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Robust Invariance in Uncertain Discrete Event Systems with Applications to Transportation Networks

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Abstract— **This paper studies a class of uncertain discrete event systems over the max-plus algebra, where system matrices are unknown but are convex combinations of known matrices. These systems model a wide range of applications, for example, transportation systems with varying vehicle travel time and queueing networks with uncertain arrival and queuing time. This paper presents computational methods for different robust invariant sets of such systems. A recursive algorithm is given to compute the supremal robust invariant sub-semimodule in a given sub-semimodule. The algorithm converges to a fixed point in a finite number of iterations under proper assumptions on the state semimodule. This paper also presents computational methods for positively robust invariant polyhedral sets. A search algorithm is presented for the positively robust invariant polyhedral sets. The main results are applied to the time table design of a public transportation network.**

Keywords: **Robust controlled invariance, discrete event systems, max-plus algebra.**

I. Introduction

In the study of linear systems over a field or a ring, the geometric approach ([5], [6], [10], [11], [15]) is used in many control problems, for example, the disturbance decoupling problem, the model matching problem, and the block decoupling problem. Controlled invariant subspaces (or sets) play a key role in these fundamental problems of geometric control theory. The geometric approach for discrete event systems over the max-plus algebra is still a new research direction [4] comparing to the geometric control theory for traditional linear systems over a field [15]. The max-plus algebra, a special semiring [9], is a set of real numbers embedded with the max operation and the plus operation. Semirings can be understood as a set of objects without inverses with respect to the corresponding operators. Systems over the max-plus algebra are used to model many discrete event systems, for instance, queueing systems [3], transportation systems [1], and communication networks [8].

Researchers have been studying different computational methods for controlled invariant spaces (or sets) for discrete event systems over the max-plus algebra ([12], [14]). However, there are many uncertain factors in the discrete event system modeling. This paper focuses on a class of uncertain discrete event systems over the max-plus algebra, in which system matrices are uncertain but can be written as linear combinations of known matrices. These system model a class of discrete event systems, such as transportation systems with varying travel time and queuing networks with uncertain arrival time and queueing time. Most of these uncertainties can be characterized by max-plus convex sets in [13]. To the author's best knowledge, this paper is the first attempt for the computational methods of different robust controlled invariant sets for this class of uncertain discrete event systems.

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The main challenge for systems over a semiring (or even a ring) is that the (A, B)-invariant sub-semimodule (or submodule) does not coincide with (A, B)-invariant sub-semimodule (or submodule) of feedback type ([5], [6], [10]). A semimodule over a semiring can be analogized to a linear space over a field. In this paper, a recursive algorithm is given to compute the supremal robust invariant sub-semimodule in a given sub-semimodule. The algorithm converges to a fixed point in a finite number of iterations under proper assumptions on the state semimodule. The fixed point is the supremal robust controlled invariant sub-semimodule of the given sub-semimodule. Another computational method is presented for robust controlled invariant sub-semimodule of feedback type in a given sub-semimodule. This paper also presents computational methods for positively robust invariant polyhedral sets in two cases, time-invariant and time-varying polyhedral sets. The reason to focus on time-varying polyhedral invariant sets is that there are many quantities in discrete event systems varying with time. For instance, in transportation networks, the control goal is that the departure time of each vehicle is under a time constraint. These constraints usually vary over time and can be used to design proper time tables for the transportation systems. A search algorithm for time-varying positively robust invariant sets is presented and is demonstrated using a public transportation network.

II. Mathematical Preliminaries

A. Semiring and Semimodule

A monoid R is a semigroup $(R, \boxplus)$ with an identity element e_R with respect to the binary operation $\boxplus$. The term *semiring* means a set, $R = (R, \boxplus, e_R, \boxtimes, 1_R)$ with two binary associative operations, $\boxplus$ and $\boxtimes$, such that $(R, \boxplus, e_R)$ is a commutative monoid and $(R, \boxtimes, 1_R)$ is a monoid, which are connected by a two-sided distributive law of $\boxtimes$ over $\boxplus$. Moreover, $e_R \boxtimes r = r \boxtimes e_R = e_R$, for all r in R. Examples of semirings include the set of natural numbers, the max-plus algebra, the min-plus algebra, and the Boolean algebra. $R = (R, \boxplus, e_R, \boxtimes, 1_R)$ is a semifield if and only if $(R \backslash \{e_R\}, \boxtimes, 1_R)$ is a group, i.e. all of its elements have inverse elements with respect to the $\boxtimes$ operator. An idempotent semifield R is a semifield satisfying $a \boxplus a = a$ for all $a \in R$. The common example for idempotent semifields is max-plus algebra, which replaces the traditional addition and multiplication into the max operation and the plus operation,

$$\text{Addition}: \quad a \oplus b \equiv \max\{a, b\},$$
$$\text{Multiplication}: \quad a \otimes b \equiv a + b.$$

In max-plus algebra literature, we usually denote it as $\mathbb{R}_{\text{Max}}$=($\mathbb{R} \cup \{\epsilon\}, \oplus, \epsilon, \otimes, e$), where $\mathbb{R}$ is the set of real numbers, $\epsilon = -\infty$, and $e = 0$. Similarly, $\mathbb{Z}_{\text{Max}} = (\mathbb{Z} \cup \{\epsilon\}, \oplus, \epsilon, \otimes, e)$ denotes the semiring of integers.

Let $(R, \boxplus, e_R, \boxtimes, 1_R)$ be a semiring, and $(M, \boxplus_M, e_M)$ be a commutative monoid, where the subscript denotes the corresponding monoid for the operator $\boxplus$. M is called a *left R-semimodule* if there exists a map $\mu : R \times M \to M$, denoted by $\mu(r, m) = rm$, for all $r \in R$ and $m \in M$, such that the following conditions are satisfied:

1) $r(m_1 \boxplus_M m_2) = rm_1 \boxplus_M rm_2$;
2) $(r_1 \boxplus r_2)m = r_1 m \boxplus_M r_2 m$;
3) $r_1(r_2 m) = (r_1 \boxtimes r_2)m$;
4) $1_R\, m = m$;
5) $r\, e_M = e_M = e_R\, m$,

for any r, r_1, $r_2 \in R$ and m, m_1, $m_2 \in M$. In this paper, e denotes the unit semimodule. A *sub-semimodule* K of M is a submonoid of M with $rk \in K$, for all $r \in R$ with $k \in K$. An R-morphism between two semimodules $(M, \boxplus_M, e_M)$ and $(N, \boxplus_N, e_N)$ is a map $f : M \to N$ satisfying

1) $f(m_1 \boxplus_M m_2) = f(m_1) \boxplus_N f(m_2)$;
2) $f(rm) = rf(m)$,

for all m, m_1, $m_2 \in M$ and $r \in R$.

Let N be a subset of a R-semimodule $(M, \boxplus_M, e_M)$. We denote N_0 as the set of all elements of the form $\underset{i}{\boxplus_M} \lambda_i n_i$ where n_i are elements in N, λ_i are elements in R, and i are elements in an index set I. The sub-semimodule N_0 is said to be generated by N, and N is called a system of generators of N_0. The subset N of an R-semimodule M is called *linearly independent* if $\underset{i}{\boxplus_M} \lambda_i n_i = \underset{i}{\boxplus_M} \beta_i n_i$ implies $\lambda_i = \beta_i$ for all $i \in I$. An R-semimodule M is called a free R-semimodule if it has a linearly independent subset N of M which generates M and then N is called a basis of M. If N has a finite number of elements, M is called a finitely generated R-semimodule, denoted as span N.

In [12], Katz defined the concept of volume for the max-plus semimodule. Let $\mathcal{K} \subset \mathbb{Z}^n_{\text{Max}}$ be the integer max-plus semimodule, the volume of $\mathcal{K}$, denoted as $vol(\mathcal{K})$ is the cardinality of the set $\{x \in \mathcal{K} | x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0\}$. This set is denoted as $\widetilde{\mathcal{K}}$. Also, if $K \in \mathbb{Z}^{n \times p}_{\text{Max}}$, the volume of the semimodule $\mathcal{K} = \text{Im } K$ is denoted as $vol(K) = vol(\text{Im } K) = vol(\mathcal{K})$.

B. Uncertain Discrete Event Systems over Max-Plus Algebra

A class of uncertain discrete event systems over the max-plus algebra is described by the following equation:

$$x(k) \;\; = \;\; \widetilde{A}\, x(k-1) \; \oplus \; B\, u(k), \tag{1}$$

where the state semimodule $X \cong \mathbb{R}^n_{\text{Max}}$ and the input semimodule $U \cong \mathbb{R}^r_{\text{Max}}$ are free. $\widetilde{A} : X \to X$ and $B : U \to X$ are R-semimodule morphisms. The system's state matrix $\widetilde{A}$ is unknown but it is the linear combination of known matrices, $A_1, A_2, \cdots, A_m$, i.e.

$$\widetilde{A} = \bigoplus_{i=1}^{m} (\lambda_i \otimes A_i), \text{ with } \bigoplus_{i=1}^{m} \lambda_i = e.$$

A convex set $\mathcal{P}$ is convex if $(a \otimes x) \oplus (b \otimes y) \in \mathcal{P}$ for any $x, y \in \mathcal{P}$ and a, b in a semiring R and $a \oplus b = e$. Then $\widetilde{A}$ is in the convex hull of $A_1, \cdots, A_m$, denoted as $\widetilde{A} \in co\{A_1, \cdots, A_m\}$.

The geometric concepts of different invariant subspaces can be generalized to systems over a semiring. Given a system of the form (1) over the max-plus algebra $\mathbb{R}_{\text{Max}}$, a sub-semimodule $\mathcal{V}$ of the state semimodule X is

- called $(\widetilde{A}, B)$-invariant or robust controlled invariant if and only if, for all $x_0 \in \mathcal{V}$ and an arbitrary matrix $\widetilde{A} \in co\{A_1, \cdots, A_m\}$, there exists a sequence of control inputs, $\underline{u} = \{u_1, u_2, \cdots\}$, such that every component in the state trajectory produced by this input, $\underline{x}(x_0; \underline{u}) = \{x_0, x_1, \cdots\}$, remains inside of $\mathcal{V}$.
- called $(\widetilde{A}, B)$-invariant of feedback type, or $(\widetilde{A} \oplus BF)$-invariant, if and only if there exists a state feedback $F : X \to U$ such that $(\widetilde{A} \oplus BF)\mathcal{V} \subset \mathcal{V}$, for an arbitrary matrix $\widetilde{A} \in co\{A_1, \cdots, A_m\}$.

Unlike systems over a field, (A, B)-invariant sub-semimodules of feedback type are not same as (A, B)-invariant sub-semimodules for systems over a semiring or even a ring. Conte and Perdon [6] proved that, for systems over a ring, if an (A, B)-invariant submodule is a direct summand of the free state semimodule X, then it is (A, B)-invariant of feedback type. However, direct summand in semimodules is far more complicated than modules, the same result in [6] is not true any more.

III. Robust Invariant Sub-Semimodules

This section presents the computational methods for robust $(\widetilde{A}, B)$-invariant sub-semimodules and $(\widetilde{A} \oplus BF)$-invariant sub-semimodules in a given sub-semimodule. These computation methods are generalizations of deterministic discrete event systems in [12].

A. $(\widetilde{A}, B)$-Invariant Sub-semimodules

A sub-semimodule $\mathcal{V}$ of the state semimodule is $(\widetilde{A}, B)$-invariant if and only if

$$\mathcal{V} \;\; = \;\; \mathcal{V} \cap \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}), \tag{2}$$

where $\mathcal{B} = B(U)$ and

$$\begin{aligned} \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}) \;\; = \;\; & \{x \in \mathbb{R}^n_{\text{Max}} | \exists u \in U, s.t. \widetilde{A}x \oplus Bu \in \mathcal{V}, \\ & \forall \widetilde{A} \in co\{A_1, \cdots, A_m\}\}. \end{aligned} \tag{3}$$

Lemma 1: For an uncertain discrete event system over the max-plus algebra of the form (1), a sub-semimodule $\mathcal{V}$ in the state semimodule is $(\widetilde{A}, B)$-invariant if and only if $\mathcal{V}$ is (A_i, B)-invariant for any $i \in \{1, 2, \cdots, m\}$.

Lemma 2: For an uncertain discrete event system over the max-plus algebra of the form (1), a sub-semimodule $\mathcal{V}$ of the state semimodule satisfies the following equality,

$$\widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B}) = \bigcap_{i=1}^{m} A_i^{-1}(\mathcal{V} \ominus \mathcal{B}).$$

The supremal (A, B)-invariant subspace $\mathcal{V}^*$ of a linear system over a field is computed using the following algorithm [15], where $A^{-1}(\mathcal{V}_i + \mathcal{B}) = \{x \in \mathbb{R}^n | Ax \in \mathcal{V}_i + \mathcal{B}\}$:

$$\begin{aligned} \mathcal{V}_1 \;\; &= \;\; \mathcal{K} \\ \mathcal{V}_{k+1} \;\; &= \;\; \mathcal{V}_k \cap A^{-1}(\mathcal{V}_k + \mathcal{B}), \; k \in \mathbb{N}. \end{aligned} \tag{4}$$

For systems over a field, this recursion converges to a fixed point, which is the supremal controlled invariant subspace in $\mathcal{K}$, after a finite number of iterations. However, for systems over a semiring or even a ring, this algorithm does not guarantee to terminate in finite steps. For uncertain systems over the max-plus algebra, to calculate the supremal controlled invariant sub-semimodule of a given sub-semimodule, we can use Lemma 2 to modify the algorithm (4) to:

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{K} \\ \mathcal{V}_{k+1} &= \mathcal{V}_k \cap \widetilde{A}^{-1}(\mathcal{V}_k \ominus \mathcal{B}) \\ &= \mathcal{V}_k \cap \bigcap_{i=1}^{m} A_i^{-1}(\mathcal{V}_k \ominus \mathcal{B}),\ k \in \mathbb{N}. \end{aligned} \tag{5}$$

The following lemma states that the algorithm (5) can be used to calculate the the supremal controlled invariant sub-semimodule in $\mathcal{K}$. This result is a generalization of Lemma 3 in [12] by Katz for the deterministic case.

Lemma 3: Let $\{\mathcal{V}_k\}_{k\geq 0}$ be the family of sub-semimodules defined by the algorithm (5). If there exists $\cap_{k\in\mathbb{N}}\mathcal{V}_k$, then any $(\widetilde{A}, B)$-invariant sub-semimodule of $\mathcal{K}$ is contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$, namely the supremal controlled invariant sub-semimodule $\mathcal{V}^*$ is also contained in $\cap_{k\in\mathbb{N}}\mathcal{V}_k$. Moreover, if the algorithm in Eq. (5) terminates in r steps, then $\mathcal{V}^* = \mathcal{V}_r$.

Notice that, the algorithm (5) does not always terminate in a finite number of steps. However, if restricting to a finite volume semimodule in the integer max-plus algebra, the algorithm (5) terminates in finite steps, which is stated in the following proposition. This result is a direct generalization of Theorem 2 in [12], so the proofs are omitted due to limited space. Proposition 1 is illustrated by Example 1.

Proposition 1: Given a sub-semimodule $\mathcal{K}$ in $\mathbb{Z}^n_{\mathrm{Max}}$ with a finite volume. The supremal robust $(\widetilde{A}, B)$-invariant sub-semimodule of $\mathcal{K}$ under the dynamics of the system (1) is finitely generated. The sequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ by the algorithm 5 terminates in a finite number r of steps, $\mathcal{V}^* = \mathcal{V}_r$ and $r \leq vol(\mathcal{K}) + 1$.

Example 1: Consider an uncertain discrete event system over $\mathbb{Z}_{\mathrm{Max}}$ with system matrix $\widetilde{A} \in co\{A_1, A_2\}$, where

$$A_1 = \begin{bmatrix} 1 & -\infty \\ -\infty & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -\infty \\ -\infty & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We need to calculate the supremal $(\widetilde{A}, B)$-invariant sub-semimodule in $\mathcal{K} = \{(x, y)^T \in \mathbb{Z}^2_{\mathrm{Max}} | x + 1 \leq y \leq x + 4\}$. The set $\widetilde{\mathcal{K}}$ is

$$\begin{aligned} \widetilde{\mathcal{K}} &= \{(x, y)^T \in \mathcal{K} | x \oplus y = 0\} \\ &= \{(-1, 0)^T, (-2, 0)^T, (-3, 0)^T, (-4, 0)^T\}. \end{aligned}$$

The volume of $\mathcal{K}$ is 4. Then the algorithm (5) to calculate the supremal controlled invariant sub-semimodule will terminate at most $vol(\mathcal{K}) + 1 = 5$ steps. We verify it by computing the algorithm (5):

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{K}, \\ \mathcal{V}_2 &= \mathcal{V}_1 \cap \bigcap_{i=1}^{2} A_i^{-1}(\mathcal{V}_1 \ominus \mathcal{B}) \\ &= \{(x, y)^T \in \mathbb{Z}^2_{\mathrm{Max}} | x + 3 \leq y \leq x + 4\}, \\ \mathcal{V}_3 &= \mathcal{V}_2 \cap \bigcap_{i=1}^{2} A_i^{-1}(\mathcal{V}_2 \ominus \mathcal{B}) = \{(-\infty, -\infty)^T\}, \\ \mathcal{V}_4 &= \mathcal{V}_3, \cdots,\ \mathcal{V}_{k+1} = \mathcal{V}_3. \end{aligned}$$

The algorithm terminates in 3 steps and the supremal$(\widetilde{A}, B)$-invariant sub-semimodule is $\mathcal{V}^* = \mathcal{V}_3 = \{(-\infty, -\infty)^T\}$. $\diamondsuit$

B. $(\widetilde{A}, B)$-Invariant Sub-semimodules of Feedback Type

Because a controlled invariant sub-semimodule is not identical with a control invariant sub-semimodule of feedback type, the computational method in the algorithm (5) cannot be used to calculate $(\widetilde{A}, B)$-invariant sub-semimodules of feedback type. For the integer max-plus algebra, if a given sub-semimodule $\mathcal{V} = \mathrm{Im}\ Q$, where $Q = \mathbb{Z}^{n\times r}_{\mathrm{Max}}$, then $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant if and only if there exists matrices $F \in \mathbb{Z}^{q\times n}_{\mathrm{Max}}$ and $G \in \mathbb{Z}^{r\times r}_{\mathrm{Max}}$, such that

$$(\widetilde{A} \oplus BF)Q = QG,\ \forall \widetilde{A} \in co\{A_1, \cdots, A_m\}.$$

Because $\widetilde{A}$ is uncertain, we actually can verify this condition using each known matrix, A_i.

Lemma 4: Given sub-semimodule $\mathcal{V} = \mathrm{Im}\ Q$, where $Q = \mathbb{Z}^{n\times r}_{\mathrm{Max}}$, then $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant if and only if there exists matrices $F_i \in \mathbb{Z}^{q\times n}_{\mathrm{Max}}$ and $G_i \in \mathbb{Z}^{r\times r}_{\mathrm{Max}}$, such that

$$(A_i \oplus BF_i)Q = QG_i,\ \forall i \in \{1, \cdots, m\}.$$

To calculate for each F_i and G_i, we can use the elimination method in [12] and the residuation theory in [1].

IV. Positively Robust Invariant Polyhedral Sets

This section presents computational methods for different positively robust invariant polyhedral sets for systems over an idempotent semiring $(R, \oplus, e_R, \otimes, 1_R)$ described by the following equation:

$$x(k) = \widetilde{A}\, x(k-1). \tag{6}$$

The system's state matrix $\widetilde{A}$ is unknown but it is the linear combination of known matrices, $A_1, A_2, \cdots, A_m$. Obviously, discrete event systems over the max-plus algebra $\mathbb{R}_{\mathrm{Max}}$ are special cases of such systems. The results in this section are motivated by Truffet [14] and Bitsoris [2].

A. Time-invariant Polyhedral Sets

If we are considering time-invariant polyhedral sets

$$\begin{aligned} \mathcal{P}(F, \phi, \psi) &= \{x \in R^n | \phi \leq F \otimes x \leq \psi\}, \\ \phi,\ \psi \in R^p, &\quad \text{and} \quad F \in R^{p\times n}. \end{aligned}$$

For deterministic linear systems over an idempotent semiring, the necessary and sufficient condition for positively invariance of the set $\mathcal{P}(I_n, \phi, \psi)$ was established in [14].

Lemma 5: [14] Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\mathcal{P}(I_n, \phi, \psi)$ is positively invariant under the dynamics of the system,

$$x(k) = Ax(k-1),\ A \in R^{n\times n}, \tag{7}$$

if and only if

$$(A \otimes \psi \leq \psi) \quad \wedge \quad (\phi \leq A \otimes \phi). \tag{8}$$

For uncertain linear systems (6) over an idempotent semiring R, the similar result as Lemma 5 can also obtained.

Proposition 2: Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\mathcal{P}(I_n, \phi, \psi)$ is positively robust invariant under the dynamics of the system (6) if and only if

$$(A_i \otimes \psi \leq \psi) \quad \wedge \quad (\phi \leq A_i \otimes \phi), \tag{9}$$

for all $i \in \{1, \cdots, m\}$.

The following proposition is a sufficient condition for a polyhedral set, $\mathcal{P}(F, \phi, \psi)$, to be positively invariant under the dynamic of the system (7).

Proposition 3: Assume $n = p$, $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics of the system (7) if there exists a matrix $H \in R^{p\times p}$ such that

$$\begin{aligned} F \otimes A &= H \otimes F \text{ and} \\ (H \otimes \psi \leq \psi) &\wedge (\phi \leq H \otimes \phi). \end{aligned} \tag{10}$$

The second condition means that $\mathcal{P}(I_n, \phi, \psi)$ is H-positively invariant.

The following proposition states a sufficient condition for a polyhedral set $\mathcal{P}(F, \phi, \psi)$ to be positively robust invariant.

Proposition 4: Assume $n = p$, $\mathcal{P}(F, \phi, \psi)$ is positively robust invariant under the dynamics of the system (6) if there exists a matrix $H_i \in R^{p\times p}$ such that

$$\begin{aligned} F \otimes A_i &= H_i \otimes F \text{ and} \\ (H_i \otimes \psi \leq \psi) &\wedge (\phi \leq H_i \otimes \phi). \end{aligned} \tag{11}$$

The second condition means that $\mathcal{P}(I_n, \phi, \psi)$ is H_i-positively invariant, where $i \in \{1, \cdots, m\}$.

B. Time-variant Polyhedral Sets

If we are considering time-varying polyhedral sets

$$\widetilde{\mathcal{P}}(F, \phi(k), \psi(k)) = \{x(k) \in R^n | \phi(k) \leq F \otimes x(k) \leq \psi(k)\},$$

$$\phi(k) = K_\phi \otimes \phi(k-1), \; \psi(k) = K_\psi \otimes \psi(k-1),$$

where $\phi, \psi \in R^p$, $k \in \mathbb{Z}^+$, and $F \in R^{p\times n}$, and $x(k)$ is governed by system (6) or system (7). Time-varying polyhedral sets are polyhedrons with time-variant boundaries. Such conditions are very common in reality, such as the public transportation networks with time-varying time tables. Due to space limit, all proofs are omitted in this subsection.

Lemma 6: Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of system (7) if and only if

$$(A \otimes \psi(k) \leq K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leq A \otimes \phi(k)), \tag{12}$$

for all $k \in \mathbb{Z}^+$.

Lemma 6 presents the necessary and sufficient conditions for a time-variant polyhedral set $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ to be positively invariant for system (7). If we are given K_ϕ and K_ψ, the searching method for $\phi^* \leq \psi^*$, such that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, $k \geq 1$, to be positively invariant for system (7), can be performed as follows.

1) We solve the following equations for $\phi^* \leq \psi^*$,

$$\begin{aligned} A \otimes \psi^* &= K_\psi \otimes \psi^*, \\ A \otimes \phi^* &= K_\phi \otimes \phi^*. \end{aligned}$$

2) For any $x(0) \in R^n$ such that let $\phi(0) = \phi^*$ and $\psi(0) = \psi^*$ and $\phi(0) \leq x(0) \leq \psi(0)$, we have

$$\begin{aligned} A \otimes \phi(0) \leq A \otimes x(0) \leq A \otimes \psi(0) \Longrightarrow \\ K_\phi \otimes \phi(0) \leq A \otimes x(0) \leq K_\psi \otimes \psi(0) \Longrightarrow \\ \phi(1) \leq x(1) \leq \psi(1). \end{aligned}$$

Continuing this process, we are able to show that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively invariant with respect to system (7). Therefore, ϕ^* and ψ^* generate the possible boundaries for a positively invariant polyhedral set.

The following proposition presents the necessary and sufficient conditions for $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ to be positively robust invariant.

Proposition 5: Assume $n = p$ and $F = I_n$, where I_n denotes the $n \times n$ identity matrix. $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of the system (6) if and only if

$$(A_i \otimes \psi(k) \leq K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leq A_i \otimes \phi(k)),$$

for all $i \in \{1, \cdots, m\}$ and $k \in \mathbb{Z}^+$.

If we are given K_ϕ and K_ψ, the searching method for $\phi^* \leq \psi^*$, such that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$, $k \geq 1$, to be positively robust invariant for system (6), is similar the positively invariance case.

1) Firstly, we solve the following equations for $\phi^* \leq \psi^*$,

$$\begin{aligned} A_i \otimes \psi^* &= K_\psi \otimes \psi^*, \\ A_i \otimes \phi^* &= K_\phi \otimes \phi^*, \end{aligned}$$

for all possible A_i, $i \in \{1, \cdots, m\}$.

2) For any $x(0) \in R^n$ such that let $\phi(0) = \phi^*$ and $\psi(0) = \psi^*$ and $\phi(0) \leq x(0) \leq \psi(0)$, we have

$$\begin{aligned} A_i \otimes \phi(0) \leq A_i \otimes x(0) \leq A_i \otimes \psi(0) \Longrightarrow \\ K_\phi \otimes \phi(0) \leq A_i \otimes x(0) \leq K_\psi \otimes \psi(0) \Longrightarrow \\ \phi(1) \leq x(1) \leq \psi(1), \end{aligned}$$

for all $i \in \{1, \cdots, m\}$. Continuing this process, we are able to show that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is positively robust invariant with respect to system (6). Therefore, ϕ^* and ψ^* generate the possible boundaries for a positively robust invariant polyhedral set.

The following proposition is a sufficient condition for a polyhedral set $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ to be positively invariant under the dynamic of the system (7).

Proposition 6: Assume $n = p$, $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ is positively invariant under the dynamics of the system (7) if there exists a matrix $H \in R^{p\times p}$ such that

$$F \otimes A = H \otimes F \text{ and}$$

$$(H \otimes \psi(k) \leq K_\psi \otimes \psi(k)) \wedge (K_\psi \otimes \phi(k) \leq H \otimes \phi(k)).$$

The second condition means that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is H-positively invariant.

The following proposition states a sufficient condition for a polyhedral set $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ to be positively robust invariant.

Proposition 7: Assume $n = p$, $\widetilde{\mathcal{P}}(F, \phi(k), \psi(k))$ is positively robust invariant under the dynamics of the system (6) if there exists a matrix $H \in R^{p\times p}$ such that

$$F \otimes A_i = H_i \otimes F \text{ and}$$

$$(H_i \otimes \psi(k) \leq K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leq H_i \otimes \phi(k)).$$

The second condition means that $\widetilde{\mathcal{P}}(I_n, \phi(k), \psi(k))$ is H_i-positively invariant, where $i \in \{1, \cdots, m\}$ and $k \in \mathbb{Z}^+$.

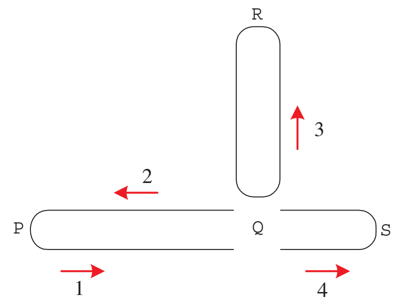

Fig. 1. A small public transportation network [7].

C. Example: A Public Transportation Network

In this section, we model a public transportation network [7] by an uncertain discrete event system over the max-plus algebra. We will use the main results in this paper to check if a polyhedral set is positively robust invariant.

In a small public transportation network [7], there are train services from P via Q to S and back and from Q to R and back. Trains from P to S have to stop at Q for the connection to trains with destination R and vice versa. If $x_i(\cdot)$ denotes the departure time of the first train in the direction i, $i = 1, \cdots, 4$. The train which is about to leave in the direction i for the k-th time cannot leave if the train has not arrived yet. This condition can be represented as

$$x_i(k) \quad \geq \quad a_{ij} \otimes x_j(k-1), \tag{13}$$

where $x_i(k)$ denotes the k-th departure time in direction i and a_{ij} is the traveling time from direction j to i, including the loading time of passengers. Another condition is that the train needs to wait the possible connecting trains, i.e.

$$x_i(k) \quad \geq \quad a_{il} \otimes x_l(k-1), \tag{14}$$

where l is the possible connecting direction and a_{il} denotes the traveling time from direction l to i, also including the loading time of passengers. For the simple transportation network in Fig. 1, the system equation is described as

$$x(k) \quad = \quad Ax(k-1), \text{ where } A = \begin{bmatrix} \epsilon & a_2 & \epsilon & \epsilon \\ \epsilon & \epsilon & a_3 & a_4 \\ a_1 & \epsilon & a_3 & a_4 \\ a_1 & \epsilon & a_3 & \epsilon \end{bmatrix},$$

where a_i denotes the traveling time on direction $i \in \{1, \cdots, 4\}$. If the traveling time a_i is deterministic, then the linear system is a deterministic discrete event system. In reality, however, the traveling time usually varies due to traffic and other emergency situations. For instance, assume $a_1 \in [10, 20]$ minutes, $a_2 \in [15, 25]$ minutes, $a_3 \in [10, 20]$ minutes and $a_4 \in [15, 25]$ minutes, then the system becomes an uncertain linear system $x(k) = \widetilde{A}x(k-1)$, where $\widetilde{A} \in co\{A_1, A_2\}$ and

$$A_1 \quad = \quad \begin{bmatrix} \epsilon & 15 & \epsilon & \epsilon \\ \epsilon & \epsilon & 10 & 15 \\ 10 & \epsilon & 10 & 15 \\ 10 & \epsilon & 10 & \epsilon \end{bmatrix}$$

and

$$A_2 \quad = \quad \begin{bmatrix} \epsilon & 25 & \epsilon & \epsilon \\ \epsilon & \epsilon & 20 & 25 \\ 20 & \epsilon & 20 & 25 \\ 20 & \epsilon & 20 & \epsilon \end{bmatrix}.$$

We are considering a polyhedral set

$$\widetilde{\mathcal{P}}(I_4, \phi(k), \psi(k)) = \{x \in \mathbb{R}^4 | \phi(k) \leq x(k) \leq \psi(k)\},$$

where $K_\phi = 13.3333$ and $K_\psi = 23.3333$ and

$$\phi(k) = K_\phi \otimes \phi(k-1) \text{ and } \psi(k) = K_\psi \otimes \psi(k-1).$$

Firstly, we solve the following equations for $\phi^* \leq \psi^*$,

$$\begin{aligned} A_i \otimes \psi^* \quad &= \quad K_\psi \otimes \psi^*, \\ A_i \otimes \phi^* \quad &= \quad K_\phi \otimes \phi^*, \end{aligned}$$

for all possible A_i, $i \in \{1, 2\}$. Because K_ϕ and K_ψ are eigenvalues for A_1 and A_2, respectively, we obtain that

$$\phi(0) = \phi^* = [30, 28.3333, 28.3333, 26.6667]^T \text{ and}$$

$$\psi(0) = \psi^* = [50, 48.3333, 48.3333, 46.6667]^T,$$

which are the eigenvectors for both A_i matrices, for $i = 1, 2$. Using Proposition 5, we can verify that

$$(A_i \otimes \psi(k) \leq K_\psi \otimes \psi(k)) \wedge (K_\phi \otimes \phi(k) \leq A_i \otimes \phi(k)),$$

for all $i \in \{1, \cdots, m\}$ and $k \in \mathbb{Z}^+$. Therefore, the given polyhedral set is positively robust invariant with respect to the public transportation network for arbitrary choice of matrix $\widetilde{A} \in co\{A_1, A_2\}$. We can understand $\phi(k)$ and $\psi(k)$ as time-variant time tables for trains in this station. Therefore, we have a feasible method to establish a reasonable time table for the traffic system.

For instance, we pick $\widetilde{A} = \lambda_1 \otimes A_1 \oplus \lambda_2 \otimes A_2$ and $\lambda_1 = 0$ and $\lambda_2 = -5$. After computing the system trajectories, the four states, $x_i(k)$, are in the set, $\widetilde{\mathcal{P}}(I_4, \phi(k), \psi(k))$, as shown in Fig. 2. These polyhedral sets can be refined to find optimal time tables.

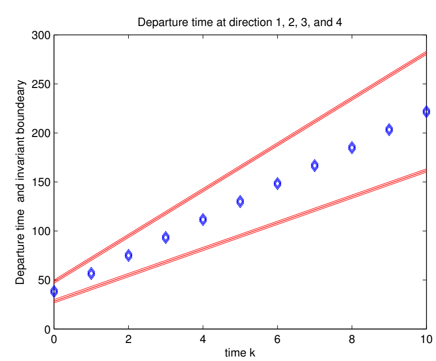

Fig. 2. The traveling time in direction 1, 2, 3, 4.

V. Conclusion

This paper studies a class of discrete event systems over the max-plus algebra, where system matrices are unknown but are convex combinations of known matrices. Various computational methods for different robust invariant sets are presented for such systems. These invariant sets are important in many control synthesis problems for discrete event systems, such as the disturbance decoupling problem, the block decoupling problem, and the model matching problem. Future research will explore different types of controlled invariant sets besides polyhedral sets, such as ellipsoidal invariant sets, in controller synthesis problems of discrete event systems.

VI. Appendix: Proofs

Proof of Lemma 1: "$\Longrightarrow$", necessity. If $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant, then, for all $x \in \mathcal{V}$, $\widetilde{A}x \oplus Bu \in \mathcal{V}$, that is, $\bigoplus_{i=1}^{m} \lambda_i A_i x \oplus Bu$ is also in $\mathcal{V}^*$. So for any special A_i, $A_i \mathcal{V} \oplus Bu \subset \mathcal{V}$ for all $i \in \{1, 2, \cdots, m\}$. Therefore, $\mathcal{V}$ is (A_i, B) -invariant for any i.

"$\Longleftarrow$", sufficiency. If $\mathcal{V}$ is (A_i, B)-invariant for any $i = 1, 2, \cdots, m$, then, for all $x \in \mathcal{V}$, there exists a $u_i \in U$, such that $A_i x \oplus Bu_i \in \mathcal{V}$. Because the sub-semimodule $\mathcal{V}$ is closed under the addition operation, $\bigoplus_{i=1}^{m} \lambda_i A_i x \oplus B \bigoplus_{i=1}^{m} \lambda_i u_i = \widetilde{A}x \oplus Bu \in \mathcal{V}$, where $u = \bigoplus_{i=1}^{m} \lambda_i u_i$. Therefore, $\mathcal{V}$ is $(\widetilde{A}, B)$-invariant. $\diamondsuit$

Proof of Lemma 2: "$\subseteq$": For any $x \in \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B})$, there exists a $u \in U$ such that $\widetilde{A}x \oplus Bu = \bigoplus_{i=1}^{m} \lambda_i A_i x \oplus Bu \in \mathcal{V}$. Therefore, $A_i x \oplus Bu \in \mathcal{V}$ for a special $i \in \{1, \cdots, m\}$. Therefore, $x \in \bigcap_{i=1}^{m} A_i^{-1}(\mathcal{V} \ominus \mathcal{B})$.

"$\supseteq$": For any $x \in \bigcap_{i=1}^{m} A_i^{-1}(\mathcal{V} \ominus \mathcal{B})$, there exists a $u_i \in U$, such that $A_i x \oplus Bu_i \in \mathcal{V}$ for all $i = \{1, \cdots, m\}$. Therefore, $\bigoplus_{i=1}^{m} \lambda_i A_i x \oplus B \bigoplus_{i=1}^{m} \lambda_i u_i = \widetilde{A}x \oplus Bu \in \mathcal{V}$, where $u = \bigoplus_{i=1}^{m} \lambda_i u_i$. Hence, $x \in \widetilde{A}^{-1}(\mathcal{V} \ominus \mathcal{B})$. $\diamondsuit$

Proof of Lemma 4: "$\Longleftarrow$": If $(A_i \oplus BF_i)Q = QG_i$, then

$$\begin{aligned} \bigoplus_{i=1}^{m} \lambda_i (A_i \oplus BF_i) Q &= \bigoplus_{i=1}^{m} \lambda_i Q G_i \\ (\bigoplus_{i=1}^{m} \lambda_i A_i \oplus B \bigoplus_{i=1}^{m} \lambda_i F_i) Q &= Q \bigoplus_{i=1}^{m} \lambda_i G_i \\ (\widetilde{A} \oplus B\widetilde{F}) Q &= Q\widetilde{G}, \end{aligned}$$

where $\widetilde{F} \in co\{F_1, \cdots, F_m\}$ and $\widetilde{G} \in co\{G_1, \cdots, G_m\}$. Then, $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant. "$\Longrightarrow$": $\mathcal{V}$ is $(\widetilde{A} \oplus BF)$-invariant, then

$$(\widetilde{A} \oplus BF)Q = QG, \ \forall \widetilde{A} \in co\{A_1, \cdots, A_m\}.$$

Therefore, $(A_i \oplus BF_i)Q = QG_i$, for all $i \in \{1, \cdots, m\}$. $\diamondsuit$

Proof of Proposition 2: "$\Longrightarrow$". If $\mathcal{P}(I_n, \phi, \psi)$ is positively invariant under the dynamics of the system (6), then for any $x \in R^n$ such that $\phi \leq x \leq \psi$, we have $\phi \leq \widetilde{A} \otimes x \leq \psi$, for any possible choice of $\widetilde{A} \in co\{A_1, \cdots, A_m\}$. Therefore, we have $\phi \leq A_i \otimes x \leq \psi$, for any $i \in \{1, \cdots, m\}$. Using Lemma 5, we have $(A_i \otimes \psi \leq \psi) \wedge (\phi \leq A_i \otimes \phi)$ for all $i \in \{1, \cdots, m\}$.

"$\Longleftarrow$". Because the equation (9) holds for all $i \in \{1, \cdots, m\}$, we have

$$\begin{aligned} &\bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes \psi \leq \bigoplus_{i=1}^{m} \lambda_i \otimes \psi \wedge \\ &\quad \bigoplus_{i=1}^{m} \lambda_i \otimes \phi \leq \bigoplus_{i=1}^{m} \lambda_i \otimes A_i \otimes \phi. \end{aligned}$$

for all $i \in \{1, \cdots, m\}$. Since $\oplus_{i=1}^{m} \lambda_i = 1_R$ and $\oplus_{i=1}^{m} \lambda_i \otimes A_i = \widetilde{A}$, the above condition becomes

$$(\widetilde{A} \otimes \psi \leq \psi) \quad \wedge \quad (\phi \leq \widetilde{A} \otimes \phi).$$

Using Lemma 5, $\mathcal{P}(I_n, \phi, \psi)$ is positively robust invariant for the discrete event system (6). $\diamondsuit$

Proof of Proposition 3: We need to prove for any $\phi \leq F \otimes x \leq \psi$, we have $\phi \leq F \otimes A \otimes x \leq \psi$. Since $F \otimes A = H \otimes F$, we only need to show that $\phi \leq H \otimes F \otimes x \leq \psi$. Since $\mathcal{P}(I_n, \phi, \psi)$ is H-positively invariant, for all $x \in R^n$, $\phi \leq x \leq \psi$, we have $\phi \leq H \otimes x \leq \psi$. Therefore, $F \otimes x \in \mathcal{P}(I_n, \phi, \psi)$. We have $\phi \leq H \otimes F \otimes x \leq \psi$. Thus, $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics of the system (7). $\diamondsuit$

Proof of Proposition 4: We need to prove for any $\phi \leq F \otimes x \leq \psi$, we have $\phi \leq F \otimes \widetilde{A} \otimes x \leq \psi$ for arbitrary $\widetilde{A} \in co\{A_1, \cdots, A_m\}$. Since $F \otimes A_i = H_i \otimes F$ and $\mathcal{P}(I_n, \phi, \psi)$ is H_i-positively invariant, for all $x \in R^n$, $\phi \leq x \leq \psi$, we have $\phi \leq H_i \otimes x \leq \psi$. Therefore, $F \otimes x \in \mathcal{P}(I_n, \phi, \psi)$. We have

$$\phi \leq H_i \otimes F \otimes x = F \otimes A_i \otimes x \leq \psi,$$

for all $i \in \{1, \cdots, m\}$. Adding them together to obtain

$$\phi = \bigoplus_{i=1}^{m} \lambda_i \phi \leq \bigoplus_{i=1}^{m} (\lambda_i \otimes F \otimes A_i \otimes x) \leq \bigoplus_{i=1}^{m} \lambda_i \psi = \psi,$$

where $\bigoplus_{i=1}^{m} (\lambda_i \otimes F \otimes A_i \otimes x) = F \otimes (\bigoplus_{i=1}^{m} \lambda_i \otimes A_i) \otimes x = F \otimes \widetilde{A} \otimes x$. Therefore, we have $\phi \leq F \otimes \widetilde{A} \otimes x \leq \psi$ for all $\phi \leq F \otimes x \leq \psi$. Thus, $\mathcal{P}(F, \phi, \psi)$ is positively invariant under the dynamics of the system (7). $\diamondsuit$

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