τ^* becomes $\tau^*=k(x_3^*-x_1^*)=mgl\sin x_1^*$ from (39d).

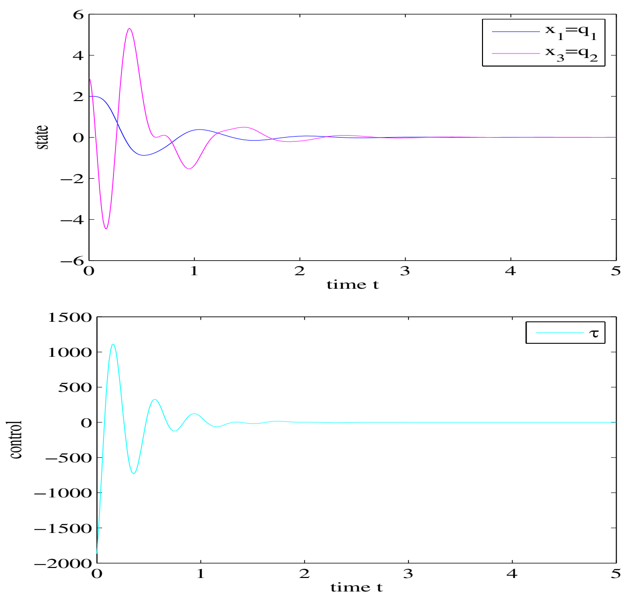

Fig.2 Elastic Joint Robot Arm (Regulation Problem)

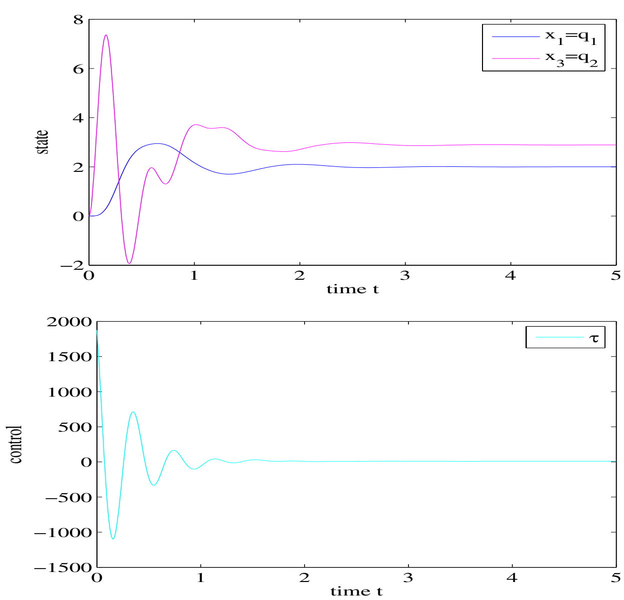

Fig.3 Elastic Joint Robot Arm (Set-Point Servo Problem)

Letting the desired value as $(x_1^*, x_3^*) = (q_1^*, q_3^*)$ and applying Theorem 2, we solve a set-point servo problem.

Since

$$\boldsymbol{y} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}, \boldsymbol{z} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \ \boldsymbol{z}^* = \begin{bmatrix} x_1^* \\ x_3^* \end{bmatrix}$$

in the elastic joint robot arm, the SPR element becomes

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = D \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} x_1^* - x_1 \\ x_3^* - x_3 \end{bmatrix} - \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

Then the control input is given from (23) as

$$\boldsymbol{u} = K_P \begin{bmatrix} x_1^* - x_1 \\ x_3^* - x_3 \end{bmatrix} + K_S \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - K_D \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \tau^* \end{bmatrix}$$

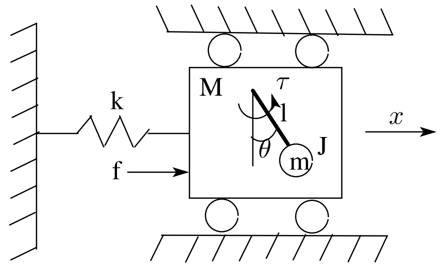

Fig.4 TORA Model

Further, as $\boldsymbol{y} = \boldsymbol{0}$, $\boldsymbol{z} = \boldsymbol{z}^*$, $\boldsymbol{\xi} = \boldsymbol{0}$, we have $\boldsymbol{u} = \boldsymbol{u}^* = (0, \tau^*)^T$ from (23). But, when the desired equilibrium is given as $\boldsymbol{x}_e = \boldsymbol{x}^* = (z_1^*, y_1^*.z_2^*, y_2^*)^T = (z_1^*, 0, z_2^*, 0)^T$, we have $\tau = \tau^* = k(x_3^* - x_1^*)$. Consequently, $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi})$ in Ω_E consists of only the desired equilibrium $(\boldsymbol{x}_e, \boldsymbol{z}_e, \boldsymbol{\xi}_e) = (\boldsymbol{x}^*, \boldsymbol{z}^*, \boldsymbol{0})$ for $\boldsymbol{u} = \boldsymbol{u}^* = (0, \tau^*)^T$. Therefore, one can apply LaSalle's invariance principle without the assumption of $\boldsymbol{x}^*$-state detectability and Theorem 2 holds.

Under the above preperation, we take $J_1 = J_2 = m = l = 1$, $k = 10$, $g = 9.804$, and set controller parameters as

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ K_P = \begin{bmatrix} * & * \\ 500 & 300 \end{bmatrix},$$
$$K_S = \begin{bmatrix} * & * \\ 1 & 1 \end{bmatrix}, \ \ K_D = \begin{bmatrix} * & * \\ 100 & 10 \end{bmatrix}$$

The simulation results of regulation problem for initial condition $\boldsymbol{x}(0) = (2, 0, 2.892, 0)$ is shown in Fig.2. And the simulation results of a set-point servo problem for $(q_1^*, q_2^*) = (2, 2.892)$ and $\boldsymbol{x}(0) = \boldsymbol{0}$ is shown in Fig.3. Very good performance is obtained in both cases.

B. TORA Model

Let us consider TORA model[10] shown in Fig.4, which is well known as a bench mark problem for nonlinear control.

In the figure, x denotes the transitional position, θ the angular position of the proof mass, and τ denotes the control torque applied to the proof mass.

By letting the state vector $\boldsymbol{x} \triangleq (x, \dot{x}, \theta, \dot{\theta})^T$, TORA model is represented with a state equation

$$\dot{x}_1 = x_2 \qquad (40a)$$
$$\dot{x}_2 = \frac{-x_1 + \epsilon x_4^2 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} - \frac{\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \tau \qquad (40b)$$
$$\dot{x}_3 = x_4 \qquad (40c)$$
$$\dot{x}_4 = \frac{\epsilon \cos^2 x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1 - \epsilon^2 \cos^2 x_3} + \frac{1}{1 - \epsilon^2 \cos^2 x_3} \tau \qquad (40d)$$

This system is clearly underactuated, and passive with respect to input τ and output $x_4 = \dot{\theta}$ (namely, dissipative with respect to suply rate $\tau\dot{\theta}$) (see Ref. [8]). Thus, it is passive with respect to input $\boldsymbol{u} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$ and output $\boldsymbol{y} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$.

Now letting the desired value as $\begin{bmatrix} x_1^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} x^* \\ \theta^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and applying Theorem 2, let us solve a set-point servo

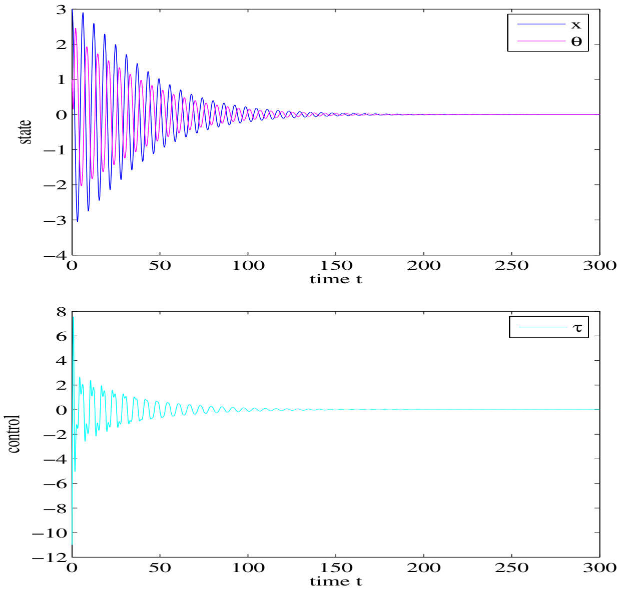

Fig.5 TORA Model

problem. Since

$$\boldsymbol{y} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}, \; \boldsymbol{z} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \; \boldsymbol{z}^* = \begin{bmatrix} x_1^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for the TORA model, the SPR element (22) becomes

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = D \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} - \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

and the control input is given from (23) as

$$\boldsymbol{u} = -K_P \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + K_S \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - K_D \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \tau^* \end{bmatrix}$$

Further, as $\boldsymbol{y} = \boldsymbol{0}$, $\boldsymbol{z} = \boldsymbol{z}^*$, $\boldsymbol{\xi} = \boldsymbol{0}$, we have $\boldsymbol{u} = \boldsymbol{u}^* = (0, \tau^*)^T$ from (23).

For the TORA model, however, when the desired value is given as $(x_1^*, x_3^*) = (x^*, \theta^*) = (0, 0)$, we have $\boldsymbol{x}_e = \boldsymbol{x}^* = \boldsymbol{0}$ and $\tau^* = 0$ at the equilibrium. Consequently, $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi})$ in Ω_E consists of only the desired equilibrium $(\boldsymbol{x}_e, \boldsymbol{z}_e, \boldsymbol{\xi}_e) = (\boldsymbol{x}^*, \boldsymbol{z}^*, \boldsymbol{0}) = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$. Therefore , we can apply LaSalle's invariance principle without the assumption of zero state detectability and Theorem 2 holds.

Under the above preparetion, we set $\epsilon = 0.1$ and contoller parameters as

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad K_P = \begin{bmatrix} * & * \\ 1 & 8 \end{bmatrix},$$

$$K_S = \begin{bmatrix} * & * \\ 2 & 2 \end{bmatrix}, \quad K_D = \begin{bmatrix} * & * \\ 6 & 2 \end{bmatrix},$$

Fig.5 shows the simulation results with initial condition $(x(0), \dot{x}(0), \theta(0), \dot{\theta}(0)) = (3, 0, 1, 0)$. The simulation results showed always accurate convergence to the origin, starting from a faraway initial state, oscillating though.

VII. Conclusion

Based on the passivity theory and LaSalle's invariance principle, we first studied the regulation problem for the affine nonlinear system by the P·SPR·D control. Next we investigated the set-point servo problem by the P·SPR·D control and P·I·SPR·D control.

The P·SPR·D and the P·I·SPR·D control are new general control schemes of output feedback and the use of SPR element as a part of the controller possesses an advantage from a passivity-based design point of view.

It was confirmed that stability of the closed-loop system and the convergence speed could be improved by the SPR element considerably. Besides, for the design of P·SPR·D control, the storage function $W(\boldsymbol{x})$ is not necessary to be known explicitly, although various existing design methods [5,6,9,10,13] require $W(\boldsymbol{x})$ concretely.

Implementation of a controller with the SPR element is not difficult by a digital processor.

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