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Stochastic Control for a Class of Overtaking Tracking Problems: Risk-Averse Feedback Design for Performance Robustness

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Abstract— **Among of important results herein is the performance information analysis of forecasting higher-order characteristics of a general criterion of performance associated with a stochastic tracking system which is closely supervised by a reference command input and a desired trajectory. Both compactness from logic of state-space model description and quantitativity from probabilistic knowledge of stochastic disturbances are exploited to therefore allow accurate prediction of the effects of Chi-squared randomness on performance distribution of the optimal tracking problem. Information about performance-measure statistics is further utilized in the synthesis of optimal cumulant-based controllers which are thus capable of shaping the distribution of tracking performance without reliance on computationally intensive Monte Carlo analysis as needed in post-design performance assessment. As a by-product, the recent results can potentially be applicable to another substantially larger class of optimal tracking systems whereby local representations with only first two statistics for non-Gaussian random distributions of exogenous disturbances and uncertain environments may be sufficient.**

I. INTRODUCTION

A class of overtaking tracking problems is central to the study of physical systems as it is to the synthesis of feedback systems that are able to track a-priori scheduling signals and target control references. For example, interested readers may consult [1], [5] and [7] to appreciate the scope of the concepts involved in designing feedback controls for deterministic systems that optimize quadratic performance indices of reference signals. The motivation in writing the present paper is to use performance information to affect achievable performance in risk-averse decision making and feedback design. The recent work proposed by the author has begun to address some key and unique aspects as follows. First, there is a recognition process that comprehends the significance of linear-quadratic structure of the stochastic tracking dynamics and incorporates this special property in the criterion of performance. Hence, the measure of performance is, in fact a random variable with Chi-squared type and thus, all random sample path realizations from the underlying stochastic process will lead to riskier and uncertain performance. The second aspect involves the linkage of a priori knowledge of probabilistic distribution of the underlying stochastic process with system performance distribution and thus describes how higher-order statistics associated with the performance-measure are exploited to project future status of performance uncertainty. The third aspect, which is distinct from the traditional average performance optimization, is a general measure of performance riskiness as being a finite linear combination of performance-measure statistics of choice that the feedback controller uses for its adaptive control decisions. Since the account [6] by the author has initially dealt with the issue of performance robustness in stochastic tracking problems, it is therefore natural to further extend the existing tracking results with additional command input references.

Notional advantages offered by the proposed paradigm are especially effective for uncertainty analysis. That is, qualitative assessment of the impact of uncertainty caused by stochastic disturbances on system performance has long been recognized as an important and indispensable consideration in reliability-based design [2] and [4]. The paper is organized as follows. In Section II the tracking system description together with the definition of performance-measure statistics and their supporting equations associated with the Chi-squared random measure of performance is presented. Problem statements for the resulting Mayer problem in dynamic programming are given in Section III. Construction of a candidate function for the value function and the calculation of optimal feedback control accounting for multiple internalized goals of performance robustness are included in Section IV, while conclusions are drawn in Section V.

II. PRELIMINARIES

Consider a general class of stochastic tracking systems, modeled on $[t_0, t_f]$ and governed by

$$dx(t) = (A(t)x(t) + B(t)u(t))dt + G(t)dw(t) \qquad (1)$$
$$x(t_0) = x_0$$

where time-continuous coefficients $A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times n})$, $B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times m})$, and $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times p})$ are deterministic, bounded matrix-valued functions. Uncertain environments and exogenous disturbances, $w(t) \in \mathbb{R}^p$ are characterized by an p-dimensional stationary Wiener process starting from t_0, independent of the known initial condition x_0, and defined with $\{\mathcal{F}_t\}_{t\geq t_0>0}$ being its filtration on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0>0}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation of independent increments $E\left\{[w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T\right\} = W|\tau - \xi|$ for all $\tau, \xi \in [t_0, t_f]$ and $W > 0$. The set of admissible controls $L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^m))$ belongs to the Hilbert space of $\mathbb{R}^m$-valued square-integrable processes on $[t_0, t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$ with $E\left\{\int_{t_0}^{t_f} u^T(\tau)u(\tau)d\tau\right\} < \infty$. Associated with admissible 2-tuple $(x(\cdot); u(\cdot)) \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n)) \times$

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$L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m))$ is a closely guided performance-measure $J:\mathbb{R}^n\times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m))\mapsto\mathbb{R}_+$

$$J(x_0;u(\cdot)) = [x(t_f)-\gamma(t_f)]^T Q_f\,[x(t_f)-\gamma(t_f)] + \int_{t_0}^{t_f}\Big\{[x(\tau)-\gamma(\tau)]^T Q(\tau)\,[x(\tau)-\gamma(\tau)] + [u(\tau)-\rho(\tau)]^T R(\tau)[u(\tau)-\rho(\tau)]\Big\}d\tau \quad (2)$$

where the desired trajectory $\gamma(\cdot)$ and reference control input $\rho(\cdot)$ are given, deterministic, bounded and piecewise-continuous functions on $[t_0,t_f]$. Design parameters $Q_f\in\mathbb{R}^{n\times n}$, $Q\in\mathcal{C}([t_0,t_f];\mathbb{R}^{n\times n})$, and invertible $R\in\mathcal{C}([t_0,t_f];\mathbb{R}^{m\times m})$ are deterministic, bounded, matrix-valued and positive semidefinite relative weightings of the terminal state, state trajectory, and control input.

Furthermore, as shown in [6], under linear, state-feedback control together with the fact of the linear-quadratic system, all higher-order statistics of the integral-quadratic performance-measure have the quadratic-affine functional form. This common form of these higher-order statistics facilitates the definition of a risk-averse performance index and the associated optimization formulation herein. Therefore, the information pattern considered in this research is a linear time-varying feedback law generated from the tracking state $x(t)$ and reference command input $\rho(t)$ by

$$u(t)=K(t)x(t)+l_f(t)+\rho(t) \quad (3)$$

where both admissible vector-valued affine input $l_f\in\mathcal{C}([t_0,t_f];\mathbb{R}^m)$ and matrix-valued feedback gain $K\in\mathcal{C}([t_0,t_f];\mathbb{R}^{m\times n})$ are yet to be determined. Hence, for the given initial condition $(t_0,x_0)\in[t_0,t_f]\times\mathbb{R}^n$ and subject to the control decision policy (3), the dynamics of the generalized tracking problem are governed by the stochastic differential equation

$$dx(t)=(A(t)+B(t)K(t))x(t)dt + B(t)(l_f(t)+\rho(t))dt+G(t)dw(t),\quad x(t_0)=x_0 \quad (4)$$

together with the realized performance-measure for a given random realization $\omega\in\Omega$

$$J(x_0;K(\cdot),l_f(\cdot)) = [x(t_f)-\gamma(t_f)]^T Q_f\,[x(t_f)-\gamma(t_f)] + \int_{t_0}^{t_f}\Big\{[x(\tau)-\gamma(\tau)]^T Q(\tau)\,[x(\tau)-\gamma(\tau)] + [K(\tau)x(\tau)+l_f(\tau)]^T R(\tau)[K(\tau)x(\tau)+l_f(\tau)]\Big\}d\tau\,. \quad (5)$$

Clearly then, the performance-measure (5) is now a random variable with Chi-squared type. Hence, the uncertainty of performance distribution must be addressed via a complete set of higher-order statistics beyond the statistical averaging. It is necessary to generate some higher-order statistics associated with (5). Such performance-measure statistics are now called cumulants for short and thus are utilized to directly target the uncertainty of tracking performance.

In general, it is suggested that the initial condition (t_0,x_0) should be replaced by any arbitrary pair (α,x_α). Then, for the given, admissible affine input l_f and feedback gain K, (5) is considered as the "performance-to-come", $J(\alpha,x_\alpha)$.

$$J(\alpha,x_\alpha)\triangleq[x(t_f)-\gamma(t_f)]^T Q_f\,[x(t_f)-\gamma(t_f)] + \int_{\alpha}^{t_f}\Big\{[x(\tau)-\gamma(\tau)]^T Q(\tau)\,[x(\tau)-\gamma(\tau)] + [K(\tau)x(\tau)+l_f(\tau)]^T R(\tau)[K(\tau)x(\tau)+l_f(\tau)]\Big\}d\tau\,. \quad (6)$$

The moment-generating function of the "performance-to-come" of (6) is defined by

$$\varphi(\alpha,x_\alpha;\theta)\triangleq E\{\exp(\theta J(\alpha,x_\alpha))\} \quad (7)$$

for all small parameters θ in an open interval about 0. Thus, the cumulant-generating function immediately follows

$$\psi(\alpha,x_\alpha;\theta)\triangleq\ln\{\varphi(\alpha,x_\alpha;\theta)\} \quad (8)$$

for all θ in some (possibly smaller) open interval about 0 while $\ln\{\cdot\}$ denotes the natural logarithmic transformation.

Theorem 1: Cumulant-Generating Function.
Suppose that $\alpha\in[t_0,t_f]$ is some running variable and θ is a small positive parameter. When $\varphi(\alpha,x_\alpha;\theta)\triangleq\varrho(\alpha;\theta)\exp\{x_\alpha^T\Upsilon(\alpha;\theta)x_\alpha+2x_\alpha^T\eta(\alpha;\theta)\}$ and $\upsilon(\alpha;\theta)\triangleq\ln\{\varrho(\alpha;\theta)\}$, the cumulant-generating function that contains all the higher-order characteristics of the performance distribution, is then given by the expression

$$\psi(\alpha,x_\alpha;\theta)=x_\alpha^T\Upsilon(\alpha;\theta)x_\alpha+2x_\alpha^T\eta(\alpha;\theta)+\upsilon(\alpha;\theta) \quad (9)$$

where the cumulant-supporting variables $\Upsilon(\alpha;\theta)$, $\eta(\alpha;\theta)$, and $\upsilon(\alpha;\theta)$ solve the time-backward differential equations

$$\frac{d}{d\alpha}\Upsilon(\alpha;\theta)=-[A(\alpha)+B(\alpha)K(\alpha)]^T\Upsilon(\alpha;\theta) - \Upsilon(\alpha;\theta)[A(\alpha)+B(\alpha)K(\alpha)] - 2\Upsilon(\alpha;\theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha;\theta) - \theta\left[Q(\alpha)+K^T(\alpha)R(\alpha)K(\alpha)\right], \quad (10)$$

$$\frac{d}{d\alpha}\eta(\alpha;\theta)=-[A(\alpha)+B(\alpha)K(\alpha)]^T\eta(\alpha;\theta) - \Upsilon(\alpha;\theta)B(\alpha)[l_f(\alpha)+\rho(\alpha)] - \theta\left[K^T(\alpha)R(\alpha)l_f(\alpha)-Q(\alpha)\gamma(\alpha)\right], \quad (11)$$

$$\frac{d}{d\alpha}\upsilon(\alpha;\theta)=-\mathrm{Tr}\left\{\Upsilon(\alpha;\theta)G(\alpha)WG^T(\alpha)\right\} - 2\eta^T(\alpha;\theta)B(\alpha)[l_f(\alpha)+\rho(\alpha)] - \theta\left[l_f^T(\alpha)R(\alpha)l_f(\alpha)+\gamma^T(\alpha)Q(\alpha)\gamma(\alpha)\right] \quad (12)$$

with the terminal-value conditions $\Upsilon(t_f;\theta)=\theta Q_f$, $\eta(t_f;\theta)=-\theta Q_f\gamma(t_f)$, and $\upsilon(t_f;\theta)=\theta\gamma^T(t_f)Q_f\gamma(t_f)$.

Proof: For notational simplicity, it is convenient to have

$$\varpi(\alpha,x_\alpha;\theta)\triangleq\exp\{\theta J(\alpha,x_\alpha)\},\quad \varphi(\alpha,x_\alpha;\theta)\triangleq E\{\varpi(\alpha,x_\alpha;\theta)\}$$

together with the time derivative of

$$\frac{d}{d\alpha}\varphi(\alpha,x_\alpha;\theta)=-\theta\Big\{x_\alpha^T[Q(\alpha)+K^T(\alpha)R(\alpha)K(\alpha)]x_\alpha + 2x_\alpha^T[K^T(\alpha)R(\alpha)l_f(\alpha)-Q(\alpha)\gamma(\alpha)]+l_f^T(\alpha)R(\alpha)l_f(\alpha) + \gamma^T(\alpha)Q(\alpha)\gamma(\alpha)\Big\}\varphi(\alpha,x_\alpha;\theta)\,. \quad (13)$$

Using the standard Ito's formula, it yields

$$\begin{aligned} d\varphi(\alpha, x_\alpha; \theta) &= E\{d\varpi(\alpha, x_\alpha; \theta)\}\ , \\ &= \varphi_{x_\alpha}(\alpha, x_\alpha; \theta)\left[A(\alpha) + B(\alpha)K(\alpha)\right] x_\alpha d\alpha \\ &+ \varphi_\alpha(\alpha, x_\alpha; \theta)\, d\alpha + \varphi_{x_\alpha}(\alpha, x_\alpha; \theta)\, B(\alpha)[l_f(\alpha) + \rho(\alpha)] d\alpha \\ &+ \frac{1}{2}\text{Tr}\left\{\varphi_{x_\alpha x_\alpha}(\alpha, x_\alpha; \theta)\, G(\alpha)\, W G^T(\alpha)\right\} d\alpha\,, \end{aligned}$$

which under the aforementioned definition $\varphi(\alpha, x_\alpha; \theta) \triangleq \varrho(\alpha; \theta)\exp\left\{x_\alpha^T \Upsilon(\alpha;\theta) x_\alpha + 2x_\alpha^T \eta(\alpha;\theta)\right\}$ and its partial derivatives, leads to the total derivative with respect to time

$$\begin{aligned} \frac{d}{d\alpha}\varphi(\alpha, x_\alpha; \theta) &= \Bigg\{\Bigg[\frac{\frac{d}{d\alpha}\varrho(\alpha;\theta)}{\varrho(\alpha,\theta)} + x_\alpha^T \frac{d}{d\alpha}\Upsilon(\alpha;\theta) x_\alpha \\ &+ 2x_\alpha^T \frac{d}{d\alpha}\eta(\alpha;\theta)\Bigg] + x_\alpha^T \left[A(\alpha) + B(\alpha)K(\alpha)\right]^T \Upsilon(\alpha;\theta) x_\alpha \\ &+ x_\alpha^T \Upsilon(\alpha;\theta)\left[A(\alpha) + B(\alpha)K(\alpha)\right] x_\alpha \\ &+ 2x_\alpha^T \left[A(\alpha) + B(\alpha)K(\alpha)\right]^T \eta(\alpha;\theta) \\ &+ 2x_\alpha^T \Upsilon(\alpha;\theta) B(\alpha)[l_f(\alpha) + \rho(\alpha)] \\ &+ 2\eta^T(\alpha;\theta) B(\alpha)[l_f(\alpha) + \rho(\alpha)] + \text{Tr}\left\{\Upsilon(\alpha;\theta) G(\alpha) W G^T(\alpha)\right\} \\ &+ 2x_\alpha^T \Upsilon(\alpha;\theta) G(\alpha) W G^T(\alpha) \Upsilon(\alpha;\theta) x_\alpha\Bigg\}\varphi(\alpha, x_\alpha; \theta)\,. \end{aligned} \quad (14)$$

Replacing (13) into (14) and having both linear and quadratic terms independent of x_α, it requires that

$$\begin{aligned} \frac{d}{d\alpha}\Upsilon(\alpha;\theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \Upsilon(\alpha;\theta) \\ &- \Upsilon(\alpha;\theta)[A(\alpha) + B(\alpha)K(\alpha)] \\ &- 2\Upsilon(\alpha;\theta) G(\alpha) W G^T(\alpha) \Upsilon(\alpha;\theta) \\ &- \theta\left[Q(\alpha) + K^T(\alpha) R(\alpha) K(\alpha)\right]\ , \\ \frac{d}{d\alpha}\eta(\alpha;\theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \eta(\alpha;\theta) \\ &- \Upsilon(\alpha;\theta) B(\alpha)\left[l_f(\alpha) + \rho(\alpha)\right] \\ &- \theta\left[K^T(\alpha) R(\alpha) l_f(\alpha) - Q(\alpha)\gamma(\alpha)\right]\ , \\ \frac{d}{d\alpha}\upsilon(\alpha;\theta) &= -\text{Tr}\left\{\Upsilon(\alpha;\theta) G(\alpha)\, W G^T(\alpha)\right\} \\ &- 2\eta^T(\alpha;\theta) B(\alpha)\left[l_f(\alpha) + \rho(\alpha)\right] \\ &- \theta\left[l_f^T(\alpha) R(\alpha) l_f(\alpha) + \gamma^T(\alpha) Q(\alpha)\gamma(\alpha)\right]\ . \end{aligned}$$

At the final time $\alpha = t_f$, it follows that $\varphi(t_f, x(t_f);\theta) = \varrho(t_f;\theta)\exp\left\{x^T(t_f)\Upsilon(t_f;\theta)x(t_f) + 2x^T(t_f)\eta(t_f;\theta)\right\} = E\left\{\exp\left\{\theta[x(t_f) - \gamma(t_f)]^T Q_f [x(t_f) - \gamma(t_f)]\right\}\right\}$ which in turn yields the terminal-value conditions as $\Upsilon(t_f;\theta) = \theta Q_f$, $\eta(t_f;\theta) = -\theta Q_f \gamma(t_f)$, $\varrho(t_f;\theta) = \exp\left\{\theta\gamma^T(t_f) Q_f \gamma(t_f)\right\}$, and $\upsilon(t_f;\theta) = \theta\gamma^T(t_f) Q_f \gamma(t_f)$. ■

Remark 1: The expression for cumulant-generating function (9) for the generalized performance-measure (5) indicates that additional affine and trailing terms take into account of dynamics mismatched in the transient responses. By definition, higher-order statistics that encapsulate the uncertain nature of tracking performance can now be generated via a MacLaurin series of (9)

$$\begin{aligned} \psi(\alpha, x_\alpha; \theta) &\triangleq \sum_{i=1}^{\infty} \kappa_i(\alpha, x_\alpha)\, \frac{\theta^i}{i!}\,, \\ &= \sum_{i=1}^{\infty} \frac{\partial^{(i)}}{\partial\theta^{(i)}}\psi(\alpha, x_\alpha;\theta)\Big|_{\theta=0}\ \frac{\theta^i}{i!} \end{aligned} \quad (15)$$

from which $\kappa_i(\alpha, x_\alpha)$ is denoted as the ith-performance-measure statistics or the ith-cumulant. Moreover, the series expansion coefficients are thus obtained by using the cumulant-generating function (9)

$$\begin{aligned} \frac{\partial^{(i)}}{\partial\theta^{(i)}}\psi(\alpha, x_\alpha;\theta)\Big|_{\theta=0} &= x_\alpha^T\ \frac{\partial^{(i)}}{\partial\theta^{(i)}}\Upsilon(\alpha;\theta)\Big|_{\theta=0} x_\alpha \\ &+ 2x_\alpha^T\ \frac{\partial^{(i)}}{\partial\theta^{(i)}}\eta(\alpha;\theta)\Big|_{\theta=0} + \frac{\partial^{(i)}}{\partial\theta^{(i)}}\upsilon(\alpha;\theta)\Big|_{\theta=0}\ . \end{aligned} \quad (16)$$

In view of the results (15) and (16), the ith-cumulant for the generalized tracking problem therefore follows

$$\begin{aligned} \kappa_i(\alpha, x_\alpha) &= x_\alpha^T\ \frac{\partial^{(i)}}{\partial\theta^{(i)}}\Upsilon(\alpha;\theta)\Big|_{\theta=0} x_\alpha \\ &+ 2x_\alpha^T\ \frac{\partial^{(i)}}{\partial\theta^{(i)}}\eta(\alpha;\theta)\Big|_{\theta=0} + \frac{\partial^{(i)}}{\partial\theta^{(i)}}\upsilon(\alpha;\theta)\Big|_{\theta=0}\ , \end{aligned} \quad (17)$$

for any finite $1 \le i < \infty$.

For notational convenience, the following definitions

$$\begin{aligned} H_i(\alpha) &\triangleq \frac{\partial^{(i)}}{\partial\theta^{(i)}}\Upsilon(\alpha;\theta)\Big|_{\theta=0}\ , \\ \breve{D}_i(\alpha) &\triangleq \frac{\partial^{(i)}}{\partial\theta^{(i)}}\eta(\alpha;\theta)\Big|_{\theta=0}\ , \\ D_i(\alpha) &\triangleq \frac{\partial^{(i)}}{\partial\theta^{(i)}}\upsilon(\alpha;\theta)\Big|_{\theta=0} \end{aligned}$$

are introduced so that the next theorem illustrates a tractable procedure of generating cumulants or performance-measure statistics in time domain. This calculation is preferred to that of (17) for the reason that the resulting cumulant-generating equations now allow the incorporation of classes of linear feedback controllers in risk-averse tracking design synthesis.

Theorem 2: Generalized Performance-Measure Statistics. The tracking dynamics governed by (4)-(5) attempt to follow the set-point signals $\gamma(t)$ and $\rho(t)$ with the generalized performance-measure (5). For $k \in \mathbb{Z}^+$, the kth-cumulant is given by the closed-form

$$\kappa_k = x_0^T H_k(t_0) x_0 + 2x_0^T \breve{D}_k(t_0) + D_k(t_0) \quad (18)$$

wherein the cumulant-generating components $\{H_i(\alpha)\}_{i=1}^k$, $\{\breve{D}_i(\alpha)\}_{i=1}^k$, and $\{D_i(\alpha)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the time-backward differential equations (with the dependence of

$H_i(\alpha)$, $\breve{D}_i(\alpha)$, and $D_i(\alpha)$ upon l_f and K suppressed)

$$\frac{d}{d\alpha}H_1(\alpha) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T H_1(\alpha) - H_1(\alpha)\left[A(\alpha)+B(\alpha)K(\alpha)\right] - Q(\alpha) - K^T(\alpha)R(\alpha)K(\alpha)\,, \quad (19)$$

$$\frac{d}{d\alpha}H_i(\alpha) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T H_i(\alpha) - H_i(\alpha)\left[A(\alpha)+B(\alpha)K(\alpha)\right] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H_j(\alpha)G(\alpha)WG^T(\alpha)H_{i-j}(\alpha)\,, \quad (20)$$

$$\frac{d}{d\alpha}\breve{D}_1(\alpha) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{D}_1(\alpha) - H_1(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right] - K^T(\alpha)R(\alpha)l_f(\alpha) + Q(\alpha)\gamma(\alpha)\,, \quad (21)$$

$$\frac{d}{d\alpha}\breve{D}_i(\alpha) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{D}_i(\alpha) - H_i(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right]\,, \quad (22)$$

$$\frac{d}{d\alpha}D_1(\alpha) = -\mathrm{Tr}\left\{H_1(\alpha)G(\alpha)WG^T(\alpha)\right\} - 2\breve{D}_1^T(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right] - l_f^T(\alpha)R(\alpha)l_f(\alpha) - \gamma^T(\alpha)Q(\alpha)\gamma(\alpha)\,, \quad (23)$$

$$\frac{d}{d\alpha}D_i(\alpha) = -\mathrm{Tr}\left\{H_i(\alpha)G(\alpha)WG^T(\alpha)\right\} - 2\breve{D}_i^T(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right] \quad (24)$$

where terminal-value conditions $H_1(t_f) = Q_f$, $H_i(t_f) = 0$ for $2 \le i \le k$; $\breve{D}_1(t_f) = -Q_f\gamma(t_f)$, $\breve{D}_i(t_f) = 0$ for $2 \le i \le k$; and $D_1(t_f) = \gamma^T(t_f)Q_f\gamma(t_f)$, $D_i(t_f) = 0$ for $2 \le i \le k$.

III. Problem Statements

Within the structure of cumulants (18), all the cumulant values depend in part of the known initial condition $x(t_0)$. Although the different states $x(t)$ will result in different values for the "performance-to-come" (5), the cumulant values are however, functions of time-backward evolutions of the cumulant-generating components $H_i(\alpha)$, $\breve{D}_i(\alpha)$ and $D_i(\alpha)$ and thus do not take into account of all the intermediate values $x(t)$. Consequently, this fact makes the new optimization problem particularly unique as compared with the more traditional dynamic programming class of investigations. In other words, the time-backward trajectories (19)-(24) are therefore considered as the "new" dynamical equations from which the resulting Mayer optimization [3] and associated value function in dynamic programming now depend on these "new" states $H_i(\alpha)$, $\breve{D}_i(\alpha)$ and $D_i(\alpha)$, not the states $x(t)$ as traditionally expected. Furthermore, it is important to see that this mathematical representation (19)-(24) underlies the conceptual structure to extract the knowledge of intrinsic performance variability introduced by the process noise stochasticity in definite terms of performance-measure statistics (18).

Next, it is convenient to introduce k-tuple variables $\mathcal{H}$, $\breve{\mathcal{D}}$, and $\mathcal{D}$ as follows $\mathcal{H}(\cdot) \triangleq (\mathcal{H}_1(\cdot),\ldots,\mathcal{H}_k(\cdot))$, $\breve{\mathcal{D}}(\cdot) \triangleq \left(\breve{\mathcal{D}}_1(\cdot),\ldots,\breve{\mathcal{D}}_k(\cdot)\right)$, and $\mathcal{D}(\cdot) \triangleq (\mathcal{D}_1(\cdot),\ldots,\mathcal{D}_k(\cdot))$ for each element $\mathcal{H}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n\times n})$ of $\mathcal{H}$, $\breve{\mathcal{D}}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^n)$ of $\breve{\mathcal{D}}$, and $\mathcal{D}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R})$ of $\mathcal{D}$ having the representations $\mathcal{H}_i(\cdot) \triangleq H_i(\cdot)$, $\breve{\mathcal{D}}_i(\cdot) \triangleq \breve{D}_i(\cdot)$, and $\mathcal{D}_i(\cdot) \triangleq D_i(\cdot)$ with the right members satisfying the dynamic equations (19)-(24) on the horizon $[t_0,t_f]$.

The problem formulation is considerably simplified if the following mappings are introduced accordingly

$$\mathcal{F}_i : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k \times \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}$$
$$\breve{\mathcal{G}}_i : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \mapsto \mathbb{R}^n$$
$$\mathcal{G}_i : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^m \mapsto \mathbb{R}$$

where the actions are given by

$$\mathcal{F}_1(\alpha,\mathcal{H},K) \triangleq -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \mathcal{H}_1(\alpha) - \mathcal{H}_1(\alpha)\left[A(\alpha)+B(\alpha)K(\alpha)\right] - Q(\alpha) - K^T(\alpha)R(\alpha)K(\alpha)$$

$$\mathcal{F}_i(\alpha,\mathcal{H},K) \triangleq -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \mathcal{H}_i(\alpha) - \mathcal{H}_i(\alpha)\left[A(\alpha)+B(\alpha)K(\alpha)\right] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{i-j}(\alpha)$$

$$\breve{\mathcal{G}}_1\left(\alpha,\mathcal{H},\breve{\mathcal{D}},K,l_f\right) \triangleq -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{\mathcal{D}}_1(\alpha) - \mathcal{H}_1(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right] - K^T(\alpha)R(\alpha)l_f(\alpha) + Q(\alpha)\gamma(\alpha)$$

$$\breve{\mathcal{G}}_i\left(\alpha,\mathcal{H},\breve{\mathcal{D}},K,l_f\right) \triangleq -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{\mathcal{D}}_i(\alpha) - \mathcal{H}_i(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right]$$

$$\mathcal{G}_1\left(\alpha,\mathcal{H},\breve{\mathcal{D}},l_f\right) \triangleq -\mathrm{Tr}\left\{\mathcal{H}_1(\alpha)G(\alpha)WG^T(\alpha)\right\} - 2\breve{\mathcal{D}}_1^T(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right] - l_f^T(\alpha)R(\alpha)l_f(\alpha) - \gamma^T(\alpha)Q(\alpha)\gamma(\alpha)$$

$$\mathcal{G}_i\left(\alpha,\mathcal{H},\breve{\mathcal{D}},l_f\right) \triangleq -\mathrm{Tr}\left\{\mathcal{H}_i(\alpha)G(\alpha)WG^T(\alpha)\right\} - 2\breve{\mathcal{D}}_i^T(\alpha)B(\alpha)\left[l_f(\alpha)+\rho(\alpha)\right]\,.$$

For even more compactness of notations, the next product mappings are further needed

$$\mathcal{F}_1\times\cdots\times\mathcal{F}_k : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times\mathbb{R}^{m\times n} \mapsto (\mathbb{R}^{n\times n})^k$$
$$\breve{\mathcal{G}}_1\times\cdots\times\breve{\mathcal{G}}_k : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^n)^k\times\mathbb{R}^{m\times n}\times\mathbb{R}^m \mapsto (\mathbb{R}^n)^k$$
$$\mathcal{G}_1\times\cdots\times\mathcal{G}_k : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^n)^k\times\mathbb{R}^m \mapsto \mathbb{R}^k$$

along with the corresponding notations $\mathcal{F} \triangleq \mathcal{F}_1\times\cdots\times\mathcal{F}_k$, $\breve{\mathcal{G}} \triangleq \breve{\mathcal{G}}_1\times\cdots\times\breve{\mathcal{G}}_k$, and $\mathcal{G} \triangleq \mathcal{G}_1\times\cdots\times\mathcal{G}_k$. Thus, the dynamic equations of motion (19)-(24) can be rewritten as follows

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\,, \quad \mathcal{H}(t_f) \equiv \mathcal{H}_f$$
$$\frac{d}{d\alpha}\breve{\mathcal{D}}(\alpha) = \breve{\mathcal{G}}(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),K(\alpha),l_f(\alpha)), \quad \breve{\mathcal{D}}(t_f) \equiv \breve{\mathcal{D}}_f$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),l_f(\alpha)), \quad \mathcal{D}(t_f) \equiv \mathcal{D}_f$$

where the k-tuple final values $\mathcal{H}_f \triangleq (Q_f,0,\ldots,0)$, $\breve{\mathcal{D}}_f \triangleq (-Q_f\gamma(t_f),0,\ldots,0)$, and $\mathcal{D}_f \triangleq (\gamma^T(t_f)Q_f\gamma(t_f),\ldots,0)$.

Note that the product system uniquely determines $\mathcal{H}$, $\breve{\mathcal{D}}$ and $\mathcal{D}$ once the admissible affine input l_f and feedback gain K are specified. Hence, they are considered as $\mathcal{H} = \mathcal{H}(\cdot, K)$, $\breve{\mathcal{D}} = \breve{\mathcal{D}}(\cdot, K, l_f)$ and $\mathcal{D} = \mathcal{D}(\cdot, K, l_f)$. The risk-averse performance index is defined by these control parameters l_f and K.

Definition 1: Performance Index.
Fix $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then, for the given (t_0, x_0), the performance index in risk-aversion, i.e., $\phi_{tk} : \{t_0\} \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}_+$ for the generalized tracking problem is defined as follows

$$\phi_{tk}\left(t_0, \mathcal{H}(t_0, K), \breve{\mathcal{D}}(t_0, K, l_f), \mathcal{D}(t_0, K, l_f)\right) \triangleq \tag{25}$$
$$\sum_{i=1}^{k} \mu_i [x_0^T \mathcal{H}_i(t_0, K) x_0 + 2x_0^T \breve{\mathcal{D}}_i(t_0, K, l_f) + \mathcal{D}_i(t_0, K, l_f)] .$$

The real constant scalars μ_i represent different degrees of freedom to shape the distribution of closed-loop tracking performance wherever they matter the most by a means of placing particular weights on any specific performance-measure statistics (i.e., mean, variance, skewness, flatness, etc.) associated with (5). The unique solutions $\{\mathcal{H}_i(t_0, K)\}_{i=1}^k$, $\left\{\breve{\mathcal{D}}_i(t_0, K, l_f)\right\}_{i=1}^k$, and $\{\mathcal{D}_i(t_0, K, l_f)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the time-backward equations of motion

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)), \quad \mathcal{H}(t_f) \tag{26}$$
$$\frac{d}{d\alpha}\breve{\mathcal{D}}(\alpha) = \breve{\mathcal{G}}\left(\alpha, \mathcal{H}(\alpha), \breve{\mathcal{D}}(\alpha), K(\alpha), l_f(\alpha)\right), \breve{\mathcal{D}}(t_f) \tag{27}$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}\left(\alpha, \mathcal{H}(\alpha), \breve{\mathcal{D}}(\alpha), l_f(\alpha)\right), \quad \mathcal{D}(t_f) . \tag{28}$$

For given terminal data $(t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f)$, the classes of admissible affine inputs and feedback gains are then defined.

Definition 2: Admissible Inputs and Feedback Gains.
Let compact subsets $\overline{L} \subset \mathbb{R}^m$ and $\overline{K} \subset \mathbb{R}^{m \times n}$ be the sets of allowable linear control inputs and gain values. For the given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the set of admissible affine inputs $\mathcal{L}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ and feedback gains $\mathcal{K}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ are respectively assumed to be the classes of $\mathcal{C}([t_0, t_f]; \mathbb{R}^m)$ and $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})$ with values $l_f(\cdot) \in \overline{L}$ and $K(\cdot) \in \overline{K}$ for which solutions to the dynamic equations (26)-(28) exist on the interval of optimization $[t_0, t_f]$.

Definition 3: Optimization Problem.
Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then, the risk-averse control optimization problem over $[t_0, t_f]$ is given by the minimization of (25) over $l_f(\cdot) \in \mathcal{L}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu}$, $K(\cdot) \in \mathcal{K}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ and subject to the dynamic equations of motion (26)-(28).

The subsequent results will then illustrate a construction of potential candidates for the value function.

Definition 4: Reachable Set.
Let reachable set $\mathcal{Q} \triangleq \left\{ \left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) \in [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k \right\}$ such that $\mathcal{L}_{\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}; \mu} \neq 0$ and $\mathcal{K}_{\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}; \mu} \neq 0$.

By adapting to the initial cost problem and the terminologies present in the risk-averse control, the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function $\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ is given as follows.

Theorem 3: HJB Equation-Mayer Problem.
Let $\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ be any interior point of the reachable set $\mathcal{Q}$ at which the value function $\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ is differentiable. If there exist optimal affine signal $l_f^* \in \mathcal{L}_{\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}; \mu}$ and feedback gain $K^* \in \mathcal{K}_{\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}; \mu}$, then the partial differential equation of dynamic programming

$$0 = \min_{l_f \in \overline{L},\, K \in \overline{K}} \left\{ \frac{\partial}{\partial \varepsilon}\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) \right. \tag{29}$$
$$+ \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) \operatorname{vec}\left(\mathcal{F}\left(\varepsilon, \mathcal{Y}, K\right)\right)$$
$$+ \frac{\partial}{\partial \operatorname{vec}\left(\breve{\mathcal{Z}}\right)}\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) \operatorname{vec}\left(\breve{\mathcal{G}}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, K, l_f\right)\right)$$
$$\left. + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) \operatorname{vec}\left(\mathcal{G}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, l_f\right)\right) \right\}$$

is satisfied together with the terminal-value condition $\mathcal{V}\left(t_0, \mathcal{H}_0, \breve{\mathcal{D}}_0, \mathcal{D}_0\right) = \phi_{tk}\left(t_0, \mathcal{H}_0, \breve{\mathcal{D}}_0, \mathcal{D}_0\right)$.

Theorem 4: Verification Theorem.
Fix $k \in \mathbb{Z}^+$ and let $\mathcal{W}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ be a continuously differentiable solution of the HJB equation (29) which satisfies the boundary condition

$$\mathcal{W}\left(t_0, \mathcal{H}_0, \breve{\mathcal{D}}_0, \mathcal{D}_0\right) = \phi_{tk}\left(t_0, \mathcal{H}_0, \breve{\mathcal{D}}_0, \mathcal{D}_0\right) . \tag{30}$$

Let $(t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f)$ be in $\mathcal{Q}$; (l_f, K) in $\mathcal{L}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu} \times \mathcal{K}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu}$; $\mathcal{H}$, $\breve{\mathcal{D}}$ and $\mathcal{D}$ the corresponding solutions of (26)-(28). Then, $\mathcal{W}(\alpha, \mathcal{H}(\alpha), \breve{\mathcal{D}}(\alpha), \mathcal{D}(\alpha))$ is a time-backward increasing function of α. If (l_f^*, K^*) is in $\mathcal{L}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu} \times \mathcal{K}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ defined on $[t_0, t_f]$ with corresponding solutions, $\mathcal{H}^*$, $\breve{\mathcal{D}}^*$, and $\mathcal{D}^*$ of (26)-(28) such that for $\alpha \in [t_0, t_f]$

$$0 = \frac{\partial}{\partial \varepsilon}\mathcal{W}\left(\alpha, \mathcal{H}^*(\alpha), \breve{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)\right)$$
$$+ \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{W}\left(\alpha, \mathcal{H}^*(\alpha), \breve{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)\right) \cdot$$
$$\cdot \operatorname{vec}\left(\mathcal{F}\left(\alpha, \mathcal{H}^*(\alpha), K^*(\alpha)\right)\right)$$
$$+ \frac{\partial}{\partial \operatorname{vec}(\breve{\mathcal{Z}})}\mathcal{W}\left(\alpha, \mathcal{H}^*(\alpha), \breve{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)\right) \cdot$$
$$\cdot \operatorname{vec}\left(\breve{\mathcal{G}}\left(\alpha, \mathcal{H}^*(\alpha), \breve{\mathcal{D}}^*(\alpha), K^*(\alpha), l_f^*(\alpha)\right)\right)$$
$$+ \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{W}\left(\alpha, \mathcal{H}^*(\alpha), \breve{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)\right) \cdot$$
$$\cdot \operatorname{vec}\left(\mathcal{G}\left(\alpha, \mathcal{H}^*(\alpha), \breve{\mathcal{D}}^*(\alpha), l_f^*(\alpha)\right)\right) , \tag{31}$$

then both l_f^* and K^* are optimal. Moreover, it follows that

$$\mathcal{W}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) = \mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) \tag{32}$$

where $\mathcal{V}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ is the value function.

IV. OPTIMAL RISK-AVERSE TRACKING SOLUTION

Because the optimization problem considered herein is in "Mayer form", it is therefore solved by applying an adaptation of the Mayer form verification theorem of dynamic programming given in [3]. Consequently, it requires to parameterize all starting times and states of a family of optimization problems as $\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$. For instance, the states of the system (26)-(28) defined on $[t_0, \varepsilon]$ with the terminal values are now denoted by $\mathcal{H}(\varepsilon) \equiv \mathcal{Y}$, $\breve{\mathcal{D}}(\varepsilon) \equiv \breve{\mathcal{Z}}$, and $\mathcal{D}(\varepsilon) \equiv \mathcal{Z}$. Furthermore, with the observation of performance index (25) being quadratic affine in terms of the arbitrarily fixed x_0, a candidate solution to the HJB equation (29) may be sought in the form of

$$\mathcal{W}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) = x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{Y}_i + \mathcal{E}_i(\varepsilon)\right) x_0 + 2x_0^T \sum_{i=1}^k \mu_i \left(\breve{\mathcal{Z}}_i + \breve{\mathcal{T}}_i(\varepsilon)\right) + \sum_{i=1}^k \mu_i \left(\mathcal{Z}_i + \mathcal{T}_i(\varepsilon)\right) \quad (33)$$

where the parametric functions $\mathcal{E}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n\times n})$, $\breve{\mathcal{T}}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^n)$ and $\mathcal{T}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ are yet to be determined. One can then obtain the derivative of $\mathcal{W}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ with respect to ε as

$$\frac{d}{d\varepsilon}\mathcal{W}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right) = x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K) + \frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)\right) x_0 + 2x_0^T \sum_{i=1}^k \mu_i \left(\breve{\mathcal{G}}_i\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, K, l_f\right) + \frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon)\right) + \sum_{i=1}^k \mu_i \left(\mathcal{G}_i\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, l_f\right) + \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon)\right) \quad (34)$$

provided that $l_f \in \overline{L}$ and $K \in \overline{K}$. Trying this candidate for the value function (33) into the HJB equation (29) yields

$$0 \equiv \min_{l_f \in \overline{L},\, K \in \overline{K}} \Bigg\{ x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K) + \frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)\right) x_0 + 2x_0^T \sum_{i=1}^k \mu_i \left(\breve{\mathcal{G}}_i\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, K, l_f\right) + \frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon)\right) + \sum_{i=1}^k \mu_i \left(\mathcal{G}_i\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, l_f\right) + \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon)\right) \Bigg\}. \quad (35)$$

Since the initial condition x_0 is an arbitrary vector, the necessary condition for an extremum of (25) on $[t_0, \varepsilon]$ is obtained by differentiating the expression within the bracket of (35) with respect to the control parameters l_f and K as follows

$$l_f(\varepsilon, \breve{\mathcal{Z}}) = -R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \breve{\mathcal{Z}}_r\,, \quad (36)$$

$$K(\varepsilon, \mathcal{Y}) = -R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_r \quad (37)$$

where the weightings $\widehat{\mu}_r \triangleq \mu_i/\mu_1$ are normalized by $\mu_1 > 0$. Replacing (36) and (37) into the HJB equation (35) leads to the value of the minimum

$$\begin{aligned}
& x_0^T \Bigg[\sum_{i=1}^k \mu_i \frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i - \sum_{i=1}^k \mu_i \mathcal{Y}_i A(\varepsilon) \\
&- \mu_1 Q(\varepsilon) + \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i \\
&+ \sum_{i=1}^k \mu_i \mathcal{Y}_i(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \widehat{\mu}_s \mathcal{Y}_s \\
&- \mu_1 \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \widehat{\mu}_s \mathcal{Y}_s \\
&- \sum_{i=2}^k \mu_i \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{Y}_j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{i-j} \Bigg] x_0 \\
&+ 2x_0^T \Bigg\{ \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^k \mu_i \breve{\mathcal{Z}}_i + \mu_1 Q(\varepsilon)\gamma(\varepsilon) \\
&+ \sum_{r=1}^k \mu_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^k \mu_i \breve{\mathcal{Z}}_i \\
&- \sum_{i=1}^k \mu_i \mathcal{Y}_i B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \breve{\mathcal{Z}}_r + \rho(\varepsilon) \right] \\
&- \mu_1 \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \widehat{\mu}_s \breve{\mathcal{Z}}_s \Bigg\} \\
&+ \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) - \sum_{i=1}^k \mu_i \mathrm{Tr}\left\{\mathcal{Y}_i G(\varepsilon) W G^T(\varepsilon)\right\} \\
&- 2\sum_{i=1}^k \mu_i \breve{\mathcal{Z}}_i^T B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \breve{\mathcal{Z}}_r + \rho(\varepsilon) \right] \\
&- \mu_1 \gamma^T(\varepsilon) Q(\varepsilon) \gamma(\varepsilon) \\
&- \mu_1 \sum_{r=1}^k \widehat{\mu}_r \breve{\mathcal{Z}}_r^T B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \widehat{\mu}_s \breve{\mathcal{Z}}_s\,. \quad (38)
\end{aligned}$$

What remains is to exhibit the time parametric functions for the candidate function $\mathcal{W}\left(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}\right)$ of the value function, i.e., $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$, $\left\{\breve{\mathcal{T}}_i(\cdot)\right\}_{i=1}^k$, and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ which yield a sufficient condition to have the left-hand side of (38) being zero for any $\varepsilon \in [t_0, t_f]$, when $\{\mathcal{Y}_i\}_{i=1}^k$ and $\left\{\breve{\mathcal{Z}}_i\right\}_{i=1}^k$ are evaluated along the solutions of the cumulant-generating equations (26)-(28).

With a careful examination of (38), one can infer that $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$, $\left\{\breve{\mathcal{T}}_i(\cdot)\right\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ may be chosen to satisfy the time-forward differential equations as follows

$$\frac{d}{d\varepsilon}\mathcal{E}_1(\varepsilon) = A^T(\varepsilon)\mathcal{H}_1(\varepsilon) + \mathcal{H}_1(\varepsilon)A(\varepsilon) + Q(\varepsilon) - \mathcal{H}_1(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \widehat{\mu}_s \mathcal{H}_s(\varepsilon)$$

$$
\begin{aligned}
&- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_1(\varepsilon) \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_s(\varepsilon)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) &= A^T(\varepsilon) \mathcal{H}_i(\varepsilon) + \mathcal{H}_i(\varepsilon) A(\varepsilon) \\
&- \mathcal{H}_i(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_s(\varepsilon) \\
&- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_i(\varepsilon) \\
&+ \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{i-j}(\varepsilon)
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \breve{\mathcal{T}}_1(\varepsilon) &= A^T(\varepsilon) \breve{\mathcal{D}}_1(\varepsilon) - Q(\varepsilon)\gamma(\varepsilon) \\
&- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \breve{\mathcal{D}}_1(\varepsilon) \\
&+ \mathcal{H}_1(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right] \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \breve{\mathcal{D}}_s(\varepsilon)
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \breve{\mathcal{T}}_i(\varepsilon) &= A^T(\varepsilon) \breve{\mathcal{D}}_i(\varepsilon) \\
&- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \breve{\mathcal{D}}_i(\varepsilon) \\
&+ \mathcal{H}_i(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right]
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{T}_1(\varepsilon) &= \mathrm{Tr}\left\{ \mathcal{H}_1(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \right\} + \gamma^T(\varepsilon) Q(\varepsilon) \gamma(\varepsilon) \\
&+ 2\breve{\mathcal{D}}_1^T(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right] \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r^T(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \breve{\mathcal{D}}_s(\varepsilon)
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) &= \mathrm{Tr}\left\{ \mathcal{H}_i(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \right\} \\
&+ 2\breve{\mathcal{D}}_i^T(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right]
\end{aligned} \tag{44}
$$

The affine control input and feedback gain specified in (36) and (37) are now applied along the solution trajectories of the time-backward Riccati-type equations (26)-(28)

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{H}_1(\varepsilon) &= -A^T(\varepsilon) \mathcal{H}_1(\varepsilon) - \mathcal{H}_1(\varepsilon) A(\varepsilon) - Q(\varepsilon) \\
&+ \mathcal{H}_1(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_s(\varepsilon) \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_1(\varepsilon) \\
&- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_s(\varepsilon)
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{H}_i(\varepsilon) &= -A^T(\varepsilon) \mathcal{H}_i(\varepsilon) - \mathcal{H}_i(\varepsilon) A(\varepsilon) \\
&+ \mathcal{H}_i(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_s(\varepsilon) \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_i(\varepsilon) \\
&- \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{i-j}(\varepsilon)
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \breve{\mathcal{D}}_1(\varepsilon) &= -A^T(\varepsilon) \breve{\mathcal{D}}_1(\varepsilon) + Q(\varepsilon)\gamma(\varepsilon) \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \breve{\mathcal{D}}_1(\varepsilon) \\
&- \mathcal{H}_1(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right] \\
&- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \breve{\mathcal{D}}_s(\varepsilon)
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \breve{\mathcal{D}}_i(\varepsilon) &= -A^T(\varepsilon) \breve{\mathcal{D}}_i(\varepsilon) \\
&+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \breve{\mathcal{D}}_i(\varepsilon) \\
&- \mathcal{H}_i(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right]
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{D}_1(\varepsilon) &= -\mathrm{Tr}\left\{ \mathcal{H}_1(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \right\} - \gamma^T(\varepsilon) Q(\varepsilon) \gamma(\varepsilon) \\
&- 2\breve{\mathcal{D}}_1^T(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right] \\
&- \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r^T(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \breve{\mathcal{D}}_s(\varepsilon)
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathcal{D}_i(\varepsilon) &= -\mathrm{Tr}\left\{ \mathcal{H}_i(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \right\} \\
&- 2\breve{\mathcal{D}}_i^T(\varepsilon) B(\varepsilon) \left[-R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \breve{\mathcal{D}}_r(\varepsilon) + \rho(\varepsilon) \right]
\end{aligned} \tag{50}
$$

where the terminal-value conditions $\mathcal{H}_1(t_f) = Q_f$, $\mathcal{H}_i(t_f) = 0$ for $2 \leq i \leq k$; $\breve{\mathcal{D}}_1(t_f) = -Q_f \gamma(t_f)$, $\breve{\mathcal{D}}_i(t_f) = 0$ for $2 \leq i \leq k$; and $\mathcal{D}_1(t_f) = \gamma^T(t_f) Q_f \gamma(t_f)$, $\mathcal{D}_i(t_f) = 0$ for $2 \leq i \leq k$.

The boundary condition of $\mathcal{W}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ implies that

$$
x_0^T \sum_{i=1}^{k} \mu_i \left(\mathcal{H}_{i0} + \mathcal{E}_i(t_0) \right) x_0 + 2x_0^T \sum_{i=1}^{k} \mu_i \left(\breve{\mathcal{D}}_{i0} + \breve{\mathcal{T}}_i(t_0) \right)
$$

$$+\sum_{i=1}^{k}\mu_i\left(\mathcal{D}_{i0}+\mathcal{T}_i(t_0)\right)$$
$$=x_0^T\sum_{i=1}^{k}\mu_i\mathcal{H}_{i0}x_0+2x_0^T\sum_{i=1}^{k}\mu_i\breve{\mathcal{D}}_{i0}+\sum_{i=1}^{k}\mu_i\mathcal{D}_{i0}\,.$$

The initial conditions for the equations (39)-(44) are given as follows $\mathcal{E}_i(t_0)=0$, $\breve{\mathcal{T}}_i(t_0)=0$, and $\mathcal{T}_i(t_0)=0$. Finally, the optimal linear input (36) and feedback gain (37) minimizing the new performance index (25) become optimal

$$l_f^*(\varepsilon)=-R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r^*(\varepsilon)\,,$$
$$K^*(\varepsilon)=-R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r^*(\varepsilon)\,.$$

Theorem 5: Optimal Risk-Averse Tracking Solution. Suppose (A,B) is uniformly stabilizable and (C,A) is uniformly detectable where $C^T(t)C(t)\triangleq Q(t)$. Assume further $k\in\mathbb{Z}^+$ and the sequence $\mu=\{\mu_i\geq 0\}_{i=1}^k$ with $\mu_1>0$ fixed. The optimal tracking solution for the generalized tracking problem whose the state dynamics $x(t)$ and control inputs $u(t)$ governed by (1) and (2) will track closely the desired trajectory $\gamma(t)$ and reference command inputs $\rho(t)$, is given by the risk-averse policy

$$u^*(t)=K^*(t)x^*(t)+l_f^*(t)+\rho(t)\,, \tag{51}$$
$$K^*(\alpha)=-R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r^*(\alpha)\,, \tag{52}$$
$$l_f^*(\alpha)=-R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r^*(\alpha) \tag{53}$$

where the normalized weightings $\widehat{\mu}_r\triangleq\mu_i/\mu_1$ emphasize on different design freedom of shaping the probability density function of the generalized performance-measure (5).

The optimal cumulant-generating solutions $\{\mathcal{H}_r^*(\alpha)\}_{r=1}^k$, and $\left\{\breve{\mathcal{D}}_r^*(\alpha)\right\}_{r=1}^k$ respectively satisfy the time-backward matrix-valued differential equations

$$\frac{d}{d\alpha}\mathcal{H}_1^*(\alpha)=-\left[A(\alpha)+B(\alpha)K^*(\alpha)\right]^T\mathcal{H}_1^*(\alpha)-\mathcal{H}_1^*(\alpha)\left[A(\alpha)+B(\alpha)K^*(\alpha)\right]-Q(\alpha)-K^{*T}(\alpha)R(\alpha)K^*(\alpha)\,;\qquad \mathcal{H}_1^*(t_f)=Q_f \tag{54}$$

and, for $2\leq r\leq k$ with $\mathcal{H}_r^*(t_f)=0$

$$\frac{d}{d\alpha}\mathcal{H}_r^*(\alpha)=-\left[A(\alpha)+B(\alpha)K^*(\alpha)\right]^T\mathcal{H}_r^*(\alpha)-\mathcal{H}_r^*(\alpha)\left[A(\alpha)+B(\alpha)K^*(\alpha)\right]-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{r-s}^*(\alpha) \tag{55}$$

finally, the time-backward vector differential equations

$$\frac{d}{d\alpha}\breve{\mathcal{D}}_1^*(\alpha)=-\left[A(\alpha)+B(\alpha)K^*(\alpha)\right]^T\breve{\mathcal{D}}_1^*(\alpha)-\mathcal{H}_1^*(\alpha)B(\alpha)\left[l_f^*(\alpha)+\rho(\alpha)\right]-K^{*T}(\alpha)R(\alpha)l_f^*(\alpha)+Q(\alpha)\gamma(\alpha) \tag{56}$$
$$\frac{d}{d\alpha}\breve{\mathcal{D}}_r^*(\alpha)=-\left[A(\alpha)+B(\alpha)K^*(\alpha)\right]^T\breve{\mathcal{D}}_r^*(\alpha)-\mathcal{H}_r^*(\alpha)B(\alpha)\left[l_f^*(\alpha)+\rho(\alpha)\right]\,,\quad 2\leq r\leq k \tag{57}$$

with the terminal-value conditions $\breve{\mathcal{D}}_1^*(t_f)=-Q_f\gamma(t_f)$ and $\breve{\mathcal{D}}_r^*(t_f)=0$ for $2\leq r\leq k$.

Remark 2: Note that the optimal feedback gain (52) and affine control input (53) operate dynamically on the time-backward histories of the cumulant-supporting equations (54)-(55) and (56)-(57) from the final to the current time. Moreover, it is important to see that these dynamical equations are functions of the noise process characteristics, i.e., second-order statistic W. Hence, the high confident tracking paradigm consisting optimal feedback gain (52) and affine input (53) has traded the certainty equivalence property, as one may normally obtain from the special case of traditional linear-quadratic tracking, for the adaptability to deal with uncertain environments and performance variations.

V. Conclusions

The present paper proposes an advanced solution concept and a novel paradigm of designing feedback controls for a class of stochastic systems to simultaneously track reference trajectory and command input in accordance of the so-called, risk-averse performance index that is now composed of multiple selective performance-measure statistics beyond the traditional statistical average. A numerical procedure of calculating higher-order statistics associated with the Chi-squared performance-measure is also obtained. The robustness and uncertainty of tracking performance is therefore, maintained compactly and robustly. The complexity of the feedback controller may however increase considerably, depending on how many performance-measure statistics of the target probability density function are to be optimized.

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