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Coherent H^∞ Control for a Class of Linear Complex Quantum Systems.

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***Abstract*—This paper considers a coherent H^∞ control problem for a class of linear quantum systems which can be defined by complex quantum stochastic differential equations in terms of annihilation operators only. For this class of quantum systems, a solution to the H^∞ control problem can be obtained in terms of a pair of complex Riccati equations. The paper also considers complex versions of the Bounded Real Lemma, the Strict Bounded Real Lemma and the Lossless Bounded Real Lemma. For the class of quantum systems under consideration, the question of physical realizability is related to the Bounded Real and Lossless Bounded Real properties.**

***Index Terms*—Quantum Feedback Control, H^∞ control, dissipativity, complex strict bounded real lemma, quantum optics, complex Riccati equations, quantum controller realization.**

I. Introduction

Robustness is an important issue in the control of quantum feedback systems; e.g. see [4], [5], [15], [13], [14] and [9]. In the recent paper [9], the problem of systematic robust control system design for quantum systems is tackled via an H^∞ approach. In [9], this problem was addressed by considering real and imaginary quadratures of the quantum system variables. This made the derivations quite complicated as the equations involving the complex annihilation operator were converted to a form involving the real quadratures. In this paper, to simplify the work of [9], we consider a class of linear quantum systems, which can be modeled purely in terms of the annihilation operator and not the creation operator. The class of quantum systems considered in this paper includes important 'passive' systems from the field of quantum optics such as interconnections of cavities, phase shifters and beam splitters; see [1]. For this class of quantum systems, the system can be described by complex linear quantum stochastic differential equations in terms of the annihilation operator. The dimension of this set of equations is half that which would be obtained by considering real quantum stochastic differential equations defined in terms of quadratures which was the case in [9]. Moreover, a solution to the quantum H^∞ control problem is obtained in terms of a pair of complex Riccati equations. The dimension of these Riccati equations is half that of the real Riccati equations obtained in [9].

In the quantum H^∞ control problem considered in this paper, we wish to construct a coherent quantum controller which is also in the class of quantum systems being considered. A connection is derived between the linear complex quantum stochastic differential equations (QSDEs) under consideration and a corresponding real QSDE defined in terms of quadrature variables. This connection is used in most of the proofs. In addition to simplifying the formulas, our derivation leads to more physical insight and better understanding of the underlying quantum systems. The paper presents complex versions of the Bounded Real, Strict Bounded Real and Lossless Bounded Real Lemmas. While the Strict Bounded Real Lemma plays an important role in the solution to the H^∞ control problem, the Bounded Real Lemma and the Lossless Bounded Real Lemma are found to be connected to the issue of physical realizability for the class of complex linear quantum systems under consideration. Also, this paper makes a contribution in considering a classical problem of complex H^∞ control which has not been investigated previously. Such a complex H^∞ control problem may have applications in other areas. This conference version of the paper only presents the results. All proofs will be presented in the full version of the paper.

II. A Class of Linear Complex Quantum Systems

The class of complex linear quantum systems under consideration can be described by using non-commutative or quantum probability theory [3]. In particular, the systems are described in terms of the complex annihilation operator by the quantum stochastic differential equations (QSDEs)

$$\begin{aligned} da(t) &= Fa(t)dt + Gdw(t); \qquad a(0) = a_0 \\ dy(t) &= Ha(t)dt + Jdw(t) \end{aligned} \tag{1}$$

where $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{n\times n_w}$, $H \in \mathbb{C}^{n_y\times n}$ and $J \in \mathbb{C}^{n_y\times n_w}$ (n, n_w, n_y are positive integers). Here $a(t) = [a_1(t)\cdots a_n(t)]^T$ is a vector of (linear combinations of) annihilation operators. The vector w represents the input signals and is assumed to admit the decomposition:

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is a self-adjoint adapted process (see [3], [11] and [8]). The noise $\tilde{w}(t)$ is a vector of quantum noises with Ito table

$$d\tilde{w}(t)d\tilde{w}^\dagger(t) = F_{\tilde{w}}dt$$

(see [2] and [11]) where F_w is a non-negative definite Hermitian matrix. Here the notation † represents the adjoint transpose

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of a vector of operators. Also, we assume the following commutation relations hold for the noise components:

$$\left[d\tilde{w}(t), d\tilde{w}^{\dagger}(t)\right] \triangleq d\tilde{w}(t)d\tilde{w}^{\dagger}(t) - (d\tilde{w}^{*}(t)d\tilde{w}^{T}(t))^{T} = T_w dt. \tag{2}$$

Here T_w is a Hermitian matrix. The noise processes can be represented as operators on an appropriate Fock space (for more details see [2] and [11]).

The process $\beta_w(t)$ represents variables of other systems which may be passed to the system (1) via an interaction. Therefore, it is required that $\beta_w(0)$ be an operator on a Hilbert space distinct from that of a_0 and the noise processes. We also assume that $\beta_w(t)$ commutes with $a(t)$ for all $t \geq 0$. Moreover, since $\beta_w(t)$ is an adapted process, we note that $\beta_w(t)$ also commutes with $d\tilde{w}(t)$ for all $t \geq 0$.

For simplicity, we make the following assumption on system (1): $n_w = n_y$. Equation (1) is a linear quantum stochastic differential equation in terms of the annihilation operator. In (1) the integral with respect to $dw(t)$ is considered to be a quantum stochastic integral. The solution $a(t)$ is adapted, and $d\tilde{w}(t)$ commutes with $a(t)$. The relation between our system defined in (1) in terms of annihilation operators and the system considered in [9] defined in terms of real quadrature states is described in the following section.

III. A Relation between real and complex linear quantum systems

It is straightforward to verify that any quantum system of the form (1) is equivalent to a real linear quantum system of the form considered in [9] defined as follows:

$$\begin{aligned} dx(t) &= Ax(t)dt + Bd\bar{w}(t); \qquad x(0) = x_0 \\ d\bar{y}(t) &= Cx(t)dt + Dd\bar{w}(t) \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{2n\times 2n}$, $B \in \mathbb{R}^{2n\times 2n_w}$, $C \in \mathbb{R}^{2n_y\times 2n}$ and $D \in \mathbb{R}^{2n_y\times 2n_w}$. Here $x(t) = [x_1(t) \cdots x_n(t)]^T$ is a vector of self-adjoint possibly non-commutative system variables. Also $d\bar{w}(t)$ is given by:

$$d\bar{w}(t) = \beta_{\bar{w}}(t)dt + d\bar{\tilde{w}}(t).$$

The variables defining the system (3) can be constructed from the variables defining the system (1) as follows where * denotes the adjoint in the case of operators.

$$x = \begin{bmatrix} a + a^* \\ -i(a - a^*) \end{bmatrix}; \quad \bar{w} = \begin{bmatrix} w + w^* \\ -i(w - w^*) \end{bmatrix};$$

$$\bar{y} = \begin{bmatrix} y + y^* \\ -i(y - y^*) \end{bmatrix}; \quad \beta_{\bar{w}} = \begin{bmatrix} \beta_w + \beta_w^* \\ -i\left(\beta_w - \beta_w^*\right) \end{bmatrix}.$$

Also, the matrices in this corresponding real linear quantum system can be constructed as follows where * denotes the complex conjugate in the case of complex matrices.

$$A = \frac{1}{2}\begin{bmatrix} F + F^* & i(F - F^*) \\ -i(F - F^*) & F + F^* \end{bmatrix};$$

$$B = \frac{1}{2}\begin{bmatrix} G + G^* & i(G - G^*) \\ -i(G - G^*) & G + G^* \end{bmatrix};$$

$$C = \frac{1}{2}\begin{bmatrix} H + H^* & i(H - H^*) \\ -i(H - H^*) & H + H^* \end{bmatrix};$$

$$D = \frac{1}{2}\begin{bmatrix} J + J^* & i(J - J^*) \\ -i(J - J^*) & J + J^* \end{bmatrix}. \tag{4}$$

From this, it follows that any linear quantum system with real quadratures of the form (3) (i.e., a system of the form considered in [9]) with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

will be equivalent to a complex linear quantum system of the form (1) if and only if $A_{11} = A_{22}$, $A_{12} = -A_{21}$, $B_{11} = B_{22}$, $B_{12} = -B_{21}$, $C_{11} = C_{22}$, $C_{12} = -C_{21}$, $D_{11} = D_{22}$ and $D_{12} = -D_{21}$. In this case, the complex matrices defining the system (1) are given by $F = A_{11} - iA_{12}$, $G = B_{11} - iB_{12}$, $H = C_{11} - iC_{12}$ and $J = D_{11} - iD_{12}$.

IV. Commutation relations

The initial system variables $a(0) = a_0$ consist of operators satisfying the commutation relations

$$[a_j(0), a_k^*(0)] = \Theta_{jk}, \qquad j, k = 1, \ldots, n. \tag{5}$$

Here the commutator is defined by $[a_j, a_k^*] = a_j a_k^* - a_k^* a_j = \Theta_{jk}$ with $\left[a_j, a_j^*\right] = a_j a_j^* - a_j^* a_j = 1$ where * denotes the adjoint and Θ is a complex matrix with elements Θ_{jk}. With $a^T = (a_1, \cdots, a_n)$, the relations (5) can be written as

$$\left[a, a^{\dagger}\right] \quad = \quad aa^{\dagger} - (a^* a^T)^T = \Theta.$$

A. Preservation of the commutation relations

The following theorem provides an algebraic characterization of when the complex linear system (1) preserves the commutation relations as time evolves. A corresponding condition was derived for the systems considered in [9].

Theorem 4.1: For system (1), we have that $[a_j(0), a_k^*(0)] = \Theta_{jk}$ implies $[a_j(t), a_k^*(t)] = \Theta_{jk}$ for all $t \geq 0$ with $j, k = 1 \ldots n$ if and only if

$$F\Theta + \Theta F^{\dagger} + GT_w G^{\dagger} = 0. \tag{6}$$

Here the notation † refers to the complex conjugate transpose of a complex matrix. We consider two special cases of the commutation relations and the quantum Wiener process Ito and commutation matrices defined in the following subsections.

B. Canonical case

In the canonical case, $\Theta = I$, $F_w = I$, $T_w = I$ and the commutation relations are preserved if and only if

$$F + F^{\dagger} + GG^{\dagger} = 0. \tag{7}$$

C. Generalized canonical case

In the generalized canonical case, Θ is a positive definite Hermitian matrix, $F_w = I$, $T_w = I$ and the commutation relations are preserved if and only if

$$F\Theta + \Theta F^\dagger + GG^\dagger = 0. \tag{8}$$

Theorem 4.2: Suppose the system (1) satisfies (8) with $\Theta = \Theta^\dagger > 0$. Then there exists a state transformation $\tilde{a} = Sa$ such that the corresponding transformed system satisfies (7) with $\Theta = I$ and $T_w = I$.

V. Open Quantum Harmonic Oscillators and Physical Realizability

In this section, we consider conditions under which a complex linear quantum system of the form (1) corresponds to a complex open quantum harmonic oscillator. The class of complex open harmonic oscillators under consideration (see [7], [11], [3], [9] and [6]) are defined by n_w measurement channels coupled via the operator $L = \Lambda a$ (Λ is a complex $n_w \times n$ matrix) and a Hamiltonian $\mathcal{H} = a^\dagger M a$ where M is a $n \times n$ complex Hermitian matrix. To derive a system of the form (1) from a complex open harmonic oscillator defined by L and $\mathcal{H}$, first note that as in [6], the Lindblad generator corresponding to the coupling operator L and the Hamiltonian $\mathcal{H}$ is given by:

$$\mathcal{L}[a] = i[\mathcal{H}, a] + \frac{1}{2}\left(L^\dagger [a, L] + \left[L^\dagger, a\right] L\right)$$

where $L^\dagger [a, L] = 0$ and $\left[L^\dagger, a\right] L = -\Theta\Lambda^\dagger\Lambda a$. Hence, $\frac{1}{2}\left(L^\dagger [a, L] + \left[L^\dagger, a\right] L\right) = -\frac{1}{2}\Theta\Lambda^\dagger\Lambda a$. Also, $i[\mathcal{H}, a] = -i\Theta M a$. Thus,

$$\mathcal{L}[a] = -\Theta\left(iM + \frac{1}{2}\Lambda^\dagger\Lambda\right) a.$$

Now, as in [6], the quantum Langevin equation corresponding to the coupling operator L and the Hamiltonian $\mathcal{H}$ is given by:

$$da = \mathcal{L}[a] \otimes dt + [a, L] \otimes dw^* - \left[a, L^\dagger\right] \otimes dw$$

where $[a, L] = 0$ and $\left[a, L^\dagger\right] = \Theta\Lambda^\dagger$.

Hence, we can write:

$$da = -\Theta\left(iM + \frac{1}{2}\Lambda^\dagger\Lambda\right) a dt - \Theta\Lambda^\dagger (dw). \tag{9}$$

Thus, by comparing (9) to (1), we can write:

$$F = -\Theta\left(iM + \frac{1}{2}\Lambda^\dagger\Lambda\right)$$

and

$$G = -\Theta\Lambda^\dagger.$$

Extending the approach of [6] to consider both quadratures of the measurement channel, the output equation is given by:

$$dy(t) = L \otimes dt + I \otimes dw$$

where $L = \Lambda a$. From this it follows that

$$dy(t) = \Lambda a dt + dw. \tag{10}$$

Thus, by comparing (10) and (1) we can write:

$$H = \Lambda$$

and

$$J = I_{n_y}.$$

In this case, we say that the complex linear quantum system defined by these matrices F, G, H, J is a representation of the open quantum harmonic oscillator defined by the coupling operator matrix Λ and the Hamiltonian matrix M with the commutation matrix Θ. The above calculations lead us to the following definitions.

A. Physical Realizability

Definition 5.1: A complex linear quantum system of the form (1) is said to be canonically physically realizable if it satisfies the canonical commutation relations and is a representation of a complex open harmonic oscillator with $\Theta = I$.

Definition 5.2: A complex linear quantum system of the form (1) is said to be physically realizable if it satisfies the generalized commutation relations with $\Theta = \Theta^\dagger > 0$, $T_w = I$ and is a representation of a complex harmonic oscillator with commutation matrix $\Theta > 0$.

These definitions represent a special class of open quantum harmonic oscillators (i.e., 'canonical'). The following theorem provides necessary and sufficient conditions for being physically realizable.

Theorem 5.1: A complex linear quantum system of the form (1) is physically realizable if and only if there exists $\Theta = \Theta^\dagger > 0$ such that

$$\begin{aligned} F\Theta + \Theta F^\dagger + GG^\dagger &= 0; \\ G &= -\Theta H^\dagger; \\ J &= I_{n_y}. \end{aligned} \tag{11}$$

In this case, the Hamiltonian matrix *M* is given by:

$$M = \frac{i}{2}\left(\Theta^{-1}F - F^\dagger\Theta^{-1}\right) \tag{12}$$

and the corresponding coupling matrix Λ is given by:

$$\Lambda = H. \tag{13}$$

Note that *M* is a complex Hermitian matrix. Furthermore, the system (1) is canonically physically realizable if and only if the condition (11) is satisfied with $\Theta = I$. In this case, the corresponding Hamiltonian matrix and coupling matrix are given as in (12) and (13) with $\Theta = I$.

Theorem 5.2: If the system (1) is physically realizable then there exists a state space transformation $\tilde{a} = Sa$ such that the resulting transformed system is canonically physically realizable with $\Theta = I$.

VI. Bounded Real Lemmas for Linear Complex Quantum Systems

In this section, we consider complex quantum systems of the form (1) and we give a complex version of the Strict Bounded Real Lemma which will be used later for our quantum H^∞ controller synthesis result. Also, we give complex versions of the Bounded Real Lemma and the Lossless Bounded Real Lemma which are found to be connected to the issue of physical realizability for the class of complex linear quantum systems considered in this paper.

Definition 6.1: Given an operator-valued quadratic form

$$r(a, \beta_w) = \frac{1}{2}\left[\begin{array}{cc} a^\dagger & \beta_w^\dagger \end{array}\right] R \left[\begin{array}{c} a \\ \beta_w \end{array}\right] + \frac{1}{2}\left[\begin{array}{cc} a^T & \beta_w^T \end{array}\right] R \left[\begin{array}{c} a^* \\ \beta_w^* \end{array}\right]$$

where

$$R = \left[\begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array}\right]$$

is a given complex Hermitian matrix, we say that the system (1) is *dissipative with supply rate* $r(a, \beta_w)$ if there exists a positive operator-valued quadratic form $V(a) = \frac{1}{2}a^\dagger X a + \frac{1}{2}a^T X a^*$ (where X is a positive definite Hermitian matrix) and a constant $\lambda > 0$ such that

$$\langle V(a(t))\rangle + \int_0^t \langle r(a(s), \beta_w(s))\rangle ds \leq \langle V(a(0))\rangle + \lambda t, \forall t > 0 \tag{14}$$

for all Gaussian states ρ for the initial variables $a(0)$. Note that we use the shorthand notation $\langle . \rangle$ for expectation over all initial variables and noises. We say that the system (1) is *strictly dissipative* if there exists a constant $\epsilon > 0$ such that inequality (14) holds with the matrix R replaced by the matrix $R + \epsilon I$.

Definition 6.2: The complex quantum stochastic system (1) is said to be *Bounded Real with disturbance attenuation g* if the system (1) is dissipative with supply rate

$$\begin{aligned} r(a, \beta_w) &= \frac{1}{2}(\beta_z^\dagger \beta_z - g^2 \beta_w^\dagger \beta_w) + \frac{1}{2}(\beta_z^T \beta_z^* - g^2 \beta_w^T \beta_w^*) \\ &= \frac{1}{2}\left[\begin{array}{cc} a^\dagger & \beta_w^\dagger \end{array}\right] \left[\begin{array}{cc} H^\dagger H & H^\dagger J \\ J^\dagger H & J^\dagger J - g^2 I \end{array}\right] \left[\begin{array}{c} a \\ \beta_w \end{array}\right] \\ &+ \frac{1}{2}\left[\begin{array}{cc} a^T & \beta_w^T \end{array}\right] \left[\begin{array}{cc} H^\dagger H & H^\dagger J \\ J^\dagger H & J^\dagger J - g^2 I \end{array}\right] \left[\begin{array}{c} a^* \\ \beta_w^* \end{array}\right] \end{aligned} \tag{15}$$

where $\beta_z(t) = Ha(t) + J\beta_w(t)$. Also, the complex quantum stochastic system (1) is said to be *Strictly Bounded Real with disturbance attenuation g* if the system (1) is strictly dissipative with this supply rate.

These definitions generalize the corresponding definitions given in [9] to the class of complex quantum systems considered here. Note, in the case that the disturbance attenuation parameter *g* is equal to one, the system is said to be *Bounded Real* or *Strictly Bounded Real* respectively. Also, note that the above definitions relate to the dissipativity properties of the quantum system. However, the following definition of Lossless Bounded Real relates to the transfer function of the system. In the sequel, we will see that all of these notions are connected.

Definition 6.3: The complex quantum stochastic system (1) is said to be *lossless bounded real* if

i) F is a Hurwitz matrix; i.e., all of its eigenvalues have strictly negative real parts.
ii) The transfer function matrix $Q(s) = H(sI-F)^{-1}G+J$ satisfies $Q(i\omega)^\dagger Q(i\omega) = I \ \forall \omega \in \mathbb{R}$

The following definition extends the standard linear systems notion of minimal realization to linear complex quantum systems of the form (1).

Definition 6.4: A complex quantum system of the form (1) is said to be minimal if the following conditions hold:

i) *Controllability.* $a^\dagger F = \lambda a^\dagger$ for some $\lambda \in \mathbb{C}$ and $a^\dagger G = 0$ implies $a = 0$;
ii) *Observability.* $Fa = \lambda a$ for some $\lambda \in \mathbb{C}$ and $Ha = 0$ implies $a = 0$.

Theorem 6.1: (Complex Bounded Real Lemma, Part I). The complex quantum system (1) is bounded real with disturbance attenuation g if and only if there exists a complex positive definite Hermitian matrix X such that the following inequality is satisfied:

$$\left(\begin{array}{cc} F^\dagger X + XF + H^\dagger H & G^\dagger X + J^\dagger H \\ H^\dagger J + XG & J^\dagger J - g^2 I_{n_w} \end{array}\right) \leq 0. \tag{16}$$

Theorem 6.2: (Complex Bounded Real Lemma, Part II.) Suppose the system (1) is minimal and satisfies $g^2 I - J^\dagger J > 0$. Then the following statements are equivalent:

i) The system is bounded real with disturbance attenuation g.
ii) F is Hurwitz and $\left\| H(sI-F)^{-1} G + J \right\|_\infty \leq g$.
iii) The Riccati equation

$$\begin{aligned} &F^\dagger X + XF + H^\dagger H \\ &+(XG + H^\dagger J)(g^2 I - J^\dagger J)^{-1}(G^\dagger X + J^\dagger H) = 0 \end{aligned}$$

has a Hermitian solution $X = X^\dagger > 0$.

Theorem 6.3: (Complex Strict Bounded Real Lemma, Part I.) The complex quantum system (1) is strictly bounded real with disturbance attenuation g if and only if $g^2 I - J^\dagger J > 0$ and there exists a complex positive definite Hermitian matrix X such that the following inequality is satisfied:

$$\left(\begin{array}{cc} F^\dagger X + XF + H^\dagger H & G^\dagger X + J^\dagger H \\ H^\dagger J + XG & J^\dagger J - g^2 I_{n_w} \end{array}\right) < 0. \tag{17}$$

Definition 6.5: A Hermitian matrix $X = X^\dagger$ is said to be a *stabilizing solution* to a complex algebraic Riccati equation (with $N \geq 0$)

$$F^\dagger X + XF + XMX + N = 0$$

if it satisfies the Riccati equation and the matrix $F + MX$ is Hurwitz.

Now combining Theorem 6.3 with the standard Strict Bounded Real Lemma (e.g., refer to [12], [16]), we obtain the following corollary.

Corollary 6.1: (Complex Strict Bounded Real Lemma, Part II.) The following statements are equivalent for the linear complex quantum system (1) :

i) The complex quantum stochastic system (1) is strictly bounded real with disturbance attenuation g.
ii) F is Hurwitz and $\left\| H(sI_n - F)^{-1}G + J \right\|_\infty < g$.
iii) $g^2 I_{n_w} - J^\dagger J > 0$ and there exists a Hermitian positive definite matrix $\tilde{X} > 0$ such that

$$\left(F^\dagger \tilde{X} + \tilde{X} F + H^\dagger H\right) + \left(\tilde{X} G + H^\dagger J\right) \times \left(g^2 I_{n_w} - J^\dagger J\right)^{-1} \left(G^\dagger \tilde{X} + J^\dagger H\right) < 0.$$

iv) $g^2 I_{n_w} - J^\dagger J > 0$ and the complex algebraic Riccati equation

$$\left(F^\dagger X + XF + H^\dagger H\right) + \left(XG + H^\dagger J\right) \times \left(g^2 I_{n_w} - J^\dagger J\right)^{-1} \left(G^\dagger X + J^\dagger H\right) = 0$$

has a stabilizing solution $X \geq 0$.

Furthermore if these statements hold then $X < \tilde{X}$.

Theorem 6.4: (Complex Lossless Bounded Real Lemma.) Suppose the system (1) is minimal. Then the system (1) is lossless bounded real if and only if there exists a complex Hermitian matrix $X > 0$ such that

$$\begin{aligned} XF + F^\dagger X + H^\dagger H &= 0; \\ H^\dagger J &= -XG; \\ J^\dagger J &= I. \end{aligned} \tag{18}$$

Combining this result with Theorem 5.1, we obtain the following theorem on the physical realizability of the linear complex quantum systems under consideration.

Theorem 6.5: A minimal quantum system of the form (1) is physically realizable if and only if $J = I$ and the system is lossless bounded real.

VII. Coherent H^∞ Controller Synthesis

In this section, we consider the problem of H^∞ controller design for complex linear quantum systems of the form (1). The closed-loop plant-controller system is defined in Subsection VII-A, and then in Subsection VII-C, we apply the complex Strict Bounded Real Lemma to the closed-loop system to obtain our main results. Subsection VII-D provides conditions under which the controller is physically realizable.

A. The Closed-Loop Plant-Controller System

We now introduce our plant and controller models, and the resulting closed-loop system.

1) The Plant Model: We consider *plants* described by non-commutative stochastic models in terms of the complex annihilation operator of the following form defined in an analogous way to the quantum system (1):

$$\begin{aligned} da(t) &= Fa(t)\,dt + \begin{bmatrix} G_0 & G_1 & G_2 \end{bmatrix} \\ &\quad \begin{bmatrix} dv(t)^T & dw(t)^T & du(t)^T \end{bmatrix}^T; a(0) = a_0; \\ dz(t) &= H_1 a(t)\,dt + J_{12} du(t); \\ dy(t) &= H_2 a(t)\,dt + \begin{bmatrix} J_{20} & J_{21} & 0_{n_y \times n_u} \end{bmatrix} \\ &\quad \begin{bmatrix} dv(t)^T & dw(t)^T & du(t)^T \end{bmatrix}^T \end{aligned} \tag{19}$$

where $F \in \mathbb{C}^{n\times n}$, $G_0 \in \mathbb{C}^{n\times n_v}$, $G_1 \in \mathbb{C}^{n\times n_w}$, $G_2 \in \mathbb{C}^{n\times n_u}$, $v \in \mathbb{C}^{n_v\times 1}$, $w \in \mathbb{C}^{n_w\times 1}$, $u \in \mathbb{C}^{n_u\times 1}$, $H_1 \in \mathbb{C}^{n_z\times n}$, $J_{12} \in \mathbb{C}^{n_z\times n_u}$, $H_2 \in \mathbb{C}^{n_y\times n}$, $J_{20} \in \mathbb{C}^{n_y\times n_v}$ and $J_{21} \in \mathbb{C}^{n_y\times n_w}$.

Here, the input $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$ represents a disturbance signal. The signal $u(t)$ is a control input of the form $du(t) = \beta_u(t)dt + d\tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is the adapted, self-adjoint finite variation part of $u(t)$. The quantity $dv(t)$ represents any additional quantum noise in the plant. The vectors $v(t)$, $\tilde{w}(t)$ and $\tilde{u}(t)$ are quantum noises with Ito matrices F_v, $F_{\tilde{w}}$ and $F_{\tilde{u}}$ and commutation matrices T_v, T_w and T_u respectively. We assume in this paper that $F_v = F_{\tilde{w}} = F_{\tilde{u}} = I$ and $T_v = T_w = T_u = I$.

2) The Controller Model: The controllers to be considered are assumed to be non-commutative linear complex stochastic quantum systems of the form (1) defined as follows:

$$\begin{aligned} d\xi(t) &= F_c \xi(t)\,dt + \begin{bmatrix} G_{c_0} & G_{c_1} & G_c \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix}; \\ &\quad \xi(0) = \xi_0; \\ du(t) &= H_c \xi(t)\,dt + dw_{c_0} \end{aligned} \tag{20}$$

where $\xi(t) = [\xi_1(t) \ldots \xi_{n_k}(t)]^T$ is a vector of annihilation operator controller variables.

The quantum noises $dw_{c_0}(t)$ and $dw_{c_1}(t)$ are vectors of non-commutative quantum Wiener processes with Ito matrices $F_{w_{c_0}} = F_{w_{c_1}} = I$ and commutation matrices $T_{w_{c_0}} = T_{w_{c_1}} = I$. At time $t = 0$, we also assume that $a(0)$ commutes with $\xi(0)$. Also, $\xi \in \mathbb{C}^{n_c\times 1}$, $F_c \in \mathbb{C}^{n_c\times n_c}$, $G_c \in \mathbb{C}^{n_c\times n_y}$, $G_{c_0} \in \mathbb{C}^{n_c\times n_{c_0}}$, $u \in \mathbb{C}^{n_u\times 1}$, $H_c \in \mathbb{C}^{n_u\times n_c}$, $G_{c_1} \in \mathbb{C}^{n_c\times n_{c_1}}$, $y \in \mathbb{C}^{n_y\times 1}$, $w_{c_0} \in \mathbb{C}^{n_{c_0}\times 1}$ and $w_{c_1} \in \mathbb{C}^{n_{c_1}\times 1}$.

3) The Closed-Loop System Equations: The closed-loop system is obtained by making the identification $\beta_u(t) = H_c\xi(t)$ and interconnecting equations (19) and (20) to give:

$$\begin{aligned} d\eta(t) &= \begin{bmatrix} F & G_2 H_c \\ G_c H_2 & F_c \end{bmatrix} \eta(t)\,dt + \\ &\quad \begin{bmatrix} G_0 & G_2 & 0 \\ G_c J_{20} & G_{c_0} & G_{c_1} \end{bmatrix} \begin{bmatrix} dv(t) \\ dw_{c_0}(t) \\ dw_{c_1}(t) \end{bmatrix} + \\ &\quad \begin{bmatrix} G_1 \\ G_c J_{21} \end{bmatrix} dw(t); \\ dz(t) &= \begin{bmatrix} H_1 & J_{12} H_c \end{bmatrix} \eta(t)\,dt + \begin{bmatrix} 0 & J_{12} & 0 \end{bmatrix} \\ &\quad \begin{bmatrix} dv(t) \\ dw_{c_0}(t) \\ dw_{c_1}(t) \end{bmatrix} \end{aligned} \tag{21}$$

where $\eta(t) = \begin{bmatrix} a^T(t) & \xi^T(t) \end{bmatrix}^T$. That is, we can write:

$$\begin{aligned} d\eta(t) &= \tilde{F}\eta(t)\,dt + \tilde{G}dw(t) + \tilde{L}d\tilde{v}(t) \\ &= \tilde{F}\eta(t)\,dt + \begin{bmatrix} \tilde{G} & \tilde{L} \end{bmatrix} \begin{bmatrix} dw(t) \\ d\tilde{v}(t) \end{bmatrix}; \\ dz(t) &= \tilde{H}\eta(t)\,dt + \tilde{N}d\tilde{v}(t) = \tilde{H}\eta(t)\,dt + \\ & \begin{bmatrix} 0 & \tilde{N} \end{bmatrix} \begin{bmatrix} dw(t) \\ d\tilde{v}(t) \end{bmatrix} \end{aligned} \quad (22)$$

where

$$\tilde{v}(t) = \begin{bmatrix} v(t) \\ w_{c_0}(t) \\ w_{c_1}(t) \end{bmatrix}; \quad \tilde{F} = \begin{bmatrix} F & G_2H_c \\ G_cH_2 & F_c \end{bmatrix};$$

$$\tilde{G} = \begin{bmatrix} G_1 \\ G_cJ_{21} \end{bmatrix}; \quad \tilde{L} = \begin{bmatrix} G_0 & G_2 & 0 \\ G_cJ_{20} & G_{c_0} & G_{c_1} \end{bmatrix};$$

$$\tilde{H} = \begin{bmatrix} H_1 & J_{12}H_c \end{bmatrix}; \quad \tilde{N} = \begin{bmatrix} 0 & J_{12} & 0 \end{bmatrix}.$$

Here, $\tilde{v} \in \mathbb{C}^{(n_v+n_{w_c})\times 1}$, $\tilde{F} \in \mathbb{C}^{(n+n_c)\times(n+n_c)}$, $\tilde{G} \in \mathbb{C}^{(n+n_c)\times n_w}$, $\tilde{L} \in \mathbb{C}^{(n+n_c)\times(n_v+n_{w_c})}$, $\tilde{H} \in \mathbb{C}^{n_z\times(n+n_c)}$ and $\tilde{N} \in \mathbb{C}^{(n_z\times(n_{w_c}+n_v))}$ with $n_{w_c} = n_{c_0} + n_{c_1}$.

B. H^∞ Control Objective

The objective in our H^∞ controller problem is to find a controller of the form (20) such that for a given disturbance attenuation parameter $g > 0$, then

$$\begin{aligned} &\int_0^t \left\langle z^\dagger(s)\,z(s) + \varepsilon a^\dagger(s)\,a(s)\right\rangle ds \\ &\leq \left(g^2 - \varepsilon^2\right)\int_0^t \left\langle \beta_w^\dagger(s)\,\beta_w(s)\right\rangle ds + \mu_1 + \mu_2 t \end{aligned} \quad (23)$$

for some real constants ε, μ_1, $\mu_2 > 0$. Thus, the controller bounds the effect of the 'energy' in the disturbance signal $\beta_w(t)$ on the 'energy' of the signal $z(t)$. In particular, we require that the closed loop system is strictly bounded real with disturbance attenuation g. Necessary and sufficient conditions for the existence of a controller which achieves this goal for a given g are given in the next section, as well as explicit formulas for F_c, G_c and H_c.

C. Necessary and Sufficient Conditions

In order to establish our results on H^∞ control for the complex quantum systems under consideration, the plant (19) is assumed to satisfy the following assumptions.

Assumption 7.1:

i) $J_{12}^\dagger J_{12} = E_1 > 0$
ii) $J_{21}J_{21}^\dagger = E_2 > 0$
iii) The matrix $\begin{bmatrix} F - i\omega I_n & G_2 \\ H_1 & J_{12} \end{bmatrix}$ is full rank for all $\omega \geq 0$
iv) The matrix $\begin{bmatrix} F - i\omega I_n & G_1 \\ H_2 & J_{21} \end{bmatrix}$ is full rank for all $\omega \geq 0$.

The results will be stated in terms of the following pair of complex algebraic Riccati equations:

$$\begin{aligned} &\left(F - G_2E_1^{-1}J_{12}^\dagger H_1\right)^\dagger X + X\left(F - G_2E_1^{-1}J_{12}^\dagger H_1\right) \\ &+X\left(G_1G_1^\dagger - g^2G_2E_1^{-1}G_2^\dagger\right)X \\ &+g^{-2}H_1^\dagger\left(I - J_{12}E_1^{-1}J_{12}^\dagger\right)H_1 = 0; \end{aligned} \quad (24)$$

$$\begin{aligned} &\left(F - G_1J_{21}^\dagger E_2^{-1}H_2\right)Y + Y\left(F - G_1J_{21}^\dagger E_2^{-1}H_2\right)^\dagger \\ &+Y\left(g^{-2}H_1^\dagger H_1 - H_2^\dagger E_2^{-1}H_2\right)Y \\ &+G_1\left(I - J_{21}^\dagger E_2^{-1}J_{21}\right)G_1^\dagger = 0. \end{aligned} \quad (25)$$

The solutions to these Riccati equations will be required to satisfy the following assumption.

Assumption 7.2:

i) The matrix $F - G_2E_1^{-1}J_{12}^\dagger H_1 + \left(G_1G_1^\dagger - g^2G_2E_1^{-1}G_2^\dagger\right)X$ is Hurwitz.
ii) The matrix $F - G_1J_{21}^\dagger E_2^{-1}H_2 + Y\left(g^{-2}H_1^\dagger H_1 - H_2^\dagger E_2^{-1}H_2\right)$ is Hurwitz.
iii) The matrix XY has a spectral radius strictly less than one.

Our results will show that if the Riccati equations (24) and (25) have solutions satisfying Assumption 7.2, then a controller of the form (20) will solve the H^∞ control problem under consideration if its system matrices are constructed from the Riccati solutions as follows:

$$\begin{aligned} F_c &= F + G_2H_c - G_cH_2 + \left(G_1 - G_cJ_{21}\right)G_1^\dagger X; \\ G_c &= (I - YX)^{-1}\left(YH_2^\dagger + G_1J_{21}^\dagger\right)E_2^{-1}; \\ H_c &= -E_1^{-1}\left(g^2G_2^\dagger X + J_{21}^\dagger H_1\right). \end{aligned} \quad (26)$$

Theorem 7.1: Necessity: Consider the system (19) and assume that Assumption 7.1 is satisfied. If there exists a controller of the form (20) such that the resulting closed-loop system is strictly bounded real with disturbance attenuation g, then the Riccati equations (24) and (25) will have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying Assumption 7.2.
Sufficiency: Suppose the Riccati equations (24) and (25) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying Assumption 7.2. If the controller (20) is such that the matrices F_c, G_c, H_c are as defined in (26), then the resulting closed-loop system (21) will be strictly bounded real with disturbance attenuation g.

D. Physical Realizability of the H^∞ Controller

The H^∞ controller defined by the matrices F_c, G_c and H_c will not always be physically realizable within the class of linear complex quantum systems under consideration. We now provide conditions under which the H^∞ controller will be physically realizable within this class.

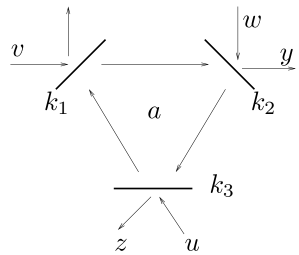

Fig. 1. An optical cavity

Definition 7.1: The matrices $\{F_c, G_c, H_c\}$ are said to define a *physically realizable controller* of the form (20) if there exists matrices G_{c_0}, G_{c_1}, H_{c_1} and H_{c_2} such that the system

$$\begin{aligned} d\xi(t) &= F_c\xi(t)\,dt + \begin{bmatrix} G_{c_0} & G_{c_1} & G_c \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix} \\ \begin{bmatrix} du \\ du_1 \\ du_2 \end{bmatrix} &= \begin{bmatrix} H_c \\ H_{c_1} \\ H_{c_2} \end{bmatrix} \xi(t)\,dt + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix} \end{aligned} \tag{27}$$

is physically realizable when $T_y = J_{20}T_vJ_{20}^\dagger + J_{21}T_wJ_{21}^\dagger = I$.

Theorem 7.2: Suppose the matrices $\{F_c, G_c, H_c\}$ are such the corresponding system is minimal. Then the matrices $\{F_c, G_c, H_c\}$ define a physically realizable controller of the form (20) if and only if F_c is Hurwitz and

$$\left\|H_c\left(sI - F_c\right)^{-1}G_c\right\|_\infty \leq 1; \tag{28}$$

i.e., the corresponding system is bounded real. In this case, the matrices G_{c_1} and H_{c_1} can be taken as zero; i.e., there is no need for the quantum noise term dw_{c_1} in the controller realization (20).

VIII. An example from quantum optics

In this section, we provide an example of H^∞ controller design for simple quantum optical plant consisting of an optical cavity coupled to optical fields; e.g., see [1], [7].

A. H^∞ controller synthesis

We consider an extension to one of the examples considered in [9] consisting of an optical cavity resonantly coupled to three optical channels v, w, u as in Figure 1. A modification of the example considered in [9] has also been implemented experimentally as described in [10]. The control objective is to attenuate the effect of the disturbance w on the output z.

The dynamics of the cavity system is described by the evolution of the complex annihilation operator (representing a traveling wave). The quantum noises v, $\tilde{w}$ have Ito matrices $F_v = F_{\tilde{w}} = 1$. This leads to a system of the form (19) with the following matrices: $F = -\frac{\gamma}{2}$; $G_0 = -\sqrt{k_1}$; $G_1 = -\sqrt{k_2}$; $G_2 = -\sqrt{k_3}$ with $\gamma = k_1+k_2+k_3$. Also, $H_1 = \sqrt{k_3}$; $J_{12} = 1$; $H_2 = \sqrt{k_2}$; $J_{21} = 1$. Here $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, are parameters of the cavity. In this model, the boson commutation relation $[a, a^*] = 1$ holds. This means that the commutation matrix for this plant is $\Theta = 1$. The Riccati equations (24) and (25) for this system are the following:

$$\left(-\frac{\gamma}{2} + k_3\right)X + X\left(-\frac{\gamma}{2} + k_3\right) + X\left(k_2 - g^2k_3\right)X = 0; \tag{29}$$

$$\left(-\frac{\gamma}{2} + k_2\right)Y + Y\left(-\frac{\gamma}{2} + k_2\right) + Y\left(g^{-2}k_3 - k_2\right)Y = 0. \tag{30}$$

A stabilizing solution for (29) satisfying Assumption 7.2 is $X = 0$ if $\gamma - 2k_3 > 0$ or $X = \frac{\gamma-2k_3}{k_2-g^2k_3}$ if $\gamma - 2k_3 < 0$. A stabilizing solution for (30) satisfying Assumption 7.2 is $Y = 0$ if $\gamma - 2k_2 > 0$ or $Y = \frac{\gamma-2k_2}{g^{-2}k_3-k_2}$ if $\gamma - 2k_2 < 0$. We note here that the inequalities $\gamma - 2k_3 < 0$ and $\gamma - 2k_2 < 0$ cannot both be satisfied because this will result $k_1 < 0$ which is not possible in this example. Thus, the matrix XY is equal to zero and has a spectral radius strictly less than one for any combination of the above values of X and Y.

It follows from Theorem 7.1 that if a controller of the form (20) is applied to this system with matrices F_c, G_c, H_c defined as in (26), then the resulting closed-loop system will be strictly bounded real with disturbance attenuation g. These matrices are given as follows.

Case 1: $X = 0$ and $Y = 0$:

$$\begin{aligned} F_c &= -\frac{\gamma}{2} + k_2 + k_3; \\ G_c &= -\sqrt{k_2}; \\ H_c &= -\sqrt{k_3}; \end{aligned}$$

This case corresponds to $k_1 > |k_3 - k_2|$. In this case, $X = 0$ and $Y = 0$ are suitable solutions to the Riccati equations for all values of the disturbance attenuation $g > 0$. Also, the H^∞ controller matrices are independent of g so that this controller achieves perfect disturbance attenuation.

Case 2: $X = 0$ and $Y = \frac{\gamma-2k_2}{g^{-2}k_3-k_2}$:

$$\begin{aligned} F_c &= \frac{-\gamma}{2} + k_3 - \frac{k_2\left(\gamma - k_2 - g^{-2}k_3\right)}{g^{-2}k_3 - k_2}; \\ G_c &= \sqrt{k_2}\left(\frac{\gamma - k_2 - g^{-2}k_3}{g^{-2}k_3 - k_2}\right); \\ H_c &= -\sqrt{k_3}. \end{aligned}$$

This case corresponds to $k_2 - k_3 \geq k_1 > 0$. In this case, the condition $Y \geq 0$ requires $g^2 > \frac{k_3}{k_2}$.

Case 3: $X = \frac{\gamma-2k_3}{k_2-g^2k_3}$ and $Y = 0$:

$$\begin{aligned} F_c &= -\frac{\gamma}{2} - k_3\left(\frac{g^2\left(\gamma - k_3\right) - k_2}{k_2 - g^2k_3}\right) + k_2; \\ G_c &= -\sqrt{k_2}; \\ H_c &= \sqrt{k_3}\left(\frac{g^2\left(\gamma - k_3\right) - k_2}{k_2 - g^2k_3}\right). \end{aligned}$$

This case corresponds to $k_3 - k_2 \geq k_1 > 0$. In this case, the condition $X \geq 0$ requires $g^2 > \frac{k_2}{k_3}$.

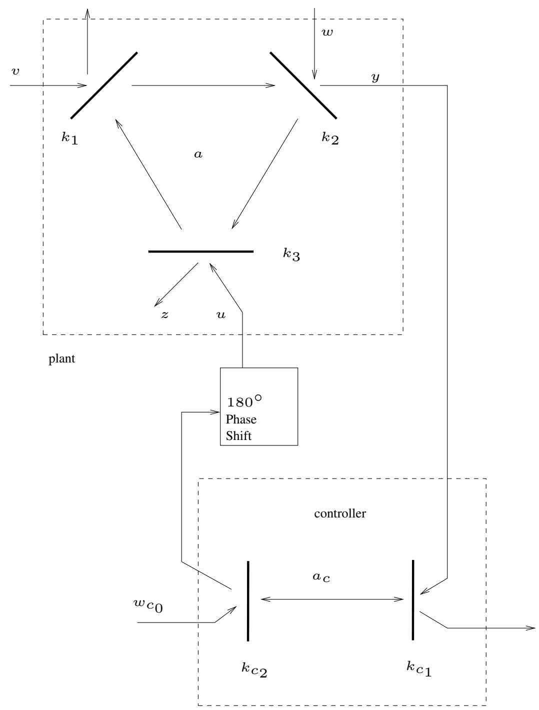

Fig. 2. An optical cavity quantum realization of the controller for the plant shown in Figure 1

B. Physical Realizability of the H^∞ Controller

We will concentrate on case 1. In this case, the condition F_c is Hurwitz is equivalent to $F_c < 0$ which holds if and only if $k_1 > k_2+k_3$. Also, the condition $\left\| H_c \left(sI - F_c\right)^{-1} G_c \right\|_\infty \leq 1$ is equivalent to $-\frac{H_c G_c}{F_c} \leq 1$ which holds if and only if $\sqrt{k_1} > \sqrt{k_2} + \sqrt{k_3}$. If this holds, the above condition for F_c Hurwitz will automatically hold.

One possible realization of the controller when $\sqrt{k_1} > \sqrt{k_2} + \sqrt{k_3}$ involves a two mirror cavity. Let

$$k_{c_1} = k_1 - k_2 - k_3 + \sqrt{\left(k_1 - k_2 - k_3\right)^2 - 4 k_2 k_3};$$

$$k_{c_2} = \frac{k_2 k_3}{k_1 - k_2 - k_3 + \sqrt{\left(k_1 - k_2 - k_3\right)^2 - 4 k_2 k_3}}.$$

This defines a two mirror cavity which is a physical implementation of a controller matrices $\tilde{F}_c = F_c = -\frac{k_{c_1}+k_{c_2}}{2}$, $\tilde{G}_c = -\sqrt{k_{c_1}}$ and $\tilde{H}_c = -\sqrt{k_{c_2}}$ provided a 180° phase shift is introduced on the output. These matrices are equivalent to the above controller matrices F_c, G_c, H_c via a state transformation and thus provides a physical realization of our H^∞ controller. Thus, the H^∞ controller can be implemented using a two mirror optical cavity as shown in Figure 2.

Remark 8.1: Even if the condition

$$\left\| H_c \left(sI - F_c\right)^{-1} G_c \right\|_\infty \leq 1$$

is not satisfied, the controller could be implemented as a quantum controller using the real 'quadrature' realization theory of [9] but 'active' optical components would be required.

Remark 8.2: If the 'central' H^∞ controller defined by the equations (26) does not satisfy this bounded real condition for physical realizability, it still may be possible that another controller solving the H^∞ control problem will satisfy the bounded real condition. Finding such a controller is a topic for future research.

IX. Conclusion

To conclude, this paper has formulated and solved an H^∞ control problem for a class of non-commutative complex stochastic models that arise in quantum technology including quantum optics. The results obtained were illustrated by an example from quantum optics.

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