2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Analysis of Periodic Motions in Relay Feedback Systems with Saturation in Plant Dynamics

N.V. Faldin, A.V. Morzhov, I.M. Boiko

Abstract—**A method of analysis of possible periodic motions in relay feedback systems and chattering in sliding-mode systems that have saturation in plant dynamics is proposed. The proposed analysis is based on the concept of the *state locus* of a relay feedback system. A few techniques of computing of the *state locus* are proposed. A methodology of finding a periodic motion that involves the *state locus* is given. The proposed method is illustrated by an example.**

I. Introduction

RELAY feedback systems are one of the most wide-spread types of nonlinear systems. The applications range from missile thrusters to numerous on-off regulators in process control systems.

The theory of relay feedback systems with linear plants is presented in a number of classical and recent publications [1]-[13]. Closely related with the former, the problem of chattering analysis in sliding-mode control systems was also considered in a number of papers (see [14]-[15] and references within). Yet, the presented results are applicable only to relay feedback systems with linear plants, while real plants in many cases feature nonlinear behavior. Very often plant dynamics involve saturation, which cannot be singled out into a simple "signal limiter" nonlinear function. This type of saturation can be a result of limited power of actuators as well as mechanical stops in motion control. Those types of saturation always exist in electrical, hydraulic and pneumatic servomechanisms and actuators. We will further refer to this type of saturation as *dynamic saturation*.

It is worth noting that due to different mathematical models, the saturation dynamics can be categorized into saturation due to limited power and due to mechanical stops. In the latter case, state trajectories are discontinuous because of the dynamics of the mechanical impact (in particular, the velocity is discontinuous). In the present paper, only the first type of saturation is considered.

The paper is organized as follows. At first, the model of *dynamic saturation* is considered. After that, the concept of the *state locus* of the relay system is given. A few techniques of computing of the *state locus* of the plant with saturation are considered. Finally, an example of analysis is presented.

N.V. Faldin and A.V. Morzhov are with Tula State University, 92 Lenina Ave., Tula, Russia (emails: faldin@sau.tsu.tula.ru and morzhov@mail.ru).
I.M. Boiko with Honeywell and University of Calgary, 2500 University Dr. NW, Calgary, Alberta, T2N 1N4, Canada (email: i.boiko@ieee.org).

II. Periodic motions in relay feedback systems

Let us consider the model of dynamic saturation. We shall denote this type of dynamics as presented in Fig. 1.

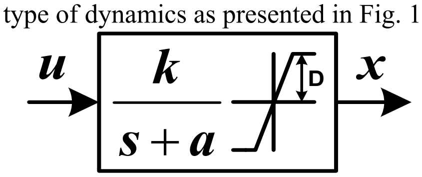

Fig. 1. First-order dynamics with saturation

These dynamics can be described as follows:

$$\dot{x} = \begin{cases} ku - ax, \text{ if } |x| < D, \text{ or } |x| = D \text{ and } (ku - ax)\cdot sign(x) \le 0; \\ 0, \text{ if } |x| = D \text{ and } (ku - ax)\cdot sign(x) > 0. \end{cases} \tag{1}$$

where $u \in \mathbf{R}^1$ is the control, $x \in \mathbf{R}^1$ is the output, k and a are parameters, s is the Laplace variable, D is the maximal value of x. In formula (1), expression $(ku - ax)\cdot sign(x)$ provides the value of the force applied to the mechanical stop (if speaking in terms of a mechanical motion). Strictly speaking, the force as a physical variable may not be present at all. Function $(ku - ax)\cdot sign(x)$ shows the possible motion from $x=D$ that would occur if there were no saturation.

It is worth noting that the dynamics of the mechanical motion with mechanical stops can also be reduced to expressions (1) if one assumes that the mass of the moving parts is negligibly small. It can be a valid assumption at the analysis of pneumatic or hydraulic double-stage servomechanisms with a small mass of the first-stage (spool) valve. Therefore, below we shall also refer to formula (1) with respect to the dynamics of mechanical stops assuming motion of a negligibly small mass.

The basis of the proposed method of analysis of periodic motions is the *state locus* of the relay system [10]. The *state locus* is the extension of the concepts of [2] and [4] to the state space representation of the plant. It offers finding all possible periodic motions in the relay feedback system.

Consider the following relay feedback system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{2}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

$$u = \Phi\left(g - y, b\right),\ y = \mathbf{C} \cdot \mathbf{x} \tag{3}$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector, $C \in \mathbf{R}^{1\times n}$ is a matrix, $f \in \mathbf{R}^n$ is a nonlinear function, $g \in \mathbf{R}^1$ is the system input, $y \in \mathbf{R}^1$ is the system output, function Φ is the relay hysteretic function with amplitude c and hysteresis $2b$. We assume that nonlinear function **f** is odd-symmetric:
$\mathbf{f}\left(-\mathbf{x}, -u\right) = -\mathbf{f}\left(\mathbf{x}, u\right)$

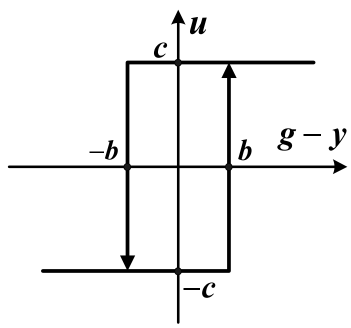

Fig. 2. The hysteretic relay nonlinearity Φ

In the autonomous mode ($g\left(t\right) \equiv 0$) the motion is defined by any single point of the closed orbit. Let us define periodic motion by the point $\mathbf{x}^*(T)$ (where $2T$ is the period) that corresponds to the relay switch from "-" to "+". Let us consider only symmetric unimodal limit cycles ($u(t)$ has a single switch within time interval $0<t<T$ and $\mathbf{x}(t+T)=-\mathbf{x}(t)$). Function $\mathbf{x}^*(T)$ defines all periodic motions that may occur in the relay feedback system. Let us call it the *state locus of the relay system* [10]. Let

$$\mathbf{x}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(0\right), c, t\right)$$

be the solution of equation (2) for $u(t)=c$ and initial state $\mathbf{x}(0)$. Due to the assumption about symmetric character of periodic motions, point $\mathbf{x}^*(T)$ belongs to the state locus if and only if the following equality holds:

$$\mathbf{x}^*\left(T\right) + \mathbf{F}\left(\mathbf{x}^*\left(T\right), c, T\right) = 0 \tag{4}$$

Let us call equation (4) the *main equation of the state locus*.

General method of obtaining the *state locus* is, therefore, the numeric solution of the *main equation* (4). All known methods of solving sets of nonlinear algebraic equations can be used for that purpose. The algorithm, which is a combination of the iterations and the shift to achieve symmetric solution, proved to be efficient for solving equation (4). It is presented below as formula (5):

$$\mathbf{v}^{k+1} = \frac{\mathbf{v}^k - \mathbf{F}\left(\mathbf{v}^k, c, T\right)}{2},\ k = 0,1,2,... \tag{5}$$

($\mathbf{v}^k \to \mathbf{x}^*\left(T\right)$ when $k \to \infty$). It is easy to implement and provides a good convergence even if the periodic motion is unstable. This algorithm is convenient for computing the *state locus* when the plant has the dynamic saturation considered above.

III. State Locus of a Plant with Saturation

In many applications, the saturation refers to the limited power of the actuator, which is located at the upstream part of the plant (Fig. 3). However, this is not a simple signal limiter but the dynamic saturation that was considered above.

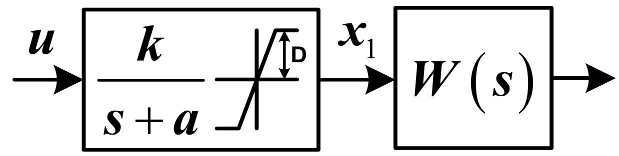

Fig. 3. Dynamic saturation at the plant input

Let us assume that $kc > aD$, so that the motion beyond the linear zone is possible.

Since the state locus is a vector function, we shall call its components the *R-characteristics* of the relay system. Fig. 4 provides the typical *R-characteristic* $x_i^*(T)$ of the relay system with saturation.

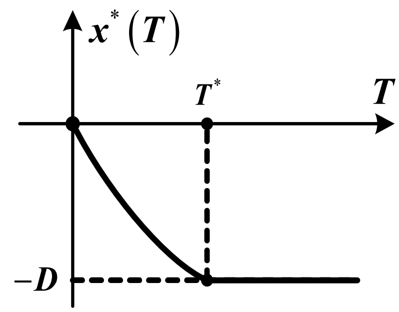

Fig. 4. Typical *R-characteristic* $x_i^*(T)$ of the relay system with dynamic saturation

If $T<T^*$ the motions do not reach the saturation value D, and the *dynamic saturation* of the plant can be disregarded. In this case, the motions occur only in the linear zone and methods [2], [4], [12], [13] can be used for computing the *state locus*. However, the most important interval of period values is $T>T^*$ when the state is saturated. Consider determination of the state locus for $T>T^*$. In periodic motion, control that is produced by the relay is as follows:

$$u\left(t\right) = c \cdot 1\left(t\right) - 2c \cdot 1\left(t - T\right)$$

Let us design a certain equivalent control that would allow us to remove the saturation from the plant dynamics and consider the plant as a linear one. Let us assume that the equivalent control $u_e(t)$ is equal to the value of control that is needed to bring state x_1 to the value of the saturation D ($x_1 = D$). The idea of the equivalent control is illustrated by Fig. 5.

Therefore, in Fig. 5 $c^* = \dfrac{aD}{k}$, where t_1 is the time of state x_1 reaching the saturation value D. The use of the equivalent control allows for removing the nonlinearity (*dynamic saturation*) from the plant model and analyzing the plant as linear.

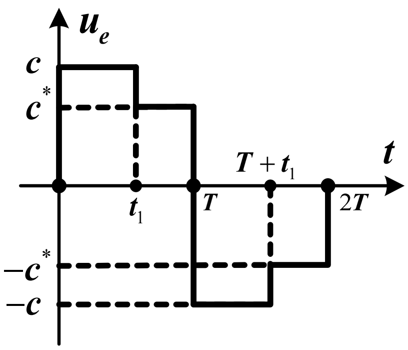

Fig. 5. The equivalent control

IV. Computing the State Locus of a Plant with Dynamic Saturation

Consider now a method of computing the *state locus* of a plant containing dynamic saturation of the actuator (presented in Fig. 3). Expand transfer function

$$W^*(s)=\frac{k}{s+a}\cdot W(s)$$

into partial fractions as follows:

$$W^*(s)=\frac{k_1}{s+a}+\frac{k_2}{s}+\frac{Hs+M}{s^2+2\alpha s+\alpha^2+\beta^2}+...$$

Derive the *state locus* formula for every partial fraction and obtain the *state locus* via summation of partial loci.

For the first-order dynamics

$$W(s)=\frac{k}{s+a}$$

the motion is described by the following equation:

$$\dot{x}=ku_e(t)-ax \tag{6}$$

it follows from (6) and Fig. 5 that

$$x(T)=x(0)\cdot e^{-aT}+\frac{kc}{a}\left(1-e^{-aT}\right)-\frac{kc^0}{a}\left(1-e^{-a(T-t_1)}\right) \tag{7}$$

where $c^0=c-c^*$. If symmetric periodic motion occurs then the following equality holds:

$$x(T)=-x(0)=-x^*(T)$$

From formula (7), the *state locus* can be found as follows:

$$x^*(T)=-\frac{kc\left(e^{at_1}-1\right)+kc^*\left(e^{aT}-e^{at_1}\right)}{a\left(1+e^{aT}\right)} \tag{8}$$

It also makes sense to consider the equation for the state derivative at time t=0- (the time of relay switch from "-" to "+"). Since the state derivative is a discontinuous variable we consider the value of the derivative "immediately before" the switch, which is indicated by the superscript "-":

$$z^-(T)=-kc^*-ax^*(T)$$

Similarly, we can show that for the integrator given by

$$W(s)=\frac{k}{s}$$

the following formulas for the state locus and its derivative will hold:

$$x^*(T)=-\frac{k\left[c^*(T-t_1)+ct_1\right]}{2}, \tag{9}$$

$$z^-(T)=-kc^*.$$

The general second-order dynamics given by the following transfer function:

$$W(s)=\frac{Hs+M}{s^2+2\alpha s+\alpha^2+\beta^2}$$

can be expanded into:

$$\frac{Hs+M}{s^2+2\alpha s+\alpha^2+\beta^2}=\frac{R_1}{s+\alpha-i\beta}+\frac{R_2}{s+\alpha+i\beta} \tag{10}$$

where

$$R_1=\frac{M-H\alpha+iH\beta}{2i\beta},\quad R_2=\frac{M-H\alpha-iH\beta}{-2i\beta}$$

Each of the transfer functions has parameters that are complex numbers. However, formula (8) can be used even in this case, as R_1 and R_2 are complex conjugate. As a result, we arrive at the following formula:

$$x^*(T)=-2\,\mathrm{Re}\left[\left(M-H\alpha+iH\beta\right)\frac{c\left(e^{(\alpha-i\beta)t_1}-1\right)+c^*\left(e^{(\alpha-i\beta)T}-e^{(\alpha-i\beta)t_1}\right)}{2i\beta\left(\alpha-i\beta\right)\left(1+e^{(\alpha-i\beta)T}\right)}\right] \tag{11}$$

Considering the real part in (11) we derive the following formula:

$$x^*(T)=-\frac{c\left[M\beta N_1(t_1)+(M\alpha-H\eta)N_2(t_1)\right]}{\eta\beta\left[1+2e^{\alpha T}\cdot\cos\beta T+e^{2\alpha T}\right]}-\\ -\frac{c^*e^{\alpha t_1}\left[M\beta N_1\left(t_1^*\right)+(M\alpha-H\eta)N_2\left(t_1^*\right)\cos\beta t_1+\left(M\beta N_2\left(t_1^*\right)-(M\alpha-H\eta)N_1\left(t_1^*\right)\right)\sin\beta t_1\right]}{\eta\beta\left[1+2e^{\alpha T}\cdot\cos\beta T+e^{2\alpha T}\right]}, \tag{12}$$

where

$$N_1(t_1)=e^{\alpha t_1}\cdot\cos\beta t_1-1+e^{\alpha(T+t_1)}\cdot\cos\beta(T-t_1)-e^{\alpha T}\cdot\cos\beta T,$$

$$N_2(t_1)=e^{\alpha(T+t_1)}\cdot\sin\beta(T-t_1)-e^{\alpha t_1}\cdot\sin\beta t_1-e^{\alpha T}\cdot\sin\beta T,$$

$$\eta=\alpha^2+\beta^2,\quad t_1^*=T-t_1$$

To find characteristic $z^-(T)$ (the derivative at the time of the switch), we represent transfer function sW(s) as follows:

$$sW(s)=H+\frac{(M-2\alpha H)s-\eta H}{s^2+2\alpha s+\eta} \tag{13}$$

It follows from (13) that for the second-order dynamics (10) the following equality holds:

$$z^-(T)=-Hc^*+L(T)$$

where $L(T)$ is given by (12) subject to the replacement H with H^*, M with M^*, where $H^*=M-2\alpha H$, $M^*=-\eta H$.

The value of t_1, which is used in formulas (8), (9), (11), should be found from the *main equation* of the *state locus* (4) - simultaneously with the locus value for every particular T. However, for first and second-order dynamics analytical formulas can be derived. Alternatively, an iterative algorithm of computing t_1, then x^* for this t_1 and after that correction of t_1 on the basis of computing the state trajectory departing x^* and reaching the saturation value, can be used.

V. Finding periodic motions via the state locus

Once computed, the state locus allows for a finding a periodic motion in the relay system in a simple way. For the system (2), (3), the period of the periodic motion can be found from the following equation:

$$\mathbf{C}\cdot\mathbf{x}^*(T) = -b \tag{14}$$

subject to:

$$\mathbf{C}\cdot\mathbf{f}\left(\mathbf{x}^*(T), -c\right) < 0 \tag{15}$$

which is the condition of the direction of the switch of the relay. Equation (14) is a direct result of the state locus definition. If the plant model can be represented as expansion into partial fractions, for computing the direction of the switch variables z^- can be used instead of formula (15).

The methodology of finding a periodic solution is as follows. At first, equation (14) is solved as per the algorithm given above. Assuming that the half-period found from (14) is T^0, inequality (15) is checked. If (15) is satisfied the periodic trajectory $\mathbf{x}(t)$ for T^0 should be computed. Then using function $y(t)=\mathbf{C}\mathbf{x}(t)$ one should check if there are no other (beside $t=T^0$) switches of the relay within interval $0<t<2T^0$. In practice, situations when the solution obtained via (14), (15) is not an actual periodic solution of the relay system happen very seldom.

The developed methodology can easily be extended to the case of several saturations in plant dynamics, which is illustrated by the example given below.

VI. Example

Consider an example of analysis of possible periodic motions in the electro-hydraulic servomechanism (Fig. 6).

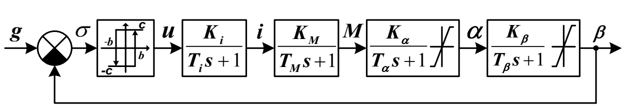

Fig. 6. Electro-hydraulic servomechanism

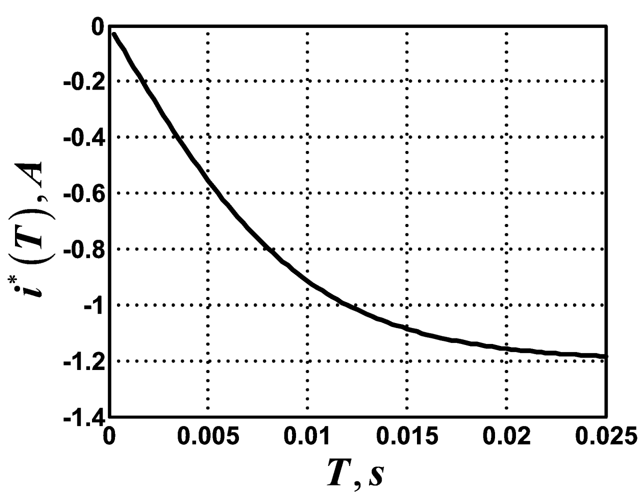

Fig. 7a. Components of the state locus

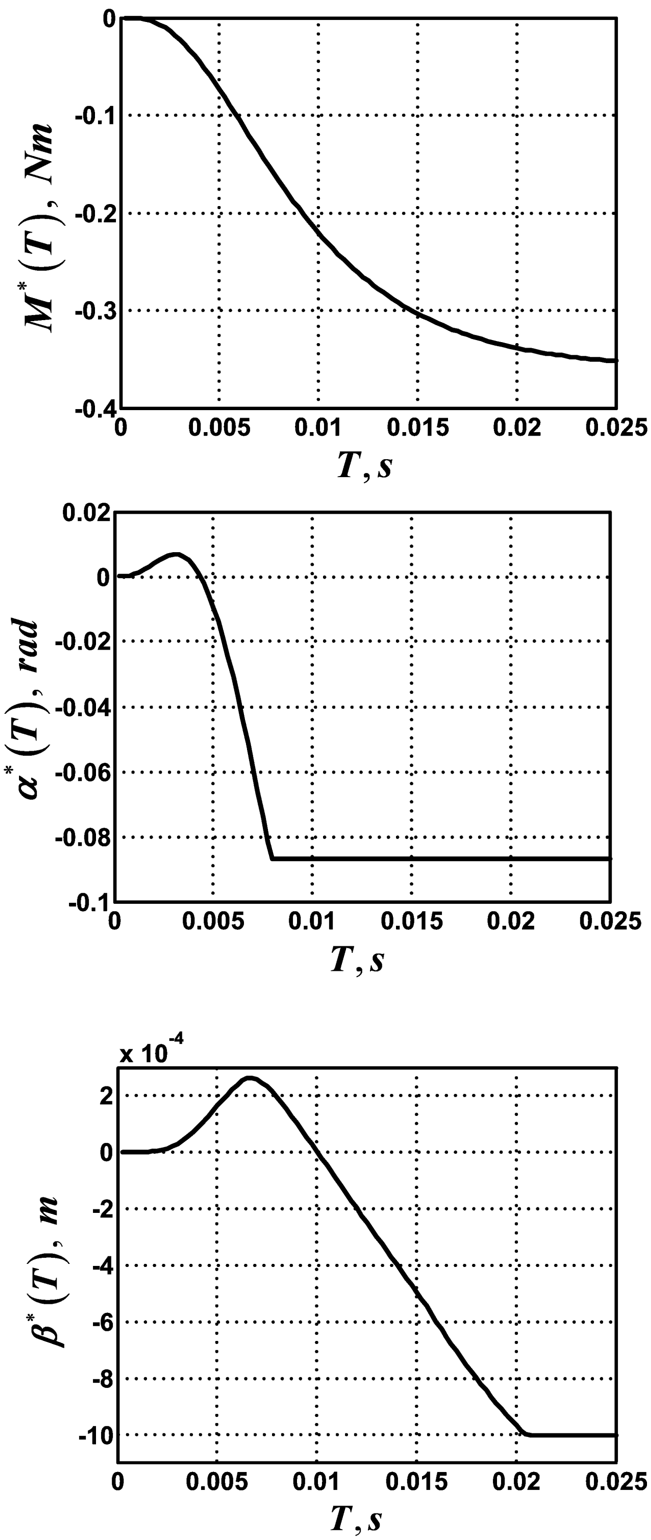

Fig. 7b. Components of the state locus (continued)

The servomechanism model has the following parameters: c=8V, b=0.008m, K_i=0.15A/V, K_M=0.3Nm/A, K_α=0.9rad/Nm, K_β=0.02m/rad, T_i=0.005s, T_M=0.002s, T_α=0.0016s, T_β=0.01s, α_{max}=0.087rad, b_{max}=0.001m. The plant variables are: u is the voltage produced by the D-class electronic amplifier, i is the electromagnet current, M is the torque developed by the electromagnet, α and β are valve positions of the first stage (spool valve) and the second stage (piston) of the two-stage electro-hydraulic servomechanism. The two valve positions have saturation in the form of the mechanical stops. The spools are fabricated of lightweight plastic and their mass is neglected in the model Fig. 6.

The *state locus* was computed for the servomechanism dynamics as per formula (5). The components of the *state locus* are presented in Fig. 7.

It was found as per (14) and (15) that the periodic motion of period $2T^0$=0.035s occurs in the system. Fig. 8 shows periodic signals $\alpha(t)$ and $\beta(t)$ on half-period found via the proposed methodology.

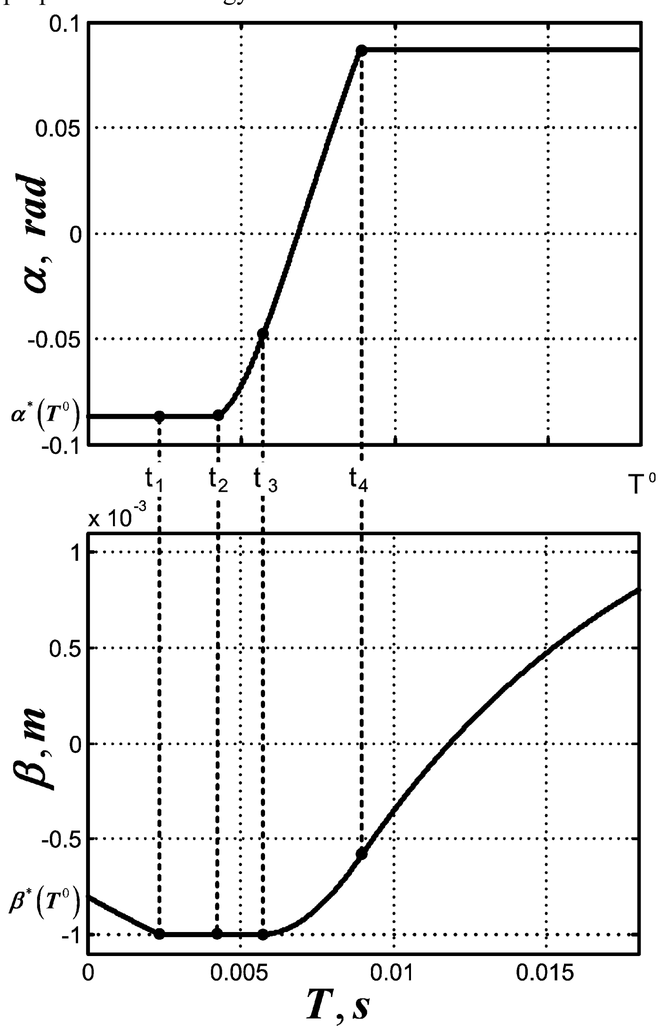

Fig. 8. Valve positions in a periodic motion

One can see from Fig. 8 that there are five intervals given by points t_1-t_4 where the motion is described by different linear equations. Those intervals correspond to the combinations of the modes of being on the two limits or in the linear zone. The first interval and the last interval correspond to the same mode.

VII. Conclusion

A method of analysis of possible periodic motions in relay feedback systems with dynamic saturation in plant dynamics is presented. The presented method is based on the concept of the *state locus*, which is an extension of the Tsypkin locus to the state space presentation of the plant dynamics and a class of nonlinear plants. In particular, the present work further extends the *state locus* concept to one of the most important nonlinearities in plant dynamics – the saturation. The saturation model used in the plant dynamics is not a simple signal limiter but *dynamic saturation*, which occurs due to actuator limited power or mechanical stops. Those types of saturation are typical of actuator dynamics. The proposed methodology allows for simple and exact solution of the problem of finding possible periodic motions in the relay feedback systems with saturation in plant dynamics.

References

[1] L.A. MacColl, *Fundamental Theory of Servomechanisms*, New York, D. Van Nostrand Co., 1945.

[2] B. Hamel, "Contribution a l'etude mathematique des systemes de reglage par tout-ou-rien," *C.E.M.V.*, No. 17, Service Technique Aeronautique, 1949.

[3] I. Flugge-Lotz, *Discontinuous Automatic Control*, Princeton University Press, 1953.

[4] Ya.Z. Tsypkin, *Relay Control Systems*, Cambridge, England, 1984.

[5] G.S. Pospelov, "Relay Control Systems," in *Automatic Control Theory*, Vol. 3, Part II, Mashinostroyenie, Moscow, 1969.

[6] D.P. Atherton, *Nonlinear Control Engineering* - Describing Function Analysis and Design, Workingham, Berks, UK: Van Nostrand Company Limited, 1975.

[7] K.J. Astrom, "Oscillations in Systems with Relay Feedback," *The IMA Volumes in Mathematics and its Applications: Adaptive Control, Filtering and Signal Processing*, 74, 1995, pp.1-25.

[8] K.J. Astrom, T.H. Lee, K.K. Tan, and K.H. Johansson, "Recent advances in relay feedback methods - a survey", *Systems, Man and Cybernetics, 1995. Intelligent Systems for the 21st Century, IEEE International Conference on* , Volume: 3 , 1995, 2616-2621.

[9] Q.-G. Wang, T.H. Lee, and C. Lin, *Relay Feedback: Analysis, Identification and Control*, Springer, 2003.

[10] N.V. Faldin, "Relay automatic control systems," in K.A. Pupkov, N.D. Egupov (eds.), *Mathematical Models, Dynamic Characteristics and Analysis of Automatic Control Systems*, MGTU, Moscow, 2004, pp. 573-636.

[11] N.V. Faldin, "Exact method of analysis of relay systems," in E.A. Fedosov (ed.), *Automatic Control* (*Encyclopedia*), Vol. 1-4, Mashinostroenie, 2000, pp.231-253.

[12] I. Boiko, "Oscillations and transfer properties of relay servo systems – the locus of a perturbed relay system approach", *Automatica*, **41**, p. 677 – 683, 2005.

[13] I. Boiko, *Discontinuous Control Systems: Frequency-Domain Analysis and Design*, Boston, Birkhauser, 2009.

[14] L. Fridman, "Singularly perturbed analysis of chattering in relay control systems*", IEEE Trans. on Automatic Control*, 47 (12), 2002, pp. 2079-2084.

[15] I. Boiko, "Analysis of closed-loop performance and frequency-domain design of compensating filters for sliding mode control systems," *IEEE Trans. Automatic Control*, Vol. 52, No. 10, 2007, pp. 1882-1891.