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Dynamics of Connected Rigid Bodies in a Perfect Fluid

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Abstract— **This paper presents an analytical model and a geometric numerical integrator for a system of rigid bodies connected by ball joints, immersed in an irrotational and incompressible fluid. The rigid bodies can translate and rotate in three-dimensional space, and each joint has three rotational degrees of freedom. This model characterizes the qualitative behavior of three-dimensional fish locomotion. A geometric numerical integrator, refereed to as a Lie group variational integrator, preserves the Hamiltonian flow structure and the Lie group configuration manifold. These properties are illustrated by a numerical simulation for a system of three connected rigid bodies.**

I. Introduction

Fish locomotion has been investigated in the fields of biomechanics and engineering [1]. This is a challenging problem as it involves interaction of a deformable fish body with an unsteady fluid, through which an internal muscular force of the fish is translated into an external propulsive force exerted on the fluid.

Various mathematical models of fish locomotion have been formulated. A quasi-static model based on a steady state flow theory is developed in [2], and an elastic plate model that treats a fish as an elongated slender body is studied in [3]. The effects of body thickness for the slender body model are considered in [4]. Numerical models involving computational fluid dynamics techniques appear in [5]. The body of a fish is modeled as a planar articulated rigid body in [6], [7].

The planar articulated rigid body model has become popular in engineering, as it depicts underwater robotic vehicles that move and steer by changing their shape [8]. Furthermore, if it is assumed that the ambient fluid is incompressible and irrotational, then equations of motion of the articulated rigid body can be derived without explicitly incorporating fluid variables [6]. The effect of the fluid is accounted by added inertia terms of the rigid body. This model is known to characterize the qualitative behavior of fish swimming [6]. Based on this assumption, optimal shape change of a planar articulated body to achieve a desired locomotion has been studied in [9], [10].

By following [6], [7], we develop an analytical model of connected rigid bodies immersed in an incompressible and irrotational fluid. The contribution of this paper is that the connected rigid bodies can freely translate and rotate in three-dimensional space, and each joint has three rotational degrees of freedom. This is important for understanding the locomotion of a fish with a blunt body and a large caudal fin.

The second part of this paper develops a geometric numerical integrator for connected rigid bodies in a perfect fluid. Geometric numerical integration is concerned with numerical integrators that preserve geometric features of a system, such as invariants, symmetry, and reversibility [11]. It is critical for numerical simulation of Hamiltonian systems that evolve on a Lie group to preserve both the symplectic property of Hamiltonian flows and the Lie group structure [12]. A geometric numerical integrator, referred to as a Lie group variational integrator, has been developed for Hamiltonian systems that evolve on an arbitrary Lie group in [13].

A system of connected rigid bodies is a Hamiltonian system, and its configuration manifold is expressed as a product of the special Euclidean group and copies of the special orthogonal group. This paper develops a Lie group variational integrator for connected rigid bodies in a perfect fluid based on the results presented in [13]. The proposed geometric numerical integrator preserves symplecticity and momentum maps, and exhibits desirable energy properties. It also respects the Lie group structure of the configuration manifold, and avoids singularities and complexities associated with local coordinates.

In summary, this paper develops an analytical model and a geometric numerical integrator for a system of connected rigid bodies in a perfect fluid. These provide a three-dimensional mathematical model and a reliable numerical simulation tool that characterizes the qualitative properties of fish locomotion.

II. Connected Rigid Bodies Immersed in a Perfect Fluid

Consider three connected rigid bodies immersed in a perfect fluid. We assume that these rigid bodies are connected by a ball joint that has three rotational degrees of freedom, and the fluid is incompressible and irrotational. We also assume each body has neutral buoyancy.

We choose a reference frame and three body-fixed frames. The origin of each body-fixed frame is located at the mass center of the rigid body. Define

$R_i \in \mathsf{SO}(3)$	Rotation matrix from the i-th body-fixed frame to the reference frame
$\Omega_i \in \mathbb{R}^3$	Angular velocity of the i-th body, represented in the i-th body-fixed frame

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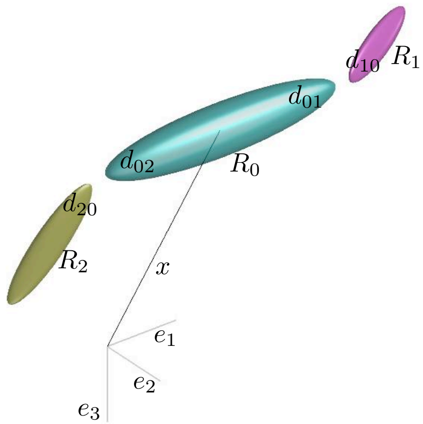


Fig. 1. Connected Rigid Bodies Immersed in a Perfect Fluid

$x \in \mathbb{R}^3$	Vector from the origin of the reference frame to the mass center of the 0-th body, represented in the reference frame
$d_{ij} \in \mathbb{R}^3$	Vector from the mass center of the i-th body to the ball joint connecting the i-th body with the j-th body, represented in the i-th body-fixed frame
$m_i^b \in \mathbb{R}$	Mass of the i-th body
$J_i^b \in \mathbb{R}^{3\times 3}$	Inertia matrix of the i-th body

for $i, j \in \{0, 1, 2\}$.

A configuration of this system can be described by the location of the mass center of the central body, and the attitude of each rigid body with respect to the reference frame. The configuration manifold is $\mathsf{G} = \mathsf{SE}(3) \times \mathsf{SO}(3) \times \mathsf{SO}(3)$, where $\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, R^T R = I, \det R = 1\}$, and $\mathsf{SE}(3) = \mathsf{SO}(3) \circledS \mathbb{R}^3$.

The attitude kinematic equation is given by

$$\dot{R}_i = R_i \hat{\Omega}_i$$

for $i \in \{0, 1, 2\}$, where the *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by the condition that $\hat{x} y = x \times y$ for any $x, y \in \mathbb{R}^3$.

III. Continuous-time Analytical Model

In this section, we develop continuous-time equations of motion. As the fluid is irrotational, they can be expressed without explicitly incorporating fluid variables, and the effects of the ambient fluid is accounted for by added inertia terms [6]. To simplify expressions for the added inertia terms, we assume each body is an ellipsoid.

A. Lagrangian

Kinetic energy of rigid bodies: Let $V_i \in \mathbb{R}^3$ be the velocity of the mass center of the i-th body represented in the i-th body-fixed frame for $i \in \{0, 1, 2\}$. Since $\dot{x}$ represents the velocity of the 0-th rigid body in the reference frame,

$$V_0 = R_0^T \dot{x}. \tag{1}$$

The location of the mass center of the i-th rigid body can be written as $x + R_0 d_{0i} - R_i d_{i0}$ for $i \in \{1, 2\}$ with respect to the reference frame. Therefore, V_i is given by

$$V_i = R_i^T \dot{x} - R_i^T R_0 \hat{d}_{0i} \Omega_0 + \hat{d}_{i0} \Omega_i. \tag{2}$$

The kinetic energy of the rigid bodies is given by

$$T_{\mathcal{B}} = \sum_{i=0}^{2} \frac{1}{2} m_i^b V_i \cdot V_i + \frac{1}{2} \Omega_i \cdot J_i^b \Omega_i. \tag{3}$$

Kinetic energy of fluid: The kinetic energy of the fluid is given by $T_{\mathcal{F}} = \frac{1}{2} \int_{\mathcal{F}} \rho_f \|u\|^2 \, dv$, where ρ_f is the density of the fluid, u is the velocity field of the fluid and dv is the standard volume element in $\mathbb{R}^3$. We assume the fluid is irrotational and the rigid bodies are ellipsoidal. Under these conditions, the kinetic energy of the fluid surrounding ellipsoidal rigid bodies is given by

$$T_{\mathcal{F}} = \sum_{i=0}^{2} \frac{1}{2} M_i^f V_i \cdot V_i + \frac{1}{2} \Omega_i \cdot J_i^f \Omega_i, \tag{5}$$

where $M_i^f, J_i^f \in \mathbb{R}^{3\times 3}$ are referred to as *added inertia matrices* [14], [15]. The resulting model captures the qualitative properties of the interaction between rigid body dynamics and fluid dynamics correctly [6], [9].

Total kinetic energy: Define total inertia matrices as $M_i = m_i^b I_{3\times 3} + M_i^f$, $J_i = J_i^b + J_i^f$ for $i = \{0, 1, 2\}$. From (3) and (5), the total kinetic energy is given by

$$T = \sum_{i=0}^{2} \frac{1}{2} M_i V_i \cdot V_i + \frac{1}{2} \Omega_i \cdot J_i \Omega_i. \tag{6}$$

Substituting (1)-(2), this can be written as

$$T = \frac{1}{2} \xi^T \mathbb{I}(R_0, R_1, R_2) \xi, \tag{7}$$

where $\xi = [\Omega_0; \dot{x}; \Omega_1; \Omega_2] \in \mathbb{R}^{12}$ and the matrix $\mathbb{I}(R_0, R_1, R_2) \in \mathbb{R}^{12\times 12}$ is given by (4). Since there is no potential field, this is equal to the Lagrangian of the connected rigid bodies immersed in a perfect fluid.

B. Euler-Lagrange Equations

Euler-Lagrange equations for a mechanical system that evolves on an arbitrary Lie group are given by

$$\frac{d}{dt} \mathbf{D}_\xi L(g, \xi) - \mathrm{ad}_\xi^* \cdot \mathbf{D}_\xi L(g, \xi) - \mathsf{T}_e^* \mathsf{L}_g \cdot \mathbf{D}_g L(g, \xi) = 0, \tag{8}$$

$$\dot{g} = g\xi, \tag{9}$$

where $L : \mathsf{TG} \simeq \mathsf{G} \times \mathfrak{g} \to \mathbb{R}$ is the Lagrangian of the system [13]. Here $\mathbf{D}_\xi L(g, \xi) \in \mathfrak{g}^*$ denotes the derivative of the Lagrangian with respect to $\xi \in \mathfrak{g}$, $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the co-adjoint operator, and $\mathsf{T}_e^* \mathsf{L}_g : \mathsf{T}^*\mathsf{G} \to \mathfrak{g}^*$ denotes the cotangent lift of the left translation map $\mathsf{L}_g : \mathsf{G} \to \mathsf{G}$ [16].

Using this result, we develop Euler-Lagrange equations of a system of connected rigid bodies in a perfect fluid. To simplify the derivation, we consider the configuration manifold given by $\mathsf{G} = \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathsf{SO}(3) \times \mathsf{SO}(3)$, left-trivialize TG to yield $\mathsf{G} \times \mathfrak{g}$, and identify its Lie algebra $\mathfrak{g}$ with $\mathbb{R}^{12}$ by the hat map. For $\xi = [\Omega_0; \dot{x}; \Omega_1; \Omega_2] \in \mathfrak{g}$ and $p = [p_0; p_x; p_1; p_2] \in \mathfrak{g}^*$, the co-adjoint operator is given by $\mathrm{ad}_\xi^* p = [-\hat{\Omega}_0 p_0; p_x; -\hat{\Omega}_1 p_1; -\hat{\Omega}_2 p_2]$.

$$\mathbb{I}=\begin{bmatrix} J_0-\hat{d}_{01}R_0^TR_1M_1R_1^TR_0\hat{d}_{01}-\hat{d}_{02}R_0^TR_2M_2R_2^TR_0\hat{d}_{02} & \hat{d}_{01}R_0^TR_1M_1R_1^T+\hat{d}_{02}R_0^TR_2M_2R_2^T & \hat{d}_{01}R_0^TR_1M_1\hat{d}_{10} & \hat{d}_{02}R_0^TR_2M_2\hat{d}_{20}\\ -R_1M_1R_1^TR_0\hat{d}_{01}-R_2M_2R_2^TR_0\hat{d}_{02} & R_0M_0R_0^T+R_1M_1R_1^T+R_2M_2R_2^T & R_1M_1\hat{d}_{10} & R_2M_2\hat{d}_{20}\\ \hat{d}_{10}M_1R_1^TR_0\hat{d}_{01} & -\hat{d}_{10}M_1R_1^T & J_1-\hat{d}_{10}M_1\hat{d}_{10} & 0\\ \hat{d}_{20}M_2R_2^TR_0\hat{d}_{02} & -\hat{d}_{20}M_2R_2^T & 0 & J_2-\hat{d}_{20}M_2\hat{d}_{20}\end{bmatrix} \tag{4}$$

Derivatives of the Lagrangian: The derivative of the Lagrangian with respect to ξ is given by

$$\mathbf{D}_\xi L(g,\xi)=\mathbb{I}(R_0,R_1,R_2)\xi. \tag{10}$$

The derivative of the Lagrangian with respect to R_0 can be found as follows. For any $\eta_0\in\mathbb{R}^3$, let $g_0^\epsilon=[R_0\exp\epsilon\eta_0,x,R_1,R_2]\in\mathsf{G}$. Then, we have

$$\begin{aligned}(\mathsf{T}_I^*\mathsf{L}_{R_0}\cdot\mathbf{D}_{R_0}L)\cdot\eta_0&=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}L(g_0^\epsilon,\xi)\\ &=-\dot{x}^TR_0M_0\hat{\eta}_0R_0^T\dot{x}+\sum_{i=1}^2\Big(-\Omega_0^T\hat{d}_{0i}R_0^TR_iM_iR_i^TR_0\hat{\eta}_0\hat{d}_{0i}\Omega_0\\ &\quad-\dot{x}^TR_iM_iR_i^TR_0\hat{\eta}_0\hat{d}_{0i}\Omega_0-\Omega_0^T\hat{d}_{0i}\hat{\eta}_0R_0^TR_iM_i\hat{d}_{i0}\Omega_i\Big)\\ &=\left(-\widehat{R_0^T\dot{x}}M_0R_0^T\dot{x}-\sum_{i=1}^2\widehat{\hat{d}_{0i}\Omega_0}R_0^TR_iM_iV_i\right)\cdot\eta_0,\end{aligned}$$

where we use identities: $x\cdot y=x^Ty=y^Tx$, $\hat{x}y=-\hat{y}x$ for any $x,y\in\mathbb{R}^3$. Since this is satisfied for any $\eta_0\in\mathbb{R}^3$, we obtain

$$\mathsf{T}_I^*\mathsf{L}_{R_0}\cdot\mathbf{D}_{R_0}L=-\widehat{R_0^T\dot{x}}M_0R_0^T\dot{x}-\sum_{i=1}^2\widehat{\hat{d}_{0i}\Omega_0}R_0^TR_iM_iV_i. \tag{11}$$

Similarly, we find

$$\mathbf{D}_xL=0, \tag{12}$$

$$\mathsf{T}_I^*\mathsf{L}_{R_i}\cdot\mathbf{D}_{R_i}L=\widehat{M_iV_i}R_i^T(\dot{x}-R_0\hat{d}_{0i}\Omega_0) \tag{13}$$

for $i\in\{1,2\}$.

Euler-Lagrange Equations: Substituting (10)-(13) into (8)-(9), and rearranging, the Euler-Lagrange equations for the connected rigid bodies immersed in a perfect fluid are given by

$$\mathbb{I}(R_0,R_1,R_2)\begin{bmatrix}\dot{\Omega}_0\\ \ddot{x}\\ \dot{\Omega}_1\\ \dot{\Omega}_2\end{bmatrix}+\begin{bmatrix}\Omega_0\times J\Omega_0+\widehat{R_0^T\dot{x}}M_0R_0^T\dot{x}+\sum_{i=1}^2\hat{d}_{0i}R_0^TR_iW_i\\ R_0(\hat{\Omega}_0M_0-M_0\hat{\Omega}_0)R_0^T\dot{x}+\sum_{i=1}^2R_iW_i\\ \Omega_1\times J_1\Omega_1+V_1\times M_1V_1-\hat{d}_{10}W_1\\ \Omega_2\times J_2\Omega_2+V_2\times M_2V_2-\hat{d}_{20}W_2\end{bmatrix}=0, \tag{14}$$

$$\dot{R}_0=R\hat{\Omega}_1,\quad \dot{R}_1=R_1\hat{\Omega}_1,\quad \dot{R}_2=R_2\hat{\Omega}_2, \tag{15}$$

where

$$V_i=R_i^T\dot{x}-R_i^TR_0\hat{d}_{0i}\Omega_0+\hat{d}_{i0}\Omega_i, \tag{16}$$

$$\begin{aligned}W_i&=(\hat{\Omega}_iM_i-M_i\hat{\Omega}_i)(R_i^T\dot{x}-R_i^TR_0\hat{d}_{0i}\Omega_0)\\ &\quad-M_iR_i^TR_0\hat{\Omega}_0\hat{d}_{0i}\Omega_0+\hat{\Omega}_iM_i\hat{d}_{i0}\Omega_i\end{aligned} \tag{17}$$

for $i\in\{1,2\}$.

Hamilton's equations: Let the momentum of the system be $\mu=[p_0;p_x;p_1;p_2]\in\mathbb{R}^{12}\simeq\mathfrak{g}^*$. The Legendre transformation is given by $\mu=\mathbf{D}_\xi L(g,\xi)=\mathbb{I}(R_0,R_1,R_2)\xi$. The corresponding Hamilton's equations can be written as

$$\dot{p}_0=-\hat{\Omega}_0p_0-\widehat{R_0^T\dot{x}}M_0R_0^T\dot{x}-\sum_{i=1}^2\widehat{\hat{d}_{0i}\Omega_0}R_0^TR_iM_iV_i, \tag{18}$$

$$\dot{p}_x=0, \tag{19}$$

$$\dot{p}_i=-\hat{\Omega}_ip_i+\widehat{M_iV_i}R_i^T(\dot{x}-R_0\hat{d}_{0i}\Omega_0) \tag{20}$$

for $i\in\{1,2\}$.

Conserved quantities: As the Lagrangian is invariant under rigid translation and rotation of the entire system, the total linear momentum $p_x\in\mathbb{R}^3$ and the total angular momentum $p_\Omega=\hat{x}p_x+\sum_{i=0}^2R_ip_i\in\mathbb{R}^3$ are preserved.

IV. Lie Group Variational Integrator

The continuous-time Euler-Lagrange equations and Hamilton's equations developed in the previous section provide analytical models of the connected rigid bodies in a perfect fluid. However, they are not suitable for a numerical study since a direct numerical integration of those equations using a general purpose numerical integrator, such as an explicit Runge Kutta method, may not preserve the geometric properties of the system accurately [11].

Variational integrators provide a systematic method of developing geometric numerical integrators for Lagrangian/Hamiltonian systems [17]. As it is derived from a discrete analogue of Hamilton's principle, it preserves symplecticity and the momentum map, and it exhibits good total energy behavior. Lie group methods conserve the structure of a Lie group configuration manifold as it updates a group element using the group operation [18].

These two methods have been unified to obtain a Lie group variational integrator for Lagrangian/Hamiltonian systems evolving on a Lie group [13]. This preserves symplecticity and group structure of those systems concurrently. It has been shown that these properties are critical for accurate and efficient simulations of rigid body dynamics [12]. In this section, we develop a Lie group variational integrator for the connected rigid bodies in a perfect fluid.

A. Discrete Lagrangian

Let $h>0$ be a fixed integration step size, and let a subscript k denote the value of a variable at the k-th time step. We define a discrete-time kinematic equation as follows. Define $f_k = (F_{0_k}, \Delta x_k, F_{1_k}, F_{2_k}) \in \mathsf{G}$ for $\Delta x_k \in \mathbb{R}^3$, $F_{0_k}, F_{1_k}, F_{2_k} \in \mathsf{SO}(3)$ such that $g_{k+1} = g_k f_k$:

$$\begin{aligned}(R_{0_{k+1}}, x_{k+1},\, R_{1_{k+1}},\, R_{2_{k+1}})\\ = (R_{0_k}F_{0_k},\, x_k + \Delta x_k,\, R_{1_k}F_{1_k},\, R_{2_k}F_{2_k}).\end{aligned} \tag{21}$$

Therefore, f_k represents the relative update between two integration steps. This ensures that the structure of the Lie group configuration manifold is numerically preserved.

A discrete Lagrangian $L_d(g_k, f_k) : \mathsf{G} \times \mathsf{G} \to \mathbb{R}$ is an approximation of the Jacobi solution of the Hamilton–Jacobi equation, which is given by the integral of the Lagrangian along the exact solution of the Euler-Lagrange equations over a single time step:

$$L_d(g_k, f_k) \approx \int_0^h L(\tilde{g}(t), \tilde{g}^{-1}(t)\dot{\tilde{g}}(t))\, dt,$$

where $\tilde{g}(t) : [0, h] \to \mathsf{G}$ satisfies Euler-Lagrange equations with boundary conditions $\tilde{g}(0) = g_k$, $\tilde{g}(h) = g_k f_k$. The resulting discrete-time Lagrangian system, referred to as a variational integrator, approximates the Euler-Lagrange equations to the same order of accuracy as the discrete Lagrangian approximates the Jacobi solution.

The kinetic energy given by (7) can be rewritten as

$$\begin{aligned}T &= \frac{1}{2}\dot{x}^T R_0 M_0 R_0^T \dot{x} + \frac{1}{2}\Omega_0^T J_0 \Omega_0\\
&+ \sum_{i=1}^{2}\Big(\frac{1}{2}\dot{x}^T R_i M_i R_i^T + \frac{1}{2}\Omega_i^T (J_i - \hat{d}_{i0} M_i \hat{d}_{i0})\Omega_i\\
&+ \frac{1}{2} d_{0i}^T \dot{R}_0^T R_i M_i R_i^T \dot{R}_0 d_{0i} + \dot{x}^T R_i M_i R_i^T \dot{R}_0 d_{0i}\\
&- \dot{x}^T R_i M_i \hat{\Omega}_i d_{i0} - d_{0i}^T \dot{R}_0^T R_i M_i \hat{\Omega}_i d_{i0}\Big).\end{aligned}$$

From this, we choose the discrete Lagrangian according to the trapezoidal rule as

$$\begin{aligned}L_{d_k} &= \frac{1}{2h}\Delta x_k^T R_{0_k} M_0 R_{0_k}^T \Delta x_k + \frac{1}{h}\mathrm{tr}[(I - F_{0_k})J_{d_0}]\\
&+ \sum_{i=1}^{2}\Big(\frac{1}{2h}\Delta x_k^T R_{i_k} M_i R_{i_k}^T \Delta x_k + \frac{1}{h}\mathrm{tr}\big[(I - F_{i_k})J'_{d_i}\big]\\
&+ \frac{1}{2h} d_{0i}^T (F_{0_k}^T - I) R_{0_k}^T R_{i_k} M_i R_{i_k}^T R_{0_k} (F_{0_k} - I) d_{0i}\\
&+ \frac{1}{h}\Delta x_k^T R_{i_k} M_i R_{i_k}^T R_{0_k}(F_{0_k} - I) d_{0i}\\
&- \frac{1}{h}\Delta x_k^T R_{i_k} M_i (F_{i_k} - I) d_{i0}\\
&- \frac{1}{h} d_{0i}^T (F_{0_k}^T - I) R_{0_k}^T R_{i_k} M_i (F_{i_k} - I) d_{i0}\Big),\end{aligned} \tag{22}$$

where nonstandard inertia matrices are defined as

$$J_{d_0} = \frac{1}{2}\mathrm{tr}[J_0]\, I - J_0, \tag{23}$$

$$J'_{d_i} = \frac{1}{2}\mathrm{tr}[J'_i]\, I - J'_i, \quad J'_i = J_i - \hat{d}_{i0} M_i \hat{d}_{i0}, \tag{24}$$

for $i \in \{1, 2\}$.

B. Discrete-time Euler-Lagrange Equations

For a discrete Lagrangian on $\mathsf{G}\times\mathsf{G}$, the following discrete-time Euler-Lagrange equations, referred to as a Lie group variational integrator, were developed in [13].

$$\begin{aligned}\mathsf{T}_e^*\mathsf{L}_{f_k} \cdot \mathbf{D}_{f_k} L_{d_k} - \mathrm{Ad}^*_{f_{k+1}^{-1}} \cdot (\mathsf{T}_e^*\mathsf{L}_{f_{k+1}} \cdot \mathbf{D}_{f_{k+1}} L_{d_{k+1}})\\
+ \mathsf{T}_e^*\mathsf{L}_{g_{k+1}} \cdot \mathbf{D}_{g_{k+1}} L_{d_{k+1}} = 0,\end{aligned} \tag{25}$$

$$g_{k+1} = g_k f_k, \tag{26}$$

where $\mathrm{Ad}^* : \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the co-Adjoint operator [16].

Using this result, we develop a Lie group variational integrator for connected rigid bodies in a perfect fluid. For $f = (F_0, \Delta x, F_1, F_2) \in \mathsf{G}$ and $p = [p_0; p_x; p_1; p_2] \in \mathfrak{g}^* \simeq \mathbb{R}^{12}$, the co-Adjoint operator is given by $\mathrm{Ad}^*_{f^{-1}} p = [F_0 p_0; p_x; F_1 p_1; F_2 p_2] = [(F_0 \hat{p}_0 F_0^T)^\vee; p_x; (F_1 \hat{p}_1 F_1^T)^\vee; (F_2 \hat{p}_2 F_2^T)^\vee]$, where the *vee map* $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ denotes the inverse of the hat map.

Derivatives of the discrete Lagrangian: We find expressions for the derivatives of the discrete Lagrangian. The derivative of the discrete Lagrangian with respect to F_{0_k} is given by

$$\mathbf{D}_{F_{0_k}} L_{d_k} \cdot \delta F_{0_k} = \frac{1}{h}\mathrm{tr}[-\delta F_{0_k} J_{d_0}] + \frac{1}{h}\sum_{i=1}^{2} A_{i_k}^T R_{0_k} \delta F_{0_k} d_{0i},$$

where we define, for $i \in \{1, 2\}$,

$$A_{i_k} = R_{i_k} M_i \left(R_{i_k}^T B_{i_k} - (F_{i_k} - I) d_{i0}\right), \tag{27}$$

$$B_{i_k} = \Delta x_k + R_{0_k}(F_{0_k} - I) d_{0i}. \tag{28}$$

The variation of F_{0_k} can be written as $\delta F_{0_k} = F_{0_k}\hat{\zeta}_{0_k}$ for $\zeta_{0_k} \in \mathbb{R}^3$. Therefore, we have

$$\begin{aligned}\mathbf{D}_{F_{0_k}} L_{d_k} \cdot (F_{0_k}\hat{\zeta}_{0_k}) &= (\mathsf{T}_I^*\mathsf{L}_{F_{0_k}} \cdot \mathbf{D}_{F_{0_k}} L_{d_k}) \cdot \zeta_{0_k}\\
&= \frac{1}{h}\mathrm{tr}\Big[-F_{0_k}\hat{\zeta}_{0_k} J_{d_0}\Big] + \frac{1}{h}\sum_{i=1}^{2} A_{i_k}^T R_{0_{k+1}} \hat{\zeta}_{0_k} d_{0i}.\end{aligned}$$

By repeatedly applying a property of the trace operator, $\mathrm{tr}[AB] = \mathrm{tr}[BA] = \mathrm{tr}[A^T B^T]$ for any $A, B \in \mathbb{R}^{3\times3}$, the first term can be written as $\mathrm{tr}\{-F_{0_k}\hat{\zeta}_{0_k} J_{d_0}\} = \mathrm{tr}\{-\hat{\zeta}_{0_k} J_{d_0} F_{0_k}\} = \mathrm{tr}\{\hat{\zeta}_{0_k} F_{0_k}^T J_{d_0}\} = -\frac{1}{2}\mathrm{tr}\{\hat{\zeta}_{0_k}(J_{d_0} F_{0_k} - F_{0_k}^T J_{d_0})\}$. Using a property of the hat map, $x^T y = -\frac{1}{2}\mathrm{tr}[\hat{x}\hat{y}]$ for any $x, y \in \mathbb{R}^3$, this can be further written as $((J_{d_0} F_{0_k} - F_{0_k}^T J_{d_0})^\vee) \cdot \zeta_{0_k}$. As $\hat{x} y = -\hat{y} x$ for any $x, y \in \mathbb{R}^3$, the second term can be written as $A_{i_k}^T R_{0_{k+1}} \hat{\zeta}_{0_k} d_{0i} = -A_{i_k}^T R_{0_{k+1}} \hat{d}_{0i} \zeta_{0_k} = (\hat{d}_{0i} R_{0_{k+1}}^T A_{i_k}) \cdot \zeta_{0_k}$. Using these, we obtain

$$\begin{aligned}&\mathsf{T}_I^*\mathsf{L}_{F_{0_k}} \cdot \mathbf{D}_{F_{0_k}} L_{d_k}\\
&\quad = \frac{1}{h}(J_{d_0} F_{0_k} - F_{0_k}^T J_{d_0})^\vee + \frac{1}{h}\sum_{i=1^2} \hat{d}_{0i} R_{0_{k+1}}^T A_{i_k}.\end{aligned} \tag{29}$$

Similarly, we can derive the derivatives of the discrete Lagrangian as follows.

$$\begin{aligned}&\mathsf{T}_I^*\mathsf{L}_{F_{i_k}} \cdot \mathbf{D}_{F_{i_k}} L_{d_k}\\
&\quad = \frac{1}{h}(J'_{i_d} F_{i_k} - F_{i_k}^T J'_{i_d})^\vee - \frac{1}{h}\hat{d}_{i0} F_{i_k}^T M_i R_{i_k}^T B_{i_k},\end{aligned} \tag{30}$$

$$\mathbf{D}_{\Delta x_k} L_{d_k} = \frac{1}{h} R_{0_k} M_0 R_{0_k}^T \Delta x_k + \frac{1}{h} A_{1_k} + \frac{1}{h} A_{2_k}, \quad (31)$$

$$\mathsf{T}_I^* \mathsf{L}_{R_{0_k}} \cdot \mathbf{D}_{R_{0_k}} L_{d_k} = \frac{1}{h} (M_0 R_{0_k}^T \Delta x_k)^\wedge R_{0_k}^T \Delta x_k + \frac{1}{h} \sum_{i=1}^{2} ((F_{0_k} - I) d_{0i})^\wedge R_{0_k}^T A_{i_k}, \quad (32)$$

$$\mathsf{T}_I^* \mathsf{L}_{R_{i_k}} \cdot \mathbf{D}_{R_{i_k}} L_{d_k} = \frac{1}{h} R_{i_k}^T \hat{A}_{i_k} B_{i_k}. \quad (33)$$

Discrete-time Euler-Lagrange Equations: Substituting (29)–(33) into (25)-(26), and rearranging, discrete-time Euler-Lagrange equations for the connected rigid bodies immersed in a perfect fluid are given by

$$\begin{aligned} &(J_{0_d} F_{0_k} - F_{0_k}^T J_{0_d})^\vee - (F_{0_{k+1}} J_{0_d} - J_{0_d} F_{0_{k+1}}^T)^\vee \\ &\quad + (M_0 R_{0_{k+1}}^T \Delta x_{k+1})^\wedge R_{0_{k+1}}^T \Delta x_{k+1} \\ &\quad + \sum_{i=1}^{2} \hat{d}_{0i} R_{0_{k+1}}^T (A_{i_k} - A_{i_{k+1}}) = 0, \end{aligned} \quad (34)$$

$$\begin{aligned} &(J'_{i_d} F_{i_k} - F_{i_k}^T J'_{i_d})^\vee - (F_{i_{k+1}} J'_{i_d} - J'_{i_d} F_{i_{k+1}}^T)^\vee \\ &\quad - \hat{d}_{i0} F_{i_k}^T M_i R_{i_k}^T B_{i_k} \\ &\quad + (\widehat{F_{i_{k+1}} d_{i0}} M_i R_{i_{k+1}}^T + R_{i_{k+1}}^T \hat{A}_{i_{k+1}}) B_{i_{k+1}} = 0, \end{aligned} \quad (35)$$

$$\begin{aligned} &R_{0_k} M_0 R_{0_k}^T \Delta x_k + A_{1_k} + A_{2_k} \\ &\quad - R_{0_{k+1}} M_0 R_{0_{k+1}}^T \Delta x_{k+1} - A_{1_{k+1}} - A_{2_{k+1}} = 0, \end{aligned} \quad (36)$$

$$R_{0_{k+1}} = R_{0_k} F_{0_k}, \quad R_{i_{k+1}} = R_{i_k} F_{i_k} \quad (37)$$

$$x_{k+1} = x_k + \Delta x_k, \quad (38)$$

where inertia matrices are given by (23), (24), and $A_{i_k}, B_{i_k} \in \mathbb{R}^3$ are given by (27), (28) for $i \in \{1, 2\}$. For given $(g_0, f_0) \in \mathsf{G} \times \mathsf{G}$, $g_1 \in \mathsf{G}$ is obtained by (37)–(38), and $f_1 \in \mathsf{G}$ is obtained by solving (34)–(36). This yields a discrete-time Lagrangian flow map $(g_0, f_0) \to (g_1, f_1)$, and this process is repeated.

Discrete-time Hamilton's Equations: The discrete-time Legendre transformation is given by $\mu_k = -\mathsf{T}_e^* \mathsf{L}_{g_k} \cdot \mathbf{D}_{g_k} L_{d_k} + \mathrm{Ad}_{f_k^{-1}}^* \cdot (\mathsf{T}_e^* \mathsf{L}_{f_k} \cdot \mathbf{D}_{f_k} L_{d_k})$. Substituting this into discrete-time Euler-Lagrange equations, we obtain discrete-time Hamilton's equations as follows.

$$hp_{0_k} = (F_{0_k} J_{0_d} - J_{0_d} F_{0_k}^T)^\vee - (M_0 R_{0_k}^T \Delta x_k)^\wedge R_{0_k}^T \Delta x_k + \sum_{i=1}^{2} \hat{d}_{0i} R_{0_k}^T A_{i_k}, \quad (39)$$

$$hp_{i_k} = (F_{i_k} J'_{i_d} - J'_{i_d} F_{i_k}^T)^\vee - \frac{1}{h} \widehat{F_{i_k} d_{i0}} M_i R_{i_k}^T B_{i_k} - \frac{1}{h} R_{i_k}^T \hat{A}_{i_k} B_{i_k}, \quad (40)$$

$$hp_{x_k} = R_{0_k} M_0 R_{0_k}^T \Delta x_k + A_{1_k} + A_{2_k}, \quad (41)$$

$$R_{0_{k+1}} = R_{0_k} F_{0_k}, \quad R_{i_{k+1}} = R_{i_k} F_{i_k}, \quad (42)$$

$$x_{k+1} = x_k + \Delta x_k, \quad (43)$$

$$hp_{0_{k+1}} = (J_{0_d} F_{0_k} - F_{0_k}^T J_{0_d})^\vee + \sum_{i=1}^{2} \hat{d}_{0i} R_{0_{k+1}}^T A_{i_k}, \quad (44)$$

$$hp_{i_{k+1}} = (J'_{i_d} F_{i_k} - F_{i_k}^T J'_{i_d})^\vee - \hat{d}_{i0} F_{i_k}^T M_i R_{i_k}^T B_{i_k}, \quad (45)$$

$$p_{x_{k+1}} = p_{x_k}, \quad (46)$$

where inertia matrices are given by (23), (24), and $A_{i_k}, B_{i_k} \in \mathbb{R}^3$ are given by (27), (28) for $i \in \{1, 2\}$. For given $(g_0, \mu_0) \in \mathsf{G} \times \mathfrak{g}^*$, $f_1 \in \mathsf{G}$ is obtained by solving (39)–(41), and $g_1 \in \mathsf{G}$ is given by (42)–(43). The momenta at the next step is obtained by (44)–(46). This yields a discrete-time Hamiltonian flow map $(g_0, \mu_0) \to (g_1, \mu_1)$, and this process is repeated.

V. Numerical Example

We show computational properties of the Lie group variational integrator developed in the previous section. The principal axes of each ellipsoid are given by

$$\text{Body 0: } l_1 = 8, \quad l_2 = 1.5, \quad l_3 = 2 \text{ (m)},$$
$$\text{Body 1,2: } l_1 = 5, \quad l_2 = 0.8, \quad l_3 = 1.5 \text{ (m)}.$$

We assume the density of the fluid is $\rho = 1\mathrm{kg/m^3}$. The corresponding inertia matrices are given by

$$\begin{aligned} M_0 &= \mathrm{diag}[1.0659,\ 2.1696,\ 1.6641],\ (\mathrm{kg}) \\ M_1 = M_2 &= \mathrm{diag}[0.2664,\ 0.6551,\ 0.3677]\ (\mathrm{kg}), \\ J_0 &= \mathrm{diag}[1.3480,\ 20.1500,\ 25.3276]\ (\mathrm{kgm^2}), \\ J_1 = J_2 &= \mathrm{diag}[0.1961,\ 1.7889,\ 2.9210]\ (\mathrm{kgm^2}). \end{aligned}$$

The location of the ball joints with respect to the mass center of each body are chosen as

$$d_{01} = -d_{02} = [8.8,\ 0,\ 0], \quad d_{10} = -d_{20} = [5.5,\ 0,\ 0]\ (\mathrm{m}).$$

The initial conditions are as follows:

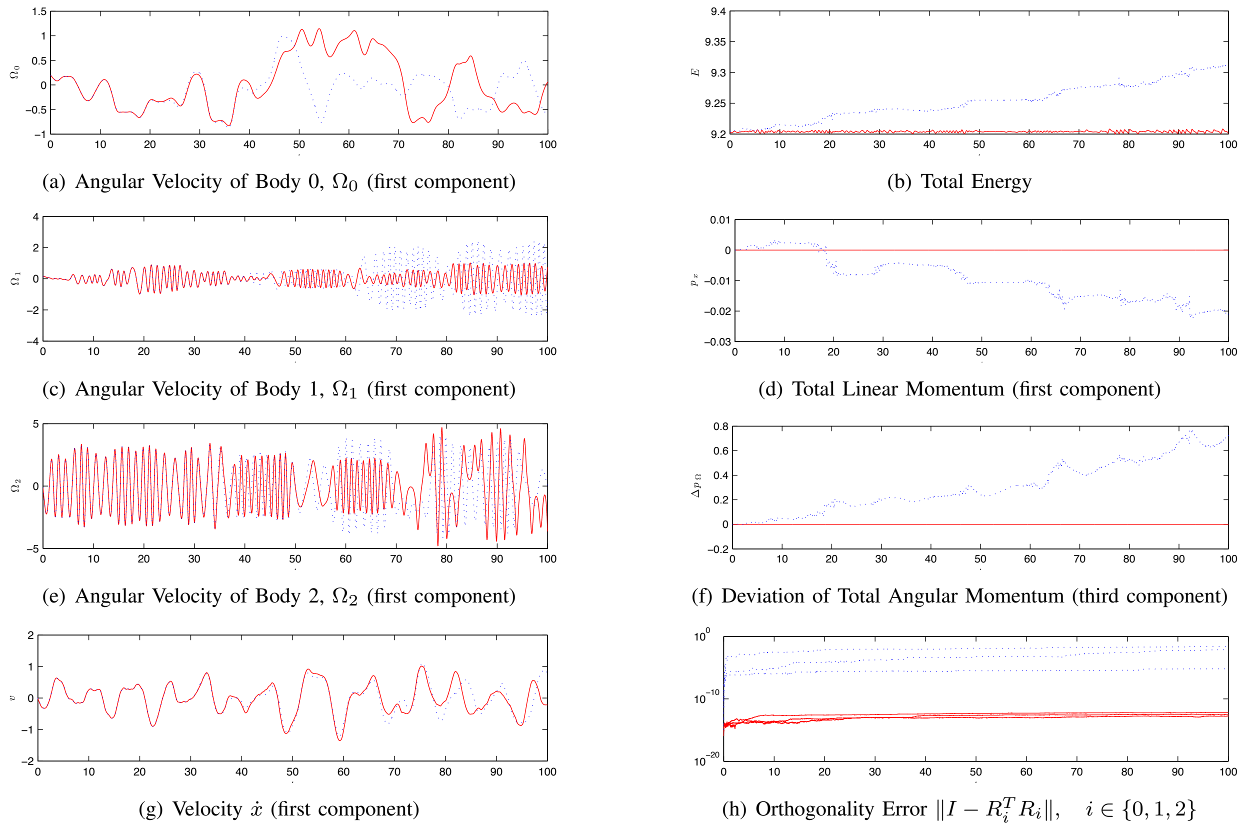
$$\begin{aligned} R_{0_0} = I, &\quad \Omega_{0_0} = [0.2,\ 0.1,\ 0.5]\ (\mathrm{rad/s}), \\ R_{1_0} = I, &\quad \Omega_{1_0} = [0.1,\ -0.3,\ -0.2]\ (\mathrm{rad/s}), \\ R_{2_0} = I, &\quad \Omega_{2_0} = [-0.1,\ 0.4,\ -0.6]\ (\mathrm{rad/s}), \\ x_0 = [0,\ 0,\ 0]\ (\mathrm{m}), &\quad \dot{x}_0 = [0,\ -0.4142,\ -0.5900]\ (\mathrm{m/s}). \end{aligned}$$

The corresponding total linear momentum is zero. These initial conditions provide a nontrivial rotational maneuver of the connected rigid bodies (an animation illustrating this maneuver is available at `http://my.fit.edu/~taeyoung`).

We compute the discrete-time Hamiltonian flow according to (39)–(46), and as comparison, we numerically integrate the continuous-time Hamilton's equations (18)–(20) using an explicit, variable step size, Runge-Kutta method. The timestep of the Lie group variational integrator is $h = 0.001$ and the maneuver time is 100 seconds.

Fig. 2 shows the resulting angular/linear velocity responses, total energy, total linear momentum, total angular momentum deviation, and orthogonality errors of the rotation matrices. The Lie group variational integrator and the Runge-Kutta method provide compatible trajectories only for a short period of time.

The computational properties of the Lie group variational integrator are as follows. As shown in Fig. 2(b), the computed total energy of the Lie group variational integrator oscillates near the initial value, but there is no increasing or decreasing drift for long time periods. This is due to the fact that the numerical solutions of symplectic numerical integrators are exponentially close to the exact solution



(a) Angular Velocity of Body 0, Ω_0 (first component)

(b) Total Energy

(c) Angular Velocity of Body 1, Ω_1 (first component)

(d) Total Linear Momentum (first component)

(e) Angular Velocity of Body 2, Ω_2 (first component)

(f) Deviation of Total Angular Momentum (third component)

(g) Velocity $\dot{x}$ (first component)

(h) Orthogonality Error $\|I - R_i^T R_i\|, \quad i \in \{0, 1, 2\}$

Fig. 2. Numerical simulation of connected rigid bodies in a perfect fluid (LGVI: red, solid, RK(4)5: blue, dotted)

of a perturbed Hamiltonian system [19]. In particular, the discrete-time flow almost exactly preserves the perturbed Hamiltonian, which is close to the original Hamiltonian. The Lie group variational integrator preserves the momentum map exactly as in Fig. 2(d) and 2(f), and it also preserves the orthogonal structure of rotation matrices accurately. The orthogonality errors, measured by $\|I - R_i^T R_i\|$ for $i \in \{0, 1, 2\}$, are less than 10^{-13} in Fig. 2(h).

These show that the structure-preserving properties of the Lie group variational integrator are important for simulating the dynamics of connected rigid bodies in a fluid accurately. A more extensive comparison study of the computational accuracy and efficiency of Lie group variational integrators can be found in [12].

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