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On Similarity Classes of Discrete-time Floquet Transformations

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Abstract— **This paper considers discrete-time Floquet transformations, by which a discrete-time periodic linear system can be transformed to a time-invariant one. The previous researches have made clear when a discrete-time periodic system has its Floquet transformations and also it is known that the system has a lot of Floquet transformations if it has one. This paper defines three new similarities between two Floquet transformations, clarifies some relations among those similarities, and also makes clear how a set of all Floquet transformations can be split into a finite number of equivalence classes.**

I. Introduction

A lot of model-based control method for periodic systems have been proposed, e.g., [2] [8] [10] and references therein. In order to establish model-based control designs for automotive engine, the authors proposed a model representation of V6 Spark Ignition SICE benchmark engine [7]: a continuous-time periodic nonlinear state space model is constructed first, then the model is discretized to get a discrete-time periodic nonlinear state space model, and finally by introducing a concept of "role state variables", the discrete-time periodic model can be transformed to a time-invariant one. It could be claimed that the last process corresponds to using the theory of Floquet in discrete-time version.

It is well known [1] [3] as *theory of Floquet* that every continuous-time periodic linear homogeneous system $\dot{x}_c(t) = A_c(t)x_c(t)$ with a period T can be transformed to a linear time-invariant system $\dot{\xi}_c(t) = A_c\xi_c(t)$ by a state transformation $\xi_c(t) = P(t)x_c(t)$ with $P(t)$ being nonsingular and periodic $P(t+T) = P(t)$.

It has been made clear in [9] when a *discrete-time* periodic linear homogeneous system can be transformed to a time-invariant one, which could be called *discrete-time version for theory of Floquet*. Those results in [9] are very interesting, in which it is shown that all discrete-time periodic systems are not necessarily transformed to time-invariant ones.

The previous research [5] also considers discrete-time Floquet transformations, where a necessary and sufficient condition for a discrete-time periodic system to have Floquet transformations is given. This condition is equivalent to one of [9], however, the proof is novel and self-contained. Furthermore, from the result of [5], it is easy to see that corresponding to every N-th root of the monodolomy matrix, you can construct a Floquet transformation. Noticing that a square matrix could have many N-th roots which are not similar each other, it is easy to guess that the system could have many Floquet transformations which are not "similar" each other. However, at this moment, some problems arises: what "similarity" between two discrete-time Floquet transformations means, and what structure the set of all Floquet transformations has.

In this paper, three kinds of similarities between two Floquet transformations are newly defined, some relations among those similarities are made clear and also it is shown how the set of all Floquet transformations can be split into a finite number of equivalence classes by using those similarities.

Section II reviews the previous results on discrete-time Floquet transformations and also gives an example to understand the problem to be considered in this paper. Section III considers similarity classes of N-th root matrices. Section IV is a main part of this paper, where similarity classes of Floquet transformations are discussed. Some conclusion remarks are stated in Section V.

Notation: $\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{C}$ a set of all complex numbers, and $\boldsymbol{Z}$ a set of all integers.

For any positive integer $n \in \boldsymbol{Z}$, $\underline{n} := \{1, 2, \cdots, n\}$ and $\underline{n}^- := \{0, 1, \cdots, n-1\}$. For any $k \in \boldsymbol{Z}$, $\mathrm{mod}(k/n)$ denotes "k modulo n" and floor(k/n) "the greatest integer less than or equal to k/n", i.e., it holds that $k = \mathrm{floor}(k/n) \times n + \mathrm{mod}(k/n)$.

Associated with $A, B \in \boldsymbol{C}^{n\times n}$, $A \simeq B$ means that A is similar to B, i.e., there exists a nonsingular matrix $S \in \boldsymbol{C}^{n\times n}$ such that $B = SAS^{-1}$.

$J_m(\lambda)$ denotes an $m \times m$ Jordan block with eigenvalue λ, and the direct sum is denoted as $\oplus$, i.e., for any square matrices C and D,

$$C \oplus D := \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}.$$

II. Discrete-Time Floquet Transformation

Consider a discrete-time periodic homogeneous system with a period N

$$x_{k+1} = A_k x_k \tag{1}$$

where $k \in \boldsymbol{Z}$ is time, $x_k \in \boldsymbol{R}^n$ state, $A_k \in \boldsymbol{R}^{n\times n}$, N a positive integer (≥ 2) and it is assumed that

$$A_k = A_{\mathrm{mod}(k/N)}. \tag{2}$$

Suppose that there exist a matrix $A \in \boldsymbol{C}^{n\times n}$ and a sequence of *nonsingular* matrices $\{P_k \in \boldsymbol{C}^{n\times n} \mid k \in \boldsymbol{Z}\}$ such that for any $k \in \boldsymbol{Z}$,

$$\begin{aligned} P_{k+1}A_k &= AP_k &(3)\\ P_k &= P_{\mathrm{mod}(k/N)}. &(4)\end{aligned}$$

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Then, by introducing a new state ξ_k as $\xi_k = P_k x_k$, the system (1) can be transformed to a time-invariant system

$$\xi_{k+1} = A\xi_k \tag{5}$$

where $\xi_k \in \boldsymbol{C}^n$ and $A \in \boldsymbol{C}^{n\times n}$.

Associated with the system (1), the set of matrices A and P_k's in (3) and (4), denoted by $\mathcal{AP} := \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in Z\}$, is called a *Floquet transformation*.

Define a monodolomy matrix $\Phi \in \boldsymbol{R}^{n\times n}$ as

$$\Phi := A_{N-1}\cdots A_1 A_0, \tag{6}$$

then it is easy to verify from (3) and (4) that $\Phi = \left(P_0^{-1}AP_0\right)^N$. This means that Φ has an N-th root $P_0^{-1}AP_0 \in \boldsymbol{C}^{n\times n}$ if the system (1) has a Floquet transformation.

A necessary and sufficient condition for the system (1) to have a Floquet transformation has been given in [9], where an algorithm is shown to calculate a Floquet transformation from the data $\{A_k \mid k \in \underline{N}^-\}$. The following theorem derived in [5] also gives a necessary and sufficient condition for the system (1) to have a Floquet transformation.

Theorem 1: [5] Given a system (1), (2) with $A_k \in R^{n\times n}$ and a period $N \geq 2$, there exists a Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ if and only if it holds that

$$\begin{gathered}\mathrm{rank}A_{k-1}A_{k-2}\cdots A_{h+1}A_h = \mathrm{rank}A^{k-h}\\ \text{for } h \in \underline{N}^- \text{ and } k-h \in \underline{n}.\end{gathered} \tag{7}$$

where $A \in \boldsymbol{C}^{n\times n}$ is any matrix similar to one of N-th roots of Φ given by (6). □

The condition (7) of Theorem 1 is equivalent to one of [9]. However, the proof given in [5] shows how to construct a Floquet transformation associated with every A similar to one of N-th roots of Φ, whereas the algorithm in [9] gives *only one* of Floquet transformations. From this point of view, Theorem 1 is more suitable to consider *"similarity classes"* in the set of discrete-time Floquet transformations.

Example 1: Consider a periodic system $x_{k+1} = A_k x_k$ with a period $N = 3$ and

$$A_0 = \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad A_k = A_{\mathrm{mod}(k/N)}.$$

According to Theorem 1, it is easy to verify that this periodic system has Floquet transformations, and also we can construct a lot of Floquet transformations; e.g., the first one is $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ where

$$A = \left[\begin{array}{c|cc} 1 & 0 & 0\\ \hline 0 & 0 & 1\\ 0 & 0 & 0\end{array}\right], \quad P_0 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$P_1 = \begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$$

The second Floquet transformation $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ is given as

$$A' = \left[\begin{array}{c|cc} 1 & 0 & 0\\ \hline 0 & 0 & 1\\ 0 & 0 & 0\end{array}\right], \quad P'_0 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$P'_1 = \begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & -1\\ 0 & 0 & 2\end{bmatrix}, \quad P'_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 3 & 2\\ 0 & 1 & 0\end{bmatrix}$$

and the third one is $\mathcal{AP}'' = \{A'', P''_k = P''_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ where

$$A'' = \left[\begin{array}{c|cc} e^{j2\pi/3} & 0 & 0\\ \hline 0 & 0 & 1\\ 0 & 0 & 0\end{array}\right], \quad P''_0 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$P''_1 = \begin{bmatrix} 0 & e^{j2\pi/3} & 0\\ 1 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}, \quad P''_2 = \begin{bmatrix} e^{j4\pi/3} & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}.$$

You can see that $A' \simeq A$ (really $A' = A$) and A'' is not similar to A. And also you can realize that $P_0 = P'_0 = I_3$, however, P'_1 and P'_2 are respectively different from P_1 and P_2 and moreover $P'_1P_1^{-1} \neq P'_2P_2^{-1}$. So, it is not easy to speculate what relation there exists between P_k's and P'_k's. □

If a discrete-time periodic system has a Floquet transformation, it has a lot of Floquet transformations as shown in the above example. It is an interesting issue to investigate some relations among those Floquet transformations. The aim of this paper is to make clear some sorts of similarities among the Floquet transformations.

First of all, we will make clear similarity classes of N-th roots of the monodolomy matrix Φ in the next section.

III. Similarity Classes of N-th root Matrices

Given a matrix $\Phi \in \boldsymbol{R}^{n\times n}$ and a positive integer $N \geq 2$, an N-th root of Φ is a matrix $A \in \boldsymbol{C}^{n\times n}$ satisfying

$$A^N = \Phi. \tag{8}$$

By the analogy with N-th roots of a real number, it is easy to imagine that there are many N-th roots if Φ has one. Note that all square matrices do not necessarily have N-th roots, whereas all real numbers do.

In this section we are interested in when Φ has N-th roots and how many similarity classes exist in a set of all N-th roots of Φ.

Suppose that (8) holds and let J_Φ and J_A be the Jordan forms of Φ and A, respectively, i.e.,

$$\Phi = S_\Phi J_\Phi S_\Phi^{-1}, \quad A = S_A J_A S_A^{-1}$$

where $S_\Phi, S_A \in \boldsymbol{C}^{n\times n}$ are nonsingular. Then it follows that $J_A^N \simeq J_\Phi$ because $S_\Phi J_\Phi S_\Phi^{-1} = \Phi = A^N = \left(S_A J_A S_A^{-1}\right)^N = S_A J_A^N S_A^{-1}$.

From the above observation, in order to characterize all the N-th roots A's of Φ, it is enough [1]to characterize all the Jordan forms J_A's of the N-th roots.

Now recall a useful fact given as Theorem 6.2.25 in [6] and set $f(t) = t^N$ in the theorem, then the following lemma is directly obtained.

Lemma 1: Let $J_m(\gamma)$ be a $m \times m$ Jordan block with eigenvalue γ.

1) In the case of $\gamma \neq 0$,

$$J_m(\gamma)^N \simeq J_m(\gamma^N) \tag{9}$$

2) In the case of $\gamma = 0$ and suppose that $\bar{m} = \text{floor}(m/N)$ and $r = \text{mod}(k/N)$, then

$$J_m(0)^N \simeq \begin{cases} \underbrace{J_{\bar{m}+1}(0) \oplus \cdots \oplus J_{\bar{m}+1}(0)}_{r} \\ \qquad \oplus \underbrace{J_{\bar{m}}(0) \oplus \cdots \oplus J_{\bar{m}}(0)}_{N-r} & \text{for } m \geq N \\ \underbrace{J_1(0) \oplus \cdots \oplus J_1(0)}_{m} = 0_{m\times m} & \text{for } m < N \end{cases} \tag{10}$$

□

From the above lemma, we get the following theorem, which is just an extension of square roots [6] to N-th roots.

Theorem 2: Let $\Phi \in \boldsymbol{R}^{n\times n}$ be given and suppose $N \geq 2$.

1) If Φ is nonsingular, Φ has N-th roots. Furthermore, if Φ has p Jordan blocks in its Jordan form as follows,

$$\Phi \simeq J_{\ell_1}(\lambda_1) \oplus J_{\ell_2}(\lambda_2) \oplus \cdots \oplus J_{\ell_p}(\lambda_p) \tag{11}$$

then every N-th root A of Φ also has p Jordan blocks in its Jordan form as follows,

$$A \simeq J_{\ell_1}(\gamma_1) \oplus J_{\ell_2}(\gamma_2) \oplus \cdots \oplus J_{\ell_p}(\gamma_p) \tag{12}$$

where $\gamma_i^N = \lambda_i$ for $i \in \underline{p}$. Therefore if there exists μ distinct eigenvalues among $\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$, a number of N-th roots of Φ is at least N^μ and at most N^p.

2) If Φ is nilpotent

$$\Phi \simeq J_{n_1}(0) \oplus J_{n_2}(0) \oplus \cdots \oplus J_{n_s}(0) \tag{13}$$

and suppose $m = \text{floor}(s/N)$ and $r = \text{mod}(s/N)$. Then Φ has an N-th root if and only if there exists a permutation ν of $\underline{s}$ such that the following conditions a) and b) hold;

a) define $\Delta_j \in R^{1\times(N-1)}$ for $j \in \underline{m}^-$ by

$$\begin{aligned}\Delta_j \quad := \quad & \big(n_{\nu(jN+1)} - n_{\nu(jN+2)}, \\ & n_{\nu(jN+2)} - n_{\nu(jN+3)}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \\ & n_{\nu(jN+N-1)} - n_{\nu(jN+N)}\big),\end{aligned} \tag{14}$$

then each Δ_j has the property that its only one element is either "0" or "1" and the other elements are all "0".

b) if $r \geq 1$, then it holds that

$$n_{\nu(s)} = n_{\nu(s-1)} = \cdots = n_{\nu(s-r+1)} = 1. \tag{15}$$

Furthermore, an N-th root A of Φ is given as

$$A \simeq J_{m_1}(0) \oplus J_{m_2}(0) \oplus \cdots \oplus J_{m_q}(0) \tag{16}$$

where $q \geq m$,

$$m_i = \sum_{h=1}^{N} n_{\nu((i-1)N+h)} \quad \text{for } 1 \leq i \leq m \tag{17}$$

and $m_i \leq N$ for $(m+1) \leq i \leq q$.

3) Φ has an N-th root if and only if the nilpotent part of Φ satisfies the above condition 2). Furthermore, if Φ has an N-th root, its set of N-th roots lies in finitely many different similarity classes.

□

Example 2: Consider a nonsingular matrix

$$\Phi \simeq J_2(2) \oplus J_2(2)$$

and list up all the similarity classes of its N-th roots.

Φ has one eigenvalue of $\lambda = 2$ and two Jordan blocks, therefore $\mu = 1$ and $p = 2$. All the N-th roots of $\lambda = 2$ are given as

$$\gamma_k = 2^{1/N} e^{j\frac{2\pi k}{N}} \quad \text{for } k \in \underline{N}^-$$

where $2^{1/N}$ denotes the *positive real* N-th root of 2. Therefore it is easy to see that any N-th root of Φ is similar to one of $J_2(\gamma_k) \oplus J_2(\gamma_{k'})$ for $k, k' \in \underline{N}^-$.

Notice

$$J_2(\gamma_k) \oplus J_2(\gamma_{k'}) \simeq J_2(\gamma_{k'}) \oplus J_2(\gamma_k).$$

This happens because the sizes of two Jordan blocks are equal. Therefore in this case, the number of similarity classes of N-th roots is equal to

$$N + {}_N\mathrm{C}_2 = \frac{N(N+1)}{2}$$

which is a number between $N(= N^\mu)$ and $N^2(= N^p)$.

□

Example 3: Consider a nilpotent matrix

$$\Phi \simeq J_2(0) \oplus J_2(0) \oplus J_1(0) \oplus J_1(0).$$

Note that $s = 4$, $n_1 = n_2 = 2$ and $n_3 = n_4 = 1$.

Φ has a square root. In fact, associated with $N = 2$, it follows that $m = \text{floor}(s/N) = 2$ and $r = \text{mod}(s/N) = 0$. Now let the permutation ν of $\underline{s}$ be $\nu(i) = i$, then we have $\Delta_0 := (n_1 - n_2) = 0$, $\Delta_1 := (n_3 - n_4) = 0$. These observations say that the condition 2) in Theorem 2 is satisfied and a square root A of Φ is similar to $J_A = J_4(0) \oplus J_2(0)$ because $m_1 = n_1 + n_2 = 4$ and $m_2 = n_3 + n_4$.

In this case, it is very interesting that Φ has another square root A' which is not similar to J_A. In fact, when the

[1]Derivation of A from J_A, J_Φ and S_Φ is straightforward. In fact, first find a nonsingular X satisfying $XJ_A^N = J_\Phi X$, then calculate $A = S_\Phi X J_A (S_\Phi X)^{-1}$.

permutation ν is set as $\nu(1) = 1, \nu(2) = 3, \nu(3) = 2, \nu(4) = 4$, then we have $\Delta_0 = (n_1 - n_3) = 1,\ \Delta_1 = (n_2 - n_4) = 1$, which also satisfy the condition 2) in Theorem 2, and $m_1 = n_1 + n_3 = 3$, $m_2 = n_2 + n_4 = 3$, so Φ has a square root A' similar to $J_{A'} = J_3(0) \oplus J_3(0)$.

Now we will decide whether the matrix Φ has a cube root or not. In this case, $N = 3$ and $m = \text{floor}(s/N) = 1, r = \text{mod}(s/N) = 1$. If we set a permutation ν as $\nu(i) = i$, then it follows that $\Delta_0 = (n_1 - n_2, n_2 - n_3) = (0, 1)$. Note that $n_{\nu(4)} = n_4 = 1$. Therefore the conditions 2), a) and b) in Theorem 2 are satisfied, $m_1 = n_1 + n_2 + n_3 = 5$, so Φ has a cube root similar to $J_A = J_5(0) \oplus J_1(0)$.

In the case of $N = 4$, it follows that $m = \text{floor}(s/N) = 1, r = \text{mod}(s/N) = 0$, and $\Delta_0 = (n_1 - n_2, n_2 - n_3, n_3 - n_4) = (0, 1, 0)$, therefore the matrix Φ has a 4th root similar to $J_A = J_6(0)$.

It is also trivial that the matrix Φ has no N-th root when $N \geq 5$.

□

IV. Similarity Classes of Floquet Transformations

Suppose the discrete-time periodic system (1) with (2) has Floquet transformations, we denote a set of all the Floquet transformations by $\mathcal{FT}$, i.e., every element of $\mathcal{FT}$ is a Floquet transformation $\mathcal{AP} := \{A, P_k = P_{\text{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$, which satisfy (3) and (4).

The matrix A of Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\text{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{FT}$ could be different from A' of another Floquet transformation $\mathcal{AP}' = \{A', P'_k = P'_{\text{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{FT}$. Collect all the matrices A's of all the Floquet transformations in $\mathcal{FT}$ and denote the set by $\mathcal{A}$.

It is easy to see that Theorem 1 implies the next lemma.

Lemma 2: $A \in \mathcal{A}$ if and only if A is similar to one of N-th roots of the monodolomy matrix Φ given as (6) and also satisfies the condition (7). □

Now suppose that the Jordan forms J_A and J_Φ of A and Φ are expressed as

$$J_A = J_{A\sigma} \oplus J_{A\nu}, \quad J_\Phi = J_{\Phi\sigma} \oplus J_{\Phi\nu} \tag{18}$$

where $J_{A\sigma}, J_{\Phi\sigma}$ are nonsingular and $J_{A\nu}, J_{\Phi\nu}$ are nilpotent.

By recalling Theorem 2, it follows that $\mathcal{A}$ has a finite number of different similarity classes. Notice that every $A \in \mathcal{A}$ satisfies the rank condition (7), so each $A \in \mathcal{A}$ has the same Jordan blocks for its nilpotent part $J_{A\nu}$. Therefore, suppose $A, A' \in \mathcal{A}$ have their Jordan forms as $J_A = J_{A\sigma} \oplus J_{A\nu}$ and $J_{A'} = J_{A'\sigma} \oplus J_{A'\nu}$ respectively, then it always holds $J_{A\nu} = J_{A'\nu}$ and also it holds that $A \simeq A'$ if and only if $J_{A\sigma} = J_{A'\sigma}$.

Associated with any $A \in \mathcal{A}$, a subset $\{A' \in \mathcal{A} \mid A' \simeq A\}$ of $\mathcal{A}$ is called a similarity class or an equivalence class.

From these observations, we get the following lemma.

Lemma 3: Suppose that the monodolomy matrix Φ given as (6) has a Jordan form $J_\Phi = J_{\Phi\sigma} \oplus J_{\Phi\nu}$ with $J_{\Phi\sigma}$ being nonsingular and $J_{\Phi\nu}$ nilpotent, and also suppose that $J_{\Phi\sigma}$ has μ distinct eigenvalues and p Jordan blocks . Then a number of similarity classes in the set $\mathcal{A}$, denoted by n_e, is at least N^μ and at most N^p, i.e., denoting each similarity class by $\mathcal{A}_i$ for $i \in \underline{n_e}$, the set $\mathcal{A}$ splits into exactly n_e similarity classes as follows.

$$\mathcal{A} = \bigcup_{i \in \underline{n_e}} \mathcal{A}_i, \quad \mathcal{A}_i \bigcap \mathcal{A}_j = \phi \text{ for } i \neq j \tag{19}$$

□

Note that the number n_e can be determined exactly from the data $\{\ell_i, \lambda_i \mid i \in \underline{p}\}$ of $J_{\Phi\sigma} = J_{\ell_1}(\lambda_1) \oplus J_{\ell_2}(\lambda_2) \oplus \cdots \oplus J_{\ell_p}(\lambda_p)$.

Now we will define a similarity between two Floquet transformations.

Definition 1: Suppose $\mathcal{AP}, \mathcal{AP}' \in \mathcal{FT}$ with $\mathcal{AP} = \{A, P_k = P_{\text{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ and $\mathcal{AP}' = \{A', P'_k = P'_{\text{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$.

1) $\mathcal{AP}'$ is *strongly similar* to $\mathcal{AP}$, denoted by $\mathcal{AP}' \simeq^s \mathcal{AP}$, when $A' \simeq A$ and there exist a nonsingular constant matrix $Q \in \boldsymbol{C}^{n \times n}$ such that
$$P'_k \;=\; QP_k \tag{20}$$
2) $\mathcal{AP}'$ is *similar* to $\mathcal{AP}$, denoted by $\mathcal{AP}' \simeq \mathcal{AP}$, when $A' \simeq A$ and there exist nonsingular matrices $Q_k = Q_{\text{mod}(k/N)} \in \boldsymbol{C}^{n \times n}$ ($k \in \boldsymbol{Z}$) such that
$$P'_k \;=\; Q_k P_k \tag{21}$$
3) $\mathcal{AP}'$ is *weakly similar* to $\mathcal{AP}$, denoted by $\mathcal{AP}' \simeq^w \mathcal{AP}$, when $(A')^N \simeq A^N$ and there exist nonsingular matrices $Q_k = Q_{\text{mod}(k/N)} \in \boldsymbol{C}^{n \times n}$ ($k \in \boldsymbol{Z}$) such that
$$P'_k = Q_k P_k \tag{22}$$

□

Note that the same notation "$\simeq$" is used for similarity both of matrix and of Floquet transformation, but it is clear from the content which similarity it means.

Each similarity defined above for Floquet transformations satisfies reflexive, symmetric, and transitive laws, therefore the similarities have right to be equivalence relations in $\mathcal{FT}$. The equivalence class of an element $\mathcal{AP} \in \mathcal{FT}$ with respect to each similarity is denoted as follows.

$$\overline{\mathcal{AP}}^s \;:=\; \{\mathcal{AP}' \in \mathcal{FT} \mid \mathcal{AP}' \simeq^s \mathcal{AP}\} \tag{23}$$
$$\overline{\mathcal{AP}} \;:=\; \{\mathcal{AP}' \in \mathcal{FT} \mid \mathcal{AP}' \simeq \mathcal{AP}\} \tag{24}$$
$$\overline{\mathcal{AP}}^w \;:=\; \{\mathcal{AP}' \in \mathcal{FT} \mid \mathcal{AP}' \simeq^w \mathcal{AP}\} \tag{25}$$

Then it is trivial that

$$\overline{\mathcal{AP}}^s \subset \overline{\mathcal{AP}} \subset \overline{\mathcal{AP}}^w \tag{26}$$

because from the definitions, $\mathcal{AP}' \simeq^s \mathcal{AP}$ implies $\mathcal{AP}' \simeq \mathcal{AP}$, which also implies $\mathcal{AP}' \simeq^w \mathcal{AP}$.

Associated with a matrix $A \in \boldsymbol{C}^{n \times n}$, we denote a set of all nonsingular matrices that are commutative with A as $\mathcal{C}(A)$, i.e.,

$$\mathcal{C}(A) := \left\{ W \in \boldsymbol{C}^{n \times n} : \text{nonsingular} \;\middle|\; WA = AW \right\}. \tag{27}$$

This set has already been investigated in detail [4] [6].

Now we extend the commutative matrix to define a new concept of an N *periodic sequence of commutative matrices with* A, which is denoted as $\mathcal{W} := \{W_k = W_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{n \times n} \mid k \in \boldsymbol{Z}\}$ where every W_k is nonsingular and satisfies

$$W_{k+1} A = A W_k. \tag{28}$$

Moreover, we denote a set of all N periodic sequences of commutative matrices with A as $\mathcal{SC}(A, N)$.

It is easy to see that $\mathcal{SC}(A,1) = \mathcal{C}(A)$ because $W_k = W_{\mathrm{mod}(k/N)} = W_0$ for any $k \in \boldsymbol{Z}$ in the case of $N = 1$.

Moreover, for any $W \in \mathcal{C}(A)$, it is easy to see that $\mathcal{W} = \{W_k = W \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, N)$ for any positive $N \in \boldsymbol{Z}$. In this sense, we could say that $\mathcal{C}(A) \subset \mathcal{SC}(A, N)$.

Under the above notation, some interesting facts on the similarity classes in $\mathcal{FT}$ are stated in the next theorem.

Theorem 3: Suppose $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{FT}$.

1) $\{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}^s$ if and only if there exist nonsingular S and $W \in \mathcal{C}(A)$ such that

$$A' = SAS^{-1}, \quad P'_k = SWP_k. \tag{29}$$

2) $\{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}$ if and only if there exist nonsingular S and $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, N)$ such that

$$A' = SAS^{-1}, \quad P'_k = SW_kP_k. \tag{30}$$

3) It holds that

$$\overline{\mathcal{AP}}^w = \mathcal{FT}. \tag{31}$$

□

(Proof)

Fact 3) It is trivial that $\overline{\mathcal{AP}}^w \subset \mathcal{FT}$. So we will prove that $\mathcal{FT} \subset \overline{\mathcal{AP}}^w$.

Consider any element $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{FT}$. Recall that $P_{k+1}A_k = AP_k$, $P'_{k+1}A_k = A'P'_k$ for $k \in \boldsymbol{Z}$. Therefore, it follows that

$$A_k = (P'_{k+1})^{-1} A' P'_k = (P_{k+1})^{-1} A P_k,$$

which implies $A' = (P'_{k+1}P_{k+1}^{-1}) A (P'_k P_k^{-1})^{-1}$. Now, define $Q_k := P'_k P_k^{-1}$. Then it is trivial that (22) holds and also noticing $Q_N = Q_0$, it follows that

$$\begin{aligned} (A')^N &= (Q_N A Q_{N-1}^{-1}) \cdots (Q_2 A Q_1^{-1})(Q_1 A Q_0^{-1}) \\ &= Q_0 A^N Q_0^{-1} \end{aligned}$$

which means $(A')^N \simeq A^N$. We have proved that $\mathcal{AP}' \in \overline{\mathcal{AP}}^w$.

Fact 2)

Sufficiency part: Suppose that there exist nonsingular S and $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, N)$ such that (30) holds, then we will prove $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{FT}$.

It is trivial that $A' \simeq A \in \mathcal{A}$, so $A' \in \mathcal{A}$. Next we can derive

$$\begin{aligned} P'_{k+1} A_k &= (SW_{k+1}P_{k+1})A_k = SW_{k+1}AP_k \\ &= SAW_kP_k = (SAS^{-1})SW_kP_k = A'P'_k \end{aligned} \tag{32}$$

where (30) is used at the first and the last equalities. The second equality comes from $P_{k+1}A_k = AP_k$ and the third equality from $\mathcal{W} \in \mathcal{SC}(A, N)$. (32) with $A' \in \mathcal{A}$ implies $\mathcal{AP}' \in \mathcal{FT}$.

Now it is easy to see that (30) implies (21) with $Q_k = SW_k$, which prove $\mathcal{AP}' \simeq \mathcal{AP}$, i.e., $\mathcal{AP}' \in \overline{\mathcal{AP}}$.

Necessity part: Suppose $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \simeq \mathcal{AP}$, then we will prove that there exist nonsingular S and $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, N)$ such that (30) holds.

By the definition, $\mathcal{AP}' \simeq \mathcal{AP}$ means that $A' \simeq A$ and there exist nonsingular Q_k's such that (21) holds.

Therefore it is trivial that there exists a nonsingular matrix S such that $A' = SAS^{-1}$.

In the same way as the proof of Fact 3), the fact that $\mathcal{AP}, \mathcal{AP}' \in \mathcal{FT}$ gives

$$A' = (P'_{k+1}P_{k+1}^{-1}) A (P'_k P_k^{-1})^{-1} = Q_{k+1} A Q_k^{-1}$$

where the second equality comes from (21). The above equation and $A' = SAS^{-1}$ derive

$$SAS^{-1} = Q_{k+1} A Q_k^{-1}$$

which means $\mathcal{W} := \{S^{-1}Q_k \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A)$. When we denote $W_k := S^{-1}Q_k$, then $Q_k = SW_k$ and (21) goes to $P'_k = Q_kP_k = SW_kP_k$. This is just (30) .

Fact 1) Recall that the similarity class becomes the strong one when all Q_k's in (21) are equal to a constant matrix Q.

And also recall that when $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, N)$ and all the W_k's are equal to a constant matrix W, then $W \in \mathcal{C}(A)$.

Therefore it is straightforward to derive (29) from (30).

(Q.E.D.)

The fact 3) of Theorem 3 demonstrates that the weak similarity can not work to distinguish one Floquet transformation from another one. In this sense, the weak similarity has no meaning in $\mathcal{FT}$.

Recall the similarity class decomposition (19) in the set $\mathcal{A}$. When $\mathcal{AP} \in \mathcal{FT}$ has $A \in \mathcal{A}_i$, we denote $\overline{\mathcal{AP}}$ by $\overline{\mathcal{AP}}_i$. Then we will have the following theorem with respect to equivalence class decomposition in the set $\mathcal{FT}$.

Theorem 4: The set $\mathcal{FT}$ splits into exactly n_e similarity classes as follows.

$$\mathcal{FT} = \bigcup_{i \in \underline{n_e}} \overline{\mathcal{AP}}_i, \quad \overline{\mathcal{AP}}_i \bigcap \overline{\mathcal{AP}}_j = \phi \text{ for } i \neq j \tag{33}$$

where the number n_e is given in Lemma 3. □

(Proof) Consider $\mathcal{AP}, \mathcal{AP}' \in \mathcal{FT}$ with $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ and $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ and suppose that $A \in \mathcal{A}_i$ and $A' \in \mathcal{A}_j$.

If $i = j$, it can be claimed that $\mathcal{AP}' \simeq \mathcal{AP}$, so $\mathcal{AP}', \mathcal{AP} \in \overline{\mathcal{AP}}_i$. In fact, if $i = j$, then $A, A' \in \mathcal{A}_i$, so $A' \simeq A$. By using the same proof of Fact 3) of Theorem 3, from $\mathcal{AP}, \mathcal{AP}' \in \mathcal{FT}$, it is derived that $P'_k = Q_kP_k$, i.e., (21). Thus we conclude $\mathcal{AP}' \simeq \mathcal{AP}$.

If $i \neq j$, this means that A' is not similar to A, so it is trivial that $\mathcal{AP}'$ is not similar to $\mathcal{AP}$, which means $\overline{\mathcal{AP}}_i \bigcap \overline{\mathcal{AP}}_j = \phi$.

The above observations and Lemma 3 gives the theorem. (Q.E.D.)

Theorem 3 and Theorem 4 say that the set of $\mathcal{SC}(A, N)$ plays a very important role in order to characterize each equivalence class $\overline{\mathcal{AP}}_i$ and so $\mathcal{FT}$ itself. Now we will here characterize $\mathcal{SC}(A, N)$.

The next lemma is trivial.

Lemma 4: Suppose that $A \simeq A'$, i.e., $A' = SAS^{-1}$ with S being nonsingular. If $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, N)$, then $\mathcal{W}' = \{SW_kS^{-1} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A', N)$. $\square$

Therefore, in order to characterize $\mathcal{SC}(A, N)$, it is enough to assume that A is in the Jordan form.

Suppose that $A = J_A \in \boldsymbol{C}^{n\times n}$ is the Jordan form given in (18), i.e., $J_A = J_{A\sigma} \oplus J_{A\nu}$ with $J_{A\sigma} \in \boldsymbol{C}^{\bar{n}\times\bar{n}}$ and $\mathcal{W} := \{W_k = W_{\mathrm{mod}(k,N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_A, N)$. Then (28) becomes

$$W_{k+1}J_A = J_AW_k. \tag{34}$$

Let W_k be expressed as

$$W_k = \begin{bmatrix} W_{11,k} & W_{12,k} \\ W_{21,k} & W_{22,k} \end{bmatrix}$$

where $W_{11,k} \in \boldsymbol{C}^{\bar{n}\times\bar{n}}$.

From (34), it is easy to see that $W_kJ_A^N = J_A^NW_k$ for any $k \in \boldsymbol{Z}$. Note that $J_A^N = J_{A\sigma}^N \oplus J_{A\nu}^N$, and also $J_{A\sigma}^N$ and $J_{A\nu}^N$ have no common eigenvalue. Therefore we can see that that $W_{12,k} = 0$ and $W_{21,k} = 0$.

Now we can suppose that

$$W_k = W_{\sigma,k} \oplus W_{\nu,k} \quad \text{for } k \in \boldsymbol{Z}$$

where $W_{\sigma,k} \in \boldsymbol{C}^{\bar{n}\times\bar{n}}$. (34) becomes

$$W_{\sigma,k+1}J_{A\sigma} = J_{A\sigma}W_{\sigma,k} \tag{35}$$

$$W_{\nu,k+1}J_{A\nu} = J_{A\nu}W_{\nu,k} \tag{36}$$

Lemma 5: The solution $W_{\sigma,k}$ to (35) is given by

$$W_{\sigma,k} = J_{A\sigma}^{\mathrm{mod}(k/N)}W_{\sigma,0}J_{A\sigma}^{-\mathrm{mod}(k/N)} \quad \text{for } k \in \boldsymbol{Z} \tag{37}$$

with any $W_{\sigma,0} \in \mathcal{C}(J_{A\sigma}^N)$. $\square$

(Proof) (35) implies $W_{\sigma,0}J_{A\sigma}^N = J_{A\sigma}^NW_{\sigma,0}$, which means $W_{\sigma,0} \in \mathcal{C}(J_{A\sigma}^N)$. (37) is derived directly from (35) because $J_{A\sigma}$ is nonsingular. (Q.E.D.)

Next we will characterize all the solutions $W_{\nu,k}$ to (36). Suppose $J_{A\nu}$ is given as

$$J_{A\nu} = J_{m_1}(0) \oplus J_{m_2}(0) \oplus \cdots \oplus J_{m_q}(0) \tag{38}$$

where $m_1 \geq m_2 \geq \cdots \geq m_q$, and let $W_{\nu,k}$ be expressed by

$$W_{\nu,k} = \begin{bmatrix} V_{1,1,k} & V_{1,2,k} & \cdots & V_{1,q,k} \\ V_{2,1,k} & V_{2,2,k} & \cdots & V_{2,q,k} \\ \vdots & \vdots & \ddots & \vdots \\ V_{q,1,k} & V_{q,2,k} & \cdots & V_{q,q,k} \end{bmatrix} \tag{39}$$

where $V_{i,j,k} \in \boldsymbol{C}^{m_i\times m_j}$. Then (36) becomes

$$V_{i,j,k+1}J_{m_j}(0) = J_{m_i}(0)V_{i,j,k}, \quad \text{for } i, j \in \underline{q} \tag{40}$$

and $V_{i,j,k} = V_{i,j,\mathrm{mod}(k,N)}$ for $k \in \boldsymbol{Z}$.

Lemma 6: The solution $V_{i,j,k}$ to (40) is given as

$$V_{i,j,k} = \begin{cases} \begin{bmatrix} V_{m_j,k} \\ 0_{(m_i-m_j)\times m_j} \end{bmatrix} & \text{for } i < j \\[2ex] V_{m_i,k} & \text{for } i = j \\[2ex] \begin{bmatrix} 0_{m_i\times(m_j-m_i)} & V_{m_i,k} \end{bmatrix} & \text{for } i > j \end{cases} \tag{41}$$

where

$$V_{m,k} = \begin{bmatrix} v_{1,k} & v_{2,k} & v_{3,k} & \cdots & v_{m,k} \\ 0 & v_{1,k+1} & v_{2,k+1} & \cdots & v_{m-1,k+1} \\ 0 & 0 & v_{1,k+2} & \cdots & v_{m-2,k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{1,k+m-1} \end{bmatrix} \tag{42}$$

$$v_{k,j} = v_{\mathrm{mod}(k/N),j} \quad \text{for } k \in \boldsymbol{Z},\ j \in \underline{m}. \tag{43}$$

and also the parameters $v_{j,k}$'s must be chosen such that $W_{\nu,k}$ is nonsingular. $\square$

(Proof) Let the matrix $V_{i,j,k} \in \boldsymbol{C}^{m_i\times m_j}$ be denoted by

$$V_{k,i,j} = \begin{bmatrix} v_{1,1,k} & v_{1,2,k} & v_{1,3,k} & \cdots & v_{1,m_j,k} \\ v_{2,1,k} & v_{2,2,k} & v_{2,3,k} & \cdots & v_{2,m_j,k} \\ v_{3,1,k} & v_{3,2,k} & v_{3,3,k} & \cdots & v_{3,m_J,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{m_i,1,k} & v_{m_i,2,k} & v_{m_i,3,k} & \cdots & v_{m_i,m_j,k} \end{bmatrix}.$$

Then (40) implies that

$$v_{s,t-1,k+1} = v_{s+1,t,k}$$
$$\text{for } 1 \leq s \leq (m_i - 1),\ 2 \leq t \leq m_j \tag{44}$$

$$v_{s,1,k} = 0 \quad \text{for } 2 \leq s \leq m_i \tag{45}$$

$$v_{m_i,t,k+1} = 0 \quad \text{for } 1 \leq t \leq (m_j - 1). \tag{46}$$

Note that the above equations hold for any k, so the last equation is equivalent to

$$v_{m_i,t,k} = 0 \quad \text{for } 1 \leq t \leq (m_j - 1). \tag{47}$$

In the case of $i \leq j$: Note that $m_i \geq m_j$. Then (44), (45) and (47) hold if and only if all $v_{s,t,k}$'s are expressed by

$$v_{s,t,k} = \bar{v}_{k+s-1,t-s+1} \tag{48}$$

where

$$\bar{v}_{k+s-1,2-s} = 0 \quad \text{for } 2 \leq s \leq m_i \tag{49}$$

$$\bar{v}_{k+m_i-1,t-m_i+1} = 0 \quad \text{for } 1 \leq t \leq (m_j - 1). \tag{50}$$

These two equations hold for any k, so we get that $\bar{v}_{s',t'} = 0$ for $t' \leq 0$. This fact and (48) imply (41) and (42) with $v_{i,j}$ being regarded as $\bar{v}_{i,j}$ in the case of $i \leq j$.

In the case of $i > j$: Note that $m_i \leq m_j$ and denote $\ell = m_j - m_i$. Then (44), (45) and (47) hold if and only if all $v_{k,s,t}$'s are expressed by

$$v_{s,t,k} = \bar{v}_{k+s-1,t-\ell-s+1} \tag{51}$$

where

$$\begin{aligned} \bar{v}_{k+s-1,2-\ell-s} &= 0 \quad \text{for } 2 \le s \le m_i & (52) \\ \bar{v}_{k+m_i-1,t-\ell-m_i+1} &= 0 \quad \text{for } 1 \le t \le m_j - 1. & (53) \end{aligned}$$

These two equations hold for any k, so we get that $\bar{v}_{s',t'} = 0$ for $t' \le$. This fact and (48) imply (41) with $v_{i,j}$ being regarded as $\bar{v}_{i,j}$ in the case of $i > j$. (Q.E.D.)

Example 4: Suppose $J_{A\nu} = J_4(0) \oplus J_2(0)$ and $N = 3$. Then the solutions $W_{\nu,k}$'s to (36) are given as

$$W_{\nu,0} = \left[\begin{array}{cccc|cc} a_{1,0} & a_{2,0} & a_{3,0} & a_{4,0} & b_{1,0} & b_{2,0} \\ 0 & a_{1,1} & a_{2,1} & a_{3,1} & 0 & b_{1,1} \\ 0 & 0 & a_{1,2} & a_{2,2} & 0 & 0 \\ 0 & 0 & 0 & a_{1,0} & 0 & 0 \\ \hline 0 & 0 & c_{1,0} & c_{2,0} & d_{1,0} & d_{2,0} \\ 0 & 0 & 0 & c_{1,1} & 0 & d_{1,1} \end{array}\right],$$

$$W_{\nu,1} = \left[\begin{array}{cccc|cc} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} & b_{1,1} & b_{2,1} \\ 0 & a_{1,2} & a_{2,2} & a_{3,2} & 0 & b_{1,2} \\ 0 & 0 & a_{1,0} & a_{2,0} & 0 & 0 \\ 0 & 0 & 0 & a_{1,1} & 0 & 0 \\ \hline 0 & 0 & c_{1,1} & c_{2,1} & d_{1,1} & d_{2,1} \\ 0 & 0 & 0 & c_{1,2} & 0 & d_{1,2} \end{array}\right],$$

$$W_{\nu,2} = \left[\begin{array}{cccc|cc} a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} & b_{1,2} & b_{2,2} \\ 0 & a_{1,0} & a_{2,0} & a_{3,0} & 0 & b_{1,0} \\ 0 & 0 & a_{1,1} & a_{2,1} & 0 & 0 \\ 0 & 0 & 0 & a_{1,2} & 0 & 0 \\ \hline 0 & 0 & c_{1,2} & c_{2,2} & d_{1,2} & d_{2,2} \\ 0 & 0 & 0 & c_{1,0} & 0 & d_{1,0} \end{array}\right],$$

where $a_{1,0}, a_{1,1}, a_{1,2}, d_{1,0}, d_{1,1}, d_{1,2}$ must be nonzero because $W_{\nu,k}$'s are nonsingular and the other parameters $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$ are free.
$\square$

We have characterized $\mathcal{SC}(J_A, N)$ completely by Lemma 5 and Lemma 6.

We can conclude from the above observations that $\mathcal{C}(A)$ is a strict subset of $\mathcal{SC}(A, N)$, so Theorem 4 claims that $\overline{\mathcal{AP}}^s$ is a strict subset of $\overline{\mathcal{AP}}$, i.e., $\overline{\mathcal{AP}}^s \subset \overline{\mathcal{AP}}$ and $\overline{\mathcal{AP}}^s \ne \overline{\mathcal{AP}}$. Figure 1 shows the structure of $\mathcal{FT}$; there exist a finite number of similarity classes $\overline{\mathcal{AP}}_1, \overline{\mathcal{AP}}_2, \cdots, \overline{\mathcal{AP}}_{n_e}$. Each similarity class $\overline{\mathcal{AP}}_i$ consists of an infinite number of strong similarity classes $\overline{\mathcal{AP}}^s_{i1}, \overline{\mathcal{AP}}^s_{i2}, \cdots$.

V. Conclusion

When a discrete-time periodic linear system has a Floquet transformation, it has a lot of Floquet transformations. This paper aimed to solve how many essentially different Floquet transformations the system has. The approach in this paper is to define a similarity between two Floquet transformations and to make clear the structure of equivalence classes in the set of all the Floquet transformations.

Three kinds of similarities have been proposed: strong similarity, similarity, and weak similarity, and then some interesting relations among those similarities have been specified. Those investigations have detected which similarity is most suitable to split the set of Floquet transformations into its equivalence classes.

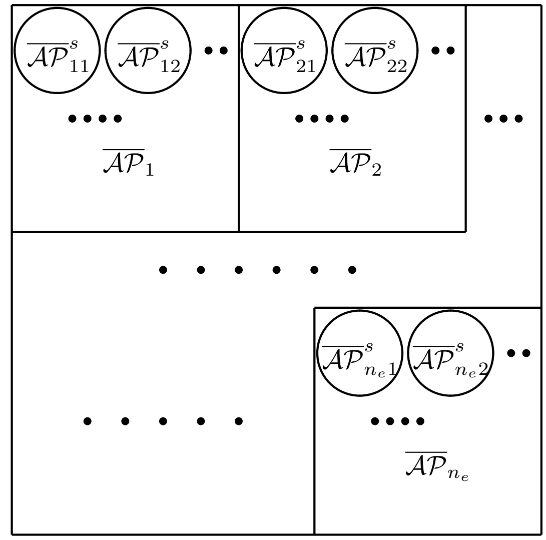

Fig. 1. The structure of $\mathcal{FT}$

Furthermore, it has been shown that in order to characterize each equivalence class, the concept of N periodic sequence of commutative matrices, which is an extension of commutative matrix, is very important and it has been done successfully to parameterize all the N periodic sequences of commutative matrices completely.

The future researches are to make clear when discrete-time periodic *non-homogeneous* systems can be transformed to time-invariant ones and to construct a theory of Floquet transformations for *nonlinear* systems.

References

[1] R.W. Brockett, "Finite Dimensional Linear System", *John Wiley and Sons,inc.*, p46-48, 1970.

[2] J. Chauvin, G. Corde, N. Petit, and P.Rouchon, "Periodic input estimation for linear periodic systems: Automotive engine applications", *Automatica*, V0l.43, pp971-980, 2007.

[3] C.T. Chen, "Linear System Theory and Design", *Holt, Rinehart and Winston*, p153-154, 1984.

[4] F.R. Gantmacher, "The Theory of Matrices", Vol.1, *Chelsea*, New York, Chapter VIII, 1959.

[5] Y. Hayakawa and T. Jimbo, "Floquet Transformations for Discrete-time Systems: Equivalence between periodic systems and time-invariant ones", the 47th IEEE CDC, 2008. (to appear)

[6] G.A. Horn and C.R. Johnson, "Topics in Matrix Analysis", *Cambridge University Press*, Chapter 6, 1991.

[7] T. Jimbo and Y. Hayakawa, "Physical-Model-Based Control of Engine Cold Start via Role State Variables", *17th World Congress of IFAC*(Seoul,Korea), pp1024-1029, 2008.

[8] P. Montagnier and R.J. Spiteri, "A Gramian-Based Controller for Linear Periodic Systems", *IEEE Trans. Automat. Contr.*, Vol.49, No.8, pp1380-1385, 2004.

[9] P. Van Dooren and J. Sreedhar, "When is a Periodic Discrete-Time System Equivalent to a Time-Invariant One?", *Linear Algebra and its Applications*, 212/213, pp131-151, 1994.

[10] J.J. Yamé and R.Hanus, "On Stabilization and Spectrum Assignment in Periodically Time-Varying Continuous-Time Systems", *IEEE Trans. Automat. Contr.*, Vol.46, No.6, pp979-983, 2001.