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On the k-switching reachability sets of single-input positive switched systems

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***Abstract*— In the paper, the k-switching reachability set $\mathbf{R}_k$ of a single-input positive switched system is introduced, and conditions for the chain of this sets $\{\mathbf{R}_k, k \in \mathbb{N}\}$, to stop increasing after some finite index k are given. For the special class of positive switched systems, which commute among n single-input n-dimensional systems, sufficient conditions that ensure that the n-switching reachability set coincides with the positive orthant $\mathbb{R}^n_+$ are given. In particular, it is proved that when the system is reachable and $n = 2, 3$, this is always true, namely $\mathbf{R}_n = \mathbb{R}^n_+$.**

I. Introduction

"Switched linear systems" are systems whose describing equations change, according to some switching law, within a (possibly infinite) family of (linear) subsystems. Research efforts in this context were first oriented to the investigation of stability and stabilizability issues [4], and it was only a few years later that structural properties, like reachability, controllability and observability, were initially addressed [3], [9], [11]. This class of systems can be viewed as a good compromise between accuracy and complexity, and, indeed, it provides a valuable alternative to complex nonlinear models. In fact, it is often preferable and more efficient to replace a single complex model with several simple and linear models, each of them suitable for describing the system evolution under specific working conditions.

On the other hand, the positivity requirement is often introduced in the system models whenever the physical nature of the describing variables constrains them to take only positive (or at least nonnegative) values. Positive linear systems naturally arise in various fields such as bioengineering (compartmental models), economic modelling, behavioral science, and stochastic processes (Markov chains or hidden Markov models), where the state variables represent quantities, like pressures, population levels, concentrations, probabilities, that have no meaning unless nonnegative [2].

In this perspective, switched positive systems are mathematical models which keep into account two different needs: the need for a system model which is obtained as a family of simple subsystems, each of them accurate enough to capture the system laws under specific operating conditions, and the need to introduce the nonnegativity constraint the physical variables are subject to. This is the case when trying to describe certain physiological and pharmacokinetic processes, like the insulin-sugar metabolism.

Of course, the need for this class of systems in specific research contexts has stimulated an interest in theoretical issues related to them, and, in particular, structural properties of continuous-time positive switched systems have been recently investigated in [6], [7], [8], [10]. In this paper, we assume a different perspective, and introduce the concept of k-switching reachability set $\mathbf{R}_k$ for the class of single-input positive switched systems. It is shown that the chain of this sets $\{\mathbf{R}_k, k \in \mathbb{N}\}$ is a non-decreasing one, and if at some stage it stops increasing then it cannot increase further. For the special class of positive switched systems, which commute among n single-input n-dimensional systems, in section III sufficient conditions that ensure that the n-switching reachability set coincides with the positive orthant $\mathbb{R}^n_+$ are given. Finally, in section IV, it is proved that when the system is reachable and $n = 2, 3$, this is always true, namely $\mathbf{R}_n = \mathbb{R}^n_+$.

Before proceeding, we introduce some notation. For every $k \in \mathbb{N}$, we set $\langle k\rangle := \{1, 2, \dots, k\}$. In the sequel, the (i, j)th entry of a matrix A is denoted by $[A]_{i,j}$. In the special case of a vector $\mathbf{v}$, we let $[\mathbf{v}]_i$ denote its ith entry. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix A with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A \ge 0$); if $A \ge 0$ and $A \ne 0$, A is a *positive matrix* ($A > 0$), while if all its entries are positive it is a *strictly positive matrix* ($A \gg 0$). The same notation is adopted for nonnegative, positive and strictly positive vectors. A *Metzler matrix*, on the other hand, is a real square matrix, whose off-diagonal entries are nonnegative. Every Metzler matrix has a real eigenvalue $\lambda_{\max}(A)$ satisfying $\lambda_{\max}(A) > \mathrm{Re}(\lambda)$ for every other $\lambda \in \sigma(A)$.

Given any matrix $A \in \mathbb{R}^{q\times r}$, by the *nonzero pattern* of A we mean the set of index pairs corresponding to its nonzero entries, namely $\overline{\mathrm{ZP}}(A) := \{(i, j) : [A]_{i,j} \ne 0\}$. Conversely, the *zero pattern* $\mathrm{ZP}(A)$ is the set of indices corresponding to the zero entries of A. The adaptation of these concepts to the vector case is straightforward.

We let $\mathbf{e}_i$ denote the ith vector of the canonical basis in $\mathbb{R}^n$ (where n is always clear from the context), whose entries are all zero except for the ith which is unitary. We say that a vector $\mathbf{v} \in \mathbb{R}^n_+$ is an *ith monomial vector* if $\overline{\mathrm{ZP}}(\mathbf{v}) = \overline{\mathrm{ZP}}(\mathbf{e}_i) = \{i\}$. For any set $\mathcal{S} \subseteq \langle n\rangle$, we set $\mathbf{e}_{\mathcal{S}} := \sum_{i\in\mathcal{S}} \mathbf{e}_i$ and we let $P_{\mathcal{S}}$ be the $n \times |\mathcal{S}|$ selection matrix that singles out the columns of the identity matrix I_n corresponding to the indices in $\mathcal{S}$. Consequently, given any vector $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, $[\mathbf{v}]_{\mathcal{S}} := P^T_{\mathcal{S}}\mathbf{v}$ is the restriction of $\mathbf{v}$ to its positive components. If $\mathcal{S} = \langle n\rangle$, $\mathbf{e}_{\mathcal{S}}$ is denoted by $\mathbf{1}_n$.

Basic definitions and results about cones may be found, for instance, in [1]. We recall here only the few facts used within

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this paper. A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* if $\alpha\mathcal{K} \subset \mathcal{K}$ for all $\alpha \geq 0$. A cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This amounts to saying that $k \in \mathbb{N}$ and $C \in \mathbb{R}^{n\times k}$ can be found, such that $\mathcal{K}$ coincides with the set of nonnegative combinations of the columns of C. In this case, we adopt the notation $\mathcal{K} := \text{Cone}(C)$.

II. Reachability property and k-switching reachability sets

A *single-input (continuous-time) positive switched system* is described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + b_{\sigma(t)}u(t), \qquad t \in \mathbb{R}_+, \tag{1}$$

where $\mathbf{x}(t)$ and $u(t)$ denote the n-dimensional state variable and the scalar input, respectively, at the time instant t, and σ is a switching sequence, taking values in a finite set $\mathcal{P} = \{1, 2, \dots, p\}$.

We assume that the switching sequence is piece-wise constant, and hence in every time interval $[0, t[$ there is a finite number of discontinuities, which corresponds to a finite number (say k, including the initial time instant) of switching instants $0 = t_0 < t_1 < \dots < t_{k-1} < t$. Also, we assume that, at each switching time t_ℓ, σ is right continuous. For each $i \in \mathcal{P}$, the pair (A_i, b_i) represents a continuous-time positive system, which means that A_i is an $n\times n$ Metzler matrix and b_i is an n-dimensional nonnegative column vector.

As a first step, we recall the definition of monomial reachability and of reachability for positive switched systems.

Definition 1: [7], [8] A state $\mathbf{x}_f \in \mathbb{R}^n_+$ is said to be *reachable* if there exist some time instant $t > 0$, a switching sequence $\sigma : [0, t[\to \mathcal{P}$ and an input $u : [0, t[\to \mathbb{R}_+$ that lead the state trajectory from $\mathbf{x}(0) = 0$ to $\mathbf{x}(t) = \mathbf{x}_f$.

A positive switched system is said to be *monomially reachable* if every monomial vector (equivalently, every vector $\mathbf{e}_i$, $i \in \langle n\rangle$) is reachable. A positive switched system is said to be *reachable* if every state $\mathbf{x}_f \in \mathbb{R}^n_+$ is reachable.

Clearly, monomial reachability is a necessary (but, unfortunately, not sufficient) condition for reachability [6]. While reachability property is hard to test [8], [10], even in the simpler case when $\mathcal{P} = \langle n\rangle$, monomial reachability admits a rather easy characterization. Indeed, system (1) is monomially reachable if and only if [6] $p \geq n$ and there exists a relabeling of the p subsystems $(A_i, b_i), i \in \mathcal{P}$, such that that the first n subsystems satisfy

$$A_i\mathbf{e}_i = \alpha_i\mathbf{e}_i \qquad \text{and} \qquad b_i = \beta_i\mathbf{e}_i, \tag{2}$$

for suitable $\alpha_i \geq 0$ and $\beta_i > 0$. Notice that if this is the case, then $e^{A_i t}b_i = e^{\alpha_i t}\beta_i\mathbf{e}_i$ for every $i \in \langle n\rangle$ and every $t > 0$.

We introduce the definition of k-switching reachability set[1].

[1]We would like to remark that, due to the assumption that the switching sequence can be arbitrarily chosen, the reachability sets obtained in this paper are in general strictly larger than the reachability sets one would obtain by imposing that the switching sequence is state-driven. This would be the case, for instance, when approximating nonlinear systems by means of switched systems, as in that case the switching would always be state-dependent.

Definition 2: Given a positive switched system (1) and a positive integer k, we define the *k-switching reachability set*, and denote it by $\mathbf{R}_k$, as the set of states that can be reached in finite time by the system, by making use (of a nonnegative input signal $u(\cdot)$ and) of a switching sequence σ that commutes no more than $k-1$ times, meaning that the switching instants of the switching sequence are no more than k (i.e., $0 = t_0 < t_1 < \dots < t_{\ell-1}$ with $\ell \leq k$).

Notice that for a non-switching system (equivalently, $\mathcal{P} = \{1\}$) the reachable set would coincide with $\mathbf{R}_1$. It is easily seen that $\mathbf{R}_k$ is a cone, since if $\mathbf{x}_f$ belongs to $\mathbf{R}_k$ then $\alpha\cdot\mathbf{x}_f$ surely does, for every $\alpha \geq 0$. However, in general, it is neither convex nor polyhedral. Of course, we are interested in investigating how the cone $\mathbf{R}_k$ varies, as k varies over the positive integers. To this end, we recall that the state at the time t, starting from the zero initial condition, under the action of the soliciting input $u(\tau), \tau \in [0, t[$, and of the switching sequence $\sigma : [0, t[$, with switching instants $0 = t_0 < t_1 < \dots < t_{k-1} < t$ and switching values $i_0, i_1, \dots, i_{k-1}$ (meaning that $i_\ell = \sigma(t)$ for $t \in [t_\ell, t_{\ell+1}[$), can be expressed as follows:

$$\begin{aligned}\mathbf{x}(t) = {} & e^{A_{i_{k-1}}(t-t_{k-1})}\dots e^{A_{i_1}(t_2-t_1)}\int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)}b_{i_0}u(\tau)\mathrm{d}\tau + \\ & + e^{A_{i_{k-1}}(t-t_{k-1})}\dots e^{A_{i_2}(t_3-t_2)}\int_{t_1}^{t_2} e^{A_{i_1}(t_2-\tau)}b_{i_1}u(\tau)\mathrm{d}\tau + \\ & + \dots + \int_{t_{k-1}}^{t} e^{A_{i_{k-1}}(t-\tau)}b_{i_{k-1}}u(\tau)\mathrm{d}\tau. \end{aligned} \tag{3}$$

Therefore, a vector $\mathbf{x}_f$ belongs to $\mathbf{R}_k$ if and only if it can be expressed as in (3), for suitable $u(\cdot) \geq 0, t, t_\ell \in \mathbb{R}_+$, and $i_\ell \in \mathcal{P}, \ell \in \{0, 1, \dots, k-1\}$. Even more, it is easily seen that $\mathbf{x}_f \in \mathbf{R}_k$ if and only if

$$\mathbf{x}_f = e^{A_{i_{k-1}}(t-t_{k-1})}\mathbf{w} + \int_{t_{k-1}}^{t} e^{A_{i_{k-1}}(t-\tau)}b_{i_{k-1}}u(\tau)\mathrm{d}\tau, \tag{4}$$

for some $0 < t_{k-1} < t$, some $i_{k-1} \in \mathcal{P}$, a nonnegative signal $u(\cdot)$ and some vector $\mathbf{w} \in \mathbf{R}_{k-1}$. Clearly, $\mathbf{R}_k \subseteq \mathbf{R}_{k+1}$, and hence

$$\mathbf{R}_1 \subseteq \mathbf{R}_2 \subseteq \dots \subseteq \mathbf{R}_k \subseteq .....$$

Moreover, if the above chain of subsets of $\mathbb{R}^n_+$ stops at some stage, namely $\mathbf{R}_k = \mathbf{R}_{k+1}$ for some $k \in \mathbb{N}$, then it cannot increase any more. The proof is omitted for the sake of brevity as it is rather straightforward.

Proposition 1: Given a positive switched system (1), if $\mathbf{R}_k = \mathbf{R}_{k+1}$ for some $k \in \mathbb{N}$, then $\mathbf{R}_{k+1} = \mathbf{R}_{k+2}$, and hence the set of all states which are reachable in finite time by the switched positive system coincides with $\mathbf{R}_k$.

We want to investigate under which conditions an index $k \in \mathbb{N}$ can be found, such that $\mathbf{R}_k = \mathbf{R}_{k+1}$. To this end, we denote by $\mathcal{R}_t(A_i, b_i)$ the cone of (positive) states which are reachable at time $t > 0$ by the single subsystem

$$\dot{\mathbf{x}}(t) = A_i\mathbf{x}(t) + b_iu(t),$$

(by means of nonnegative inputs). Notice that, differently from what happens with standard linear systems, $\mathcal{R}_t(A_i, b_i)$ typically grows with t [2], [5]. As an immediate consequence of equation (4), we obtain the following identity

$$\mathbf{R}_{k+1} = \cup_{i\in\mathcal{P}} \cup_{t>0} \left(e^{A_i t}\mathbf{R}_k + \mathcal{R}_t(A_i, b_i)\right), \qquad (5)$$

which leads to the following result.

Proposition 2: Given a positive switched system (1), commuting among p subsystems, the following facts are equivalent:

i) there exists $k \in \mathbb{N}$ such that $\mathbf{R}_k = \mathbf{R}_{k+1}$;
ii) there exists $k \in \mathbb{N}$ such that $\mathbf{R}_k$ is $e^{A_i t}$-invariant for every $t > 0$ and every $i \in \mathcal{P}$.

Proof: ii) $\Rightarrow$ i) If $\mathbf{R}_k$ is $e^{A_i t}$-invariant for every $i \in \mathcal{P}$ and every $t > 0$, then for every $t > 0$, $e^{A_i t}\mathbf{R}_k + \mathcal{R}_t(A_i, B_i) \subseteq \mathbf{R}_k + \mathcal{R}_t(A_i, B_i)$. On the other hand, it is easily seen that $\mathcal{R}_t(A_i, B_i) \subseteq \mathbf{R}_1 \subseteq \mathbf{R}_k$ for every $t > 0$, every index $i \in \mathcal{P}$ and every $k \in \mathbb{N}$. So, if ii) holds, then (by (5)), $\mathbf{R}_{k+1} \subseteq \cup_{i\in\mathcal{P}} \cup_{t>0} \mathbf{R}_k = \mathbf{R}_k$, and since the converse inclusion is always true, this implies that i) holds.

i) $\Rightarrow$ ii) If $\mathbf{R}_k = \mathbf{R}_{k+1}$, then the set of states which are reachable (in finite time) coincides with $\mathbf{R}_k$. Clearly, if a state $\mathbf{x}_f > 0$ is reachable, then $e^{A_i t}\mathbf{x}_f$ is reachable, too, for every $i \in \mathcal{P}$ and every $t > 0$. Indeed, once $\mathbf{x}_f$ has been reached, it is sufficient to switch to the ith subsystem and leave the system freely evolve for a lapse of time equal to t. This ensures that $\mathbf{R}_k$ is $e^{A_i t}$-invariant for every $i \in \mathcal{P}$ and every $t > 0$. ∎

At this stage of our research, it is not clear, yet, whether, for a reachable system (1) an index k can always be found such that $\mathbf{R}_k = \mathbb{R}^n_+$. There are classes of systems, however, for which this is surely true and it turns out that reachability ensures that $\mathbf{R}_n = \mathbb{R}^n_+$. This analysis will be the object of the next section. Before moving to this investigation, we conclude the section by showing that if a system is monomially reachable, then all strictly positive states belong to $\mathbf{R}_n$. This preliminarily requires to recall a couple of technical results proved in [6] and in [10], respectively.

Lemma 1: [6] Given an $n \times n$ Metzler matrix A, for every $\varepsilon > 0$ there exists $\tau > 0$ such that $\forall\, i, j \in \langle n\rangle$,

$$I_n \leq e^{A\tau} \leq I_n + \varepsilon \mathbf{1}_n \mathbf{1}_n^T, \qquad (6)$$

which amounts to saying that

$$\begin{array}{rcccll} 1 & \leq & [e^{A\tau}]_{ii} & \leq & (1+\varepsilon), & \\ 0 & \leq & [e^{A\tau}]_{ij} & \leq & \varepsilon, & \text{for } i \neq j. \end{array}$$

Consequently, for every $\mathbf{x}_f \gg 0$ there exists $\tau > 0$ such that $\mathbf{x}_f$ is an internal point of $\mathrm{Cone}(e^{A\tau})$, namely $\mathbf{x}_f = e^{A\tau}\mathbf{z}$ for some $\mathbf{z} \gg 0$.

Proposition 3: [10] Consider a monomially reachable positive switched system (1), and assume that its first n subsystems $(A_i, b_i), i \in \langle n\rangle$, satisfy (2) for suitable $\alpha_i \geq 0$ and $\beta_i > 0$. Given a time instant $t > 0$, a positive vector $\mathbf{x}_f \in \mathbb{R}^n_+$, $k \in \mathbb{Z}_+$, $k+1$ time instants $0 = t_0 < t_1 < \ldots < t_{k-1} < t$ and k indices $i_0, i_1, \ldots, i_{k-1} \in \langle n\rangle$, the following facts are equivalent ones:

i) there exists a nonnegative input $u(\cdot)$ such that:

$$\mathbf{x}_f = e^{A_{i_{k-1}}(t-t_{k-1})} \ldots e^{A_{i_1}(t_2-t_1)} \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)} b_{i_0} u(\tau)\mathrm{d}\tau + \ldots + \int_{t_{k-1}}^{t} e^{A_{i_{k-1}}(t-\tau)} b_{i_{k-1}} u(\tau)\mathrm{d}\tau.$$

ii) $\mathbf{x}_f$ belongs to

$$\mathrm{Cone}[e^{A_{i_{k-1}}(t-t_{k-1})} b_{i_{k-1}} | e^{A_{i_{k-1}}(t-t_{k-1})} e^{A_{i_{k-2}}(t_{k-1}-t_{k-2})} \cdot b_{i_{k-2}} | \ldots | e^{A_{i_{k-1}}(t-t_{k-1})} \ldots e^{A_{i_1}(t_2-t_1)} e^{A_{i_0}(t_1-t_0)} b_{i_0}].$$

Proposition 4: Consider an n-dimensional positive switched system (1), commuting among p single-input subsystems $(A_i, b_i), i \in \mathcal{P}$. If the system is monomially reachable, then every strictly positive vector belongs to $\mathbf{R}_n$.

Proof: If the system is monomially reachable, then we have seen that, possibly after a suitable relabelling, its first n subsystems $(A_i, b_i), i \in \langle n\rangle$, satisfy (2) for suitable coefficients $\alpha_i \geq 0$ and $\beta_i > 0$. Under this assumption, by suitably adjusting Proposition 3, we can claim that every positive vector $\mathbf{x}_f \in \mathbb{R}^n_+$, with $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| \geq 2$, is reachable if (but, if $p > n$, not necessarily "if and only if") we can find $k \in \mathbb{N}$, indices $i_0, i_1, \ldots, i_{k-1} \in \langle n\rangle \subseteq \mathcal{P}$ and positive time intervals $\tau_1, \tau_2, \ldots, \tau_{k-1}$ such that such that $\mathbf{x}_f$ belongs to

$$\mathrm{Cone}[\mathbf{e}_{i_{k-1}} | e^{A_{i_{k-1}}\tau_{k-1}} \mathbf{e}_{i_{k-2}} | \ldots | e^{A_{i_{k-1}}\tau_{k-1}} \ldots e^{A_{i_1}\tau_1} \mathbf{e}_{i_0}].$$

We want to prove that, when $\mathbf{x}_f$ is a strictly positive vector, then $\mathbf{x}_f \in \mathbf{R}_n$. This amounts to saying that n indices and $n-1$ time intervals can always be found such that the previous condition holds. Indeed, we will show that by choosing $k = n$ and $i_h = h+1, h = 0, 1, \ldots, n-1$, positive time intervals $\tau_1, \tau_2, \ldots, \tau_{n-1}$ can be found such that $\mathbf{x}_f$ belongs to

$$\mathrm{Cone}[\mathbf{e}_n | e^{A_n\tau_{n-1}} \mathbf{e}_{n-1} | \ldots | e^{A_n\tau_{n-1}} \ldots e^{A_2\tau_1} \mathbf{e}_1].$$

By Lemma 1, once a positive number ε has been chosen, for every Metzler matrix A_i there exists $\tau_{i-1} > 0$ such that

$$I_n \leq e^{A_i\tau_{i-1}} \leq I_n + \varepsilon \mathbf{1}_n \mathbf{1}_n^T,$$

This implies that

$$\begin{array}{rcccl} \mathbf{e}_{n-1} & \leq & e^{A_n\tau_{n-1}} \mathbf{e}_{n-1} & \leq & (I_n + \varepsilon \mathbf{1}_n \mathbf{1}_n^T) \mathbf{e}_{n-1}, \\ \mathbf{e}_{n-2} & \leq & e^{A_n\tau_{n-1}} e^{A_{n-1}\tau_{n-2}} \mathbf{e}_{n-2} & \leq & (I_n + \varepsilon \mathbf{1}_n \mathbf{1}_n^T)^2 \mathbf{e}_{n-2}, \\ & \vdots & & & \\ \mathbf{e}_1 & \leq & e^{A_n\tau_n} e^{A_{n-1}\tau_{n-2}} \ldots e^{A_2\tau_1} \mathbf{e}_1 & \leq & (I_n + \varepsilon \mathbf{1}_n \mathbf{1}_n^T)^{n-1} \mathbf{e}_1. \end{array}$$

We may notice that, for every $k \in \langle n-1\rangle$,

$$\begin{aligned} (I_n + \varepsilon \mathbf{1}_n \mathbf{1}_n^T)^k \mathbf{e}_{n-k} &= \left[\sum_{j=0}^{k} \binom{k}{j} \varepsilon^j (\mathbf{1}_n \mathbf{1}_n^T)^j\right] \mathbf{e}_{n-k} \\ &= \mathbf{e}_{n-k} + \left[\sum_{j=1}^{k} \binom{k}{j} \varepsilon^j n^{j-1} \mathbf{1}_n \mathbf{1}_n^T\right] \mathbf{e}_{n-k} \\ &= \mathbf{e}_{n-k} + \frac{(1+\varepsilon n)^k - 1}{n} \mathbf{1}_n. \end{aligned}$$

Consequently, once we set

$$p_k := \frac{(1+\varepsilon n)^k - 1}{n},$$

we get $\mathbf{e}_{n-k} \le e^{A_n\tau_{n-1}} e^{A_{n-1}\tau_{n-2}} \dots e^{A_{n-k+1}\tau_{n-k}} \mathbf{e}_{n-k} \le \mathbf{e}_{n-k} + p_k \mathbf{1}_n$. Even more, it is easily seen that $p_1 < p_2 < \dots < p_{n-1}$, and that, when ε is sufficiently small

$$\begin{aligned} & \text{Cone}[\mathbf{e}_n | e^{A_n\tau_{n-1}}\mathbf{e}_{n-1} | \dots | e^{A_n\tau_{n-1}} \dots e^{A_2\tau_1}\mathbf{e}_1] \\ \approx\ & \text{Cone}[\mathbf{e}_n | \mathbf{e}_{n-1} + p_1\mathbf{1}_n | \dots | \mathbf{e}_1 + p_{n-1}\mathbf{1}_n] \\ \supseteq\ & \text{Cone}[\mathbf{e}_n + p_{n-1}\mathbf{1}_n | \mathbf{e}_{n-1} + p_{n-1}\mathbf{1}_n | \dots | \mathbf{e}_1 + p_{n-1}\mathbf{1}_n]. \end{aligned}$$

So, we are remained to prove that for every vector $\mathbf{x}_f \gg 0$ there exists $\varepsilon > 0$ such that

$$\mathbf{x}_f \in \text{Cone}[\mathbf{e}_n + p_{n-1}\mathbf{1}_n | \mathbf{e}_{n-1} + p_{n-1}\mathbf{1}_n | \dots | \mathbf{e}_1 + p_{n-1}\mathbf{1}_n].$$

But this is obvious, since by suitably choosing ε, $p_{n-1} := \frac{(1+\varepsilon n)^{n-1}-1}{n}$ can be made arbitrarily small. ∎

III. Single-input n-dimensional systems switching among n subsystems

The goal of this section is that of investigating under which conditions, given a reachable single-input positive switched system (1), commuting among n subsystems of size n, a positive integer k can be found such that $\mathbf{R}_k = \mathbb{R}_+$. Clearly, as previously remarked, monomial reachability is a necessary condition for reachability. So, in this section we will steadily assume that the n subsystems $(A_i, b_i), i \in \langle n \rangle$, satisfy (2), for suitable $\alpha_i \ge 0$ and $\beta_i > 0$. Under this assumption, $\mathcal{R}_t(A_i, b_i) \equiv \text{Cone}(\mathbf{e}_i) = \{\alpha \cdot \mathbf{e}_i, \alpha \ge 0\}$ for every $i \in \langle n \rangle$ and every $t > 0$, so that

$$\mathbf{R}_1 = \cup_{t>0} \cup_{i \in \langle n \rangle} \mathcal{R}_t(A_i, b_i) = \cup_{i \in \langle n \rangle} \text{Cone}(\mathbf{e}_i).$$

Also, for every $\mathcal{S} \subsetneq \langle n \rangle$, we introduce the index set

$$\mathcal{I}_\mathcal{S} := \{i \in \langle n \rangle : \overline{\text{ZP}}(e^{A_i}\mathbf{e}_\mathcal{S}) = \mathcal{S}\}.$$

As it has been shown in [10], a necessary condition for a vector $\mathbf{x}_f$, with $\overline{\text{ZP}}(\mathbf{x}_f) = \mathcal{S}$, to be reachable, is that $\mathcal{I}_\mathcal{S} \ne \emptyset$. Of course, this condition needs to be verified only for the proper subsets of $\langle n \rangle$, since for $\mathcal{S} = \langle n \rangle$ its is always true (with $\mathcal{I}_\mathcal{S} = \langle n \rangle$). In order to prove the first two results, we need the following technical lemma.

Lemma 2: Let $\mathcal{A} := \{A_1, A_2, \dots, A_n\}$ be a set of $n \times n$ Metzler matrices. Let $\mathbf{x}_f$ be a positive vector in $\mathbb{R}^n_+$ and set $\mathcal{S} := \overline{\text{ZP}}(\mathbf{x}_f)$. For every $i \in \mathcal{I}_\mathcal{S}$ there exists $\tau = \tau(i) > 0$ and a positive vector $\mathbf{w}$, with $\overline{\text{ZP}}(\mathbf{w}) = \mathcal{S}$, such that $\mathbf{x}_f = e^{A_i\tau}\mathbf{w}$.

Proof: We preliminarily notice that if $i \in \mathcal{I}_\mathcal{S}$ then the following facts are equivalent ones[2]:

- $\mathbf{x}_f = e^{A_i\tau}\mathbf{w}$, for some $\mathbf{w}$, with $\overline{\text{ZP}}(\mathbf{w}) = \mathcal{S}$;
- $[\mathbf{x}_f]_\mathcal{S} = e^{P_\mathcal{S}^T A_i P_\mathcal{S} \tau}[\mathbf{w}]_\mathcal{S}$, for some $[\mathbf{w}]_\mathcal{S} \gg 0$,

where $[\mathbf{v}]_\mathcal{S}$ is the restriction of the vector $\mathbf{v}$ to the entries corresponding to the indices in $\mathcal{S}$, and $P_\mathcal{S}$ is the selection matrix corresponding to $\mathcal{S}$. So, in order to prove the lemma, we have to simply show that every strictly positive vector is an internal point of $\text{Cone}(e^{\tilde{A}_i\tau})$, the exponential cone of any Metzler matrix $\tilde{A}_i$, provided that $\tau > 0$ is small enough. But this is just stated in Lemma 1. ∎

Proposition 5: Consider a single-input positive switched system (1), commuting among n subsystems (A_i, b_i) of size n, that satisfy (2) for suitable $\alpha_i \ge 0$ and $\beta_i > 0$. Also, suppose that for every set $\mathcal{S} \subsetneq \langle n \rangle$, there exists[3] $i \in \mathcal{I}_\mathcal{S}$ such that $i \in \mathcal{S}$. Then every vector $\mathbf{x}_f > 0$ with $|\overline{\text{ZP}}(\mathbf{x}_f)| = k$ belongs to $\mathbf{R}_k$ and it can be reached by resorting to a switching sequence σ taking values only in $\overline{\text{ZP}}(\mathbf{x}_f)$. As a consequence, every positive vector can be reached by using at most n switching values and commuting no more than $n-1$ times (i.e., $\mathbf{R}_n = \mathbb{R}^n_+$).

Proof: We prove the result by induction on the cardinality k of $\overline{\text{ZP}}(\mathbf{x}_f)$. If $k = 1$, then $\mathbf{x}_f$ is a monomial vector, say $\overline{\text{ZP}}(\mathbf{x}_f) = \{i\}$, and hence it can be reached by resorting to the subsystem (A_i, b_i) alone. So, it can be reached by a switching sequence taking a single value which is just the unique element, i, of $\overline{\text{ZP}}(\mathbf{x}_f)$.

Suppose, now, that the result is true for every vector $\mathbf{w}' > 0$, with $|\overline{\text{ZP}}(\mathbf{w}')| < k$. We want to prove that the result holds for every vector $\mathbf{x}_f > 0$ with $\mathcal{S} := \overline{\text{ZP}}(\mathbf{x}_f)$ of cardinality $|\mathcal{S}| = k > 1$. By exploiting the proposition's assumptions, we may find an index $i \in \mathcal{I}_\mathcal{S}$ such that $i \in \mathcal{S}$. Now, by applying Lemma 2, we can find $\tau > 0$ and $\mathbf{w} > 0$, with $\overline{\text{ZP}}(\mathbf{w}) = \mathcal{S}$, such that $\mathbf{x}_f = e^{A_i\tau}\mathbf{w}$.

Express, now, $\mathbf{w}$ as $\mathbf{w} = \mathbf{w}' + \mathbf{w}'_i$, where $\overline{\text{ZP}}(\mathbf{w}') = \mathcal{S}\setminus\{i\}$ and $\mathbf{w}'_i := \mathbf{w} - \mathbf{w}'$ satisfies the constraint $\overline{\text{ZP}}(\mathbf{w}'_i) = \{i\} \subsetneq \mathcal{S}$. Consequently,

$$\mathbf{x}_f = e^{A_i\tau}\mathbf{w}' + e^{A_i\tau}\mathbf{w}'_i = e^{A_i\tau}\mathbf{w}' + \mathbf{v}_i.$$

By the reachability of the system and the inductive assumption, $\mathbf{w}'$ belongs to $\mathbf{R}_{k-1}$ and it can be reached by means of a switching sequence taking values in $\overline{\text{ZP}}(\mathbf{w}')$, while $\mathbf{v}_i$ has nonzero pattern $\{i\}$ and hence it can be reached at time τ by making use of subsystem (A_i, b_i) alone (i.e. belongs to $\mathcal{R}_\tau(A_i, b_i)$). Consequently, (see (4)), $\mathbf{x}_f$ belongs to $\mathbf{R}_k$ and it can be reached by means of a switching sequence taking values in $\overline{\text{ZP}}(\mathbf{w}') \cup \{i\} = \mathcal{S}$. ∎

The next result requires, in turn, a technical lemma.

Lemma 3: Consider a single-input positive switched system (1), commuting among n subsystems of size n, that satisfy (2) for suitable $\alpha_i \ge 0$ and $\beta_i > 0$. If a positive vector $\mathbf{x}_f$ can be reached through the switching sequence σ, ordinately taking the values $i_0, i_1, \dots, i_{k-1} \in \langle n \rangle$, then upon setting $\mathcal{S} := \overline{\text{ZP}}(\mathbf{x}_f)$, we have $i_{k-1} \in \mathcal{I}_\mathcal{S}$.

Proof: If $\mathbf{x}_f$ is reachable by means of the aforementioned switching sequence, then (see (4)),

$$\mathbf{x}_f = e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{w} + \mathbf{w}_{i_{k-1}},$$

[2]Notice that the former statement always implies the latter, while the converse is true only if $i \in \mathcal{I}_\mathcal{S}$.

[3]Notice that surely e^{A_i} preserves the set $\{i\}$, meaning that $\overline{\text{ZP}}(e^{A_i}\mathbf{e}_i) = \{i\}$. This assumption amounts to saying that for every set $\mathcal{S}$ there is at least one index i such that both e^{A_i} preserves $\mathcal{S}$, i.e. $\mathcal{S} = \overline{\text{ZP}}(e^{A_i}\mathbf{e}_\mathcal{S})$, and $i \in \mathcal{S}$.

where $i_{k-1} \in \langle n \rangle$, $\tau_{k-1} > 0$, $\mathbf{w}_{i_{k-1}}$ is a state reachable by the subsystem $(A_{i_{k-1}}, b_{i_{k-1}})$, and $\mathbf{w}$ is a reachable state. Clearly, both $\overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{w})$ and $\overline{\mathrm{ZP}}(\mathbf{w}_{i_{k-1}})$ are subsets of $\mathcal{S}$. If $\overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{w}) = \mathcal{S}$, then [8] $i_{k-1} \in \mathcal{I}_{\mathcal{S}}$. On the other hand, if $\overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{w}) \subsetneq \mathcal{S}$, it must be $\mathbf{w}_{i_{k-1}} \neq 0, \overline{\mathrm{ZP}}(\mathbf{w}_{i_{k-1}}) = \{i_{k-1}\}$, and $\overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{w}) = \mathcal{S} \setminus \{i_{k-1}\} =: \mathcal{S}^*$. Consequently, [8], again, $\overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{e}_{\mathcal{S}}) = \overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}(\mathbf{e}_{\mathcal{S}^*} + \mathbf{e}_{i_{k-1}})) = \overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{e}_{\mathcal{S}^*}) \cup \overline{\mathrm{ZP}}(e^{A_{i_{k-1}}\tau_{k-1}}\mathbf{e}_{i_{k-1}}) = \mathcal{S}^* \cup \{i_{k-1}\} = \mathcal{S}$, which ensures, again, that $i_{k-1} \in \mathcal{I}_{\mathcal{S}}$. ∎

Proposition 6: Consider a single-input positive switched system (1), commuting among n subsystems (A_i, b_i) of size n, that satisfy (2) for suitable $\alpha_i \geq 0$ and $\beta_i > 0$. If the system is reachable and for every $\mathcal{S} \subsetneq \langle n \rangle$, $|\mathcal{I}_{\mathcal{S}}| = 1$, then every vector $\mathbf{x}_f > 0$ with $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| = k$ belongs to $\mathbf{R}_k$ and it can be reached by resorting to a switching sequence σ taking values only in $\overline{\mathrm{ZP}}(\mathbf{x}_f)$. As a consequence, $\mathbf{R}_n = \mathbb{R}^n_+$.

Proof: We prove the result by induction on the cardinality k of $\overline{\mathrm{ZP}}(\mathbf{x}_f)$. If $k = 1$, then $\mathbf{x}_f$ is a monomial vector, and, by resorting to the same reasoning adopted within the proof of Proposition 5, we can claim that it can be reached by a switching sequence taking the single value $\overline{\mathrm{ZP}}(\mathbf{x}_f)$.

Suppose, now, that the result is true for every vector $\mathbf{v}$ with $|\overline{\mathrm{ZP}}(\mathbf{v})| < k$. We want to prove that the result holds for every vector $\mathbf{x}_f > 0$ with $\mathcal{S} := \overline{\mathrm{ZP}}(\mathbf{x}_f)$ of cardinality k, with $1 < k < n$. Since the system, and hence $\mathbf{x}_f$, is reachable, then

$$\mathbf{x}_f = e^{A_\ell \tau_\ell}\mathbf{v} + \mathbf{v}_\ell,$$

where $\tau_\ell > 0$, $\mathbf{v}_\ell$ is a (possibly zero) state reachable by the subsystem (A_ℓ, b_ℓ), and $\mathbf{v}$ is a reachable state. Clearly, both $\overline{\mathrm{ZP}}(e^{A_\ell \tau_\ell}\mathbf{v})$ and $\overline{\mathrm{ZP}}(\mathbf{v}_\ell)$ are subsets of $\mathcal{S}$.

Two cases may occur:

i) $\mathbf{v}_\ell \neq 0$;
ii) $\mathbf{v}_\ell = 0$.

Case i): if $\mathbf{v}_\ell \neq 0$, then $\{\ell\} = \overline{\mathrm{ZP}}(\mathbf{v}_\ell)$. On the other hand, as a consequence of (2), $\overline{\mathrm{ZP}}(e^{A_\ell}\mathbf{e}_\ell) = \{\ell\}$. So, we can express $\mathbf{v}$ as $\mathbf{v} = \mathbf{v}' + \mathbf{v}'_\ell$, where $\overline{\mathrm{ZP}}(\mathbf{v}') = \mathcal{S} \setminus \{\ell\}$ and $\mathbf{v}'_\ell := \mathbf{v} - \mathbf{v}'$ is either the zero vector or, if not, it satisfies the constraint $\overline{\mathrm{ZP}}(\mathbf{v}'_\ell) = \{\ell\}$. Consequently,

$$\mathbf{x}_f = e^{A_\ell \tau_\ell}\mathbf{v}' + (e^{A_\ell \tau_\ell}\mathbf{v}'_\ell + \mathbf{v}_\ell).$$

By the reachability of the system and the inductive assumption, $\mathbf{v}'$ belongs to $\mathbf{R}_{k-1}$ and it can be reached by means of a switching sequence taking values in $\overline{\mathrm{ZP}}(\mathbf{v}')$, while $e^{A_\ell \tau_\ell}\mathbf{v}'_\ell + \mathbf{v}_\ell$ has nonzero pattern $\{\ell\}$ and hence belongs to $\mathcal{R}_{\tau_\ell}(A_\ell, b_\ell)$. Consequently, $\mathbf{x}_f$ belongs to $\mathbf{R}_k$ and it can be reached by means of a switching sequence taking values in $\overline{\mathrm{ZP}}(\mathbf{v}') \cup \{\ell\} = \mathcal{S}$.

Case ii): if $\mathbf{v}_\ell = 0$ and $\overline{\mathrm{ZP}}(\mathbf{v}) \subsetneq \mathcal{S}$, then we can resort, as for Case i), to the inductive assumption and claim that $\mathbf{x}_f \in \mathbf{R}_k$. So, the only hypothetical case remaining is the case when $\mathbf{v}_\ell = 0$ and $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, so that $\ell \in \mathcal{I}_{\mathcal{S}}$. We want to show that this is not possible, under the proposition's assumptions[4]. Indeed, if this were the case, then vector $\mathbf{v}$, being reachable, could, in turn, be expressed as

$$\mathbf{v} = e^{A_j \tau_j}\mathbf{w} + \mathbf{w}_j,$$

where $j \in \mathcal{P}, j \neq \ell$, $\tau_j > 0$, $\mathbf{w}_j$ is a state reachable by the subsystem (A_j, b_j), and $\mathbf{w}$ is a reachable state. By Lemma 3, however, this would imply that $j \in \mathcal{I}_{\mathcal{S}}$, but since $j \neq \ell$ this is not possible.

Finally, if $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| = n$, namely $\mathbf{x}_f \gg 0$, then $\mathbf{x}_f \in \mathbf{R}_n$, as a consequence of Proposition 4. ∎

Remark 1: We would like to underline similarities and differences between Propositions 5 and 6. In both propositions we assume that the system is monomially reachable, and hence its n subsystems obey (2) for suitable values of the coefficients, and that for every proper subset $\mathcal{S}$ of $\langle n \rangle$ the index set $\mathcal{I}_{\mathcal{S}}$ is not empty. However, in Proposition 5, we suppose that for every $\mathcal{S}$ the intersection $\mathcal{I}_{\mathcal{S}} \cap \mathcal{S}$ is not empty, and we deduce that the system is reachable and that every vector $\mathbf{x}_f > 0$ belongs to $\mathbf{R}_{|\overline{\mathrm{ZP}}(\mathbf{x}_f)|}$. In Proposition 6 we assume reachability and that for every proper subset $\mathcal{S}$ of $\langle n \rangle$ the index set $\mathcal{I}_{\mathcal{S}}$ consists of a single element. This allows to say that every vector $\mathbf{x}_f > 0$ belongs to $\mathbf{R}_{|\overline{\mathrm{ZP}}(\mathbf{x}_f)|}$. It is thus worth to remark that reachability is a consequence in the former proposition, and an assumption in the latter.

IV. Single-Input Positive Switched Systems of Dimension 2 or 3

To conclude the paper, we want to show that, at least for the class of 2-dimensional and 3-dimensional single-input positive switched systems, reachability always implies that every vector $\mathbf{x}_f$ belongs to $\mathbf{R}_{|\overline{\mathrm{ZP}}(\mathbf{x}_f)|}$ and hence $\mathbf{R}_n = \mathbb{R}^n_+$, where n is either 2 or 3, depending on the class of systems we are considering.

Specifically, if we are dealing with 2-dimensional systems commuting among an arbitrary number of single-input systems, we can show that reachability is equivalent to monomial reachability and that $\mathbf{R}_2 = \mathbb{R}^2_+$.

Proposition 7: Consider a positive switched system (1), commuting among p single-input subsystems of size 2. If the system is monomially reachable, then it is reachable and $\mathbf{R}_2 = \mathbb{R}^2_+$.

Proof: If the system is monomially reachable, we can assume that (2) holds and hence that

$$(A_1, b_1) = \left(\begin{bmatrix} \alpha_1 & \star \\ 0 & \star \end{bmatrix}, \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix} \right),$$
$$(A_2, b_2) = \left(\begin{bmatrix} \star & 0 \\ \star & \alpha_2 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} \right),$$

where $\alpha_1, \alpha_2 \geq 0$, $\beta_1, \beta_2 > 0$, while $\star$ denotes a non-specified entry (nonnegative if off-diagonal).

[4]We want to underline that this is not a contradiction w.r.t the result of Lemma 2. Indeed, it is possible to have $\mathbf{x}_f = e^{A_\ell \tau_\ell}\mathbf{v}$, for some $\mathbf{v}$ with $\overline{\mathrm{ZP}}(\mathbf{v}) = \overline{\mathrm{ZP}}(\mathbf{x}_f)$. What we are denying here is the possibility that $\mathbf{v}$ may be reachable.

Clearly, all monomial vectors belong to $\mathbf{R}_1$. Consider now any vector $\mathbf{x}_f \gg 0$. In order to show that it is reachable, by Proposition 3, we can simply show that there exist indices $i_0, i_1 \in \langle 2 \rangle$ and a time interval $\tau_1 > 0$ such that

$$\mathbf{x}_f \in \mathrm{Cone}[\mathbf{e}_{i_1} \quad e^{A_{i_1}\tau_1}\mathbf{e}_{i_0}]. \qquad (7)$$

Set $i_0 = 1$ and $i_1 = 2$. Since $e^{A_{i_1}\tau_1}\mathbf{e}_{i_0} = e^{A_2\tau_1}\mathbf{e}_1$, and, by Lemma 1, the first column of $e^{A_2\tau_1}$ can be made arbitrarily close to the direction of the coordinate axis associated with $\mathbf{e}_1$, it follows that by suitable choosing τ_1 we can always ensure that (7) holds. ■

We now address the specific case $n = 3$, namely when we have a 3-dimensional positive switched system, commuting among 3 single-input positive subsystems. We want to show that if the system is reachable, then $\mathbf{R}_3 = \mathbb{R}^3_+$.

Proposition 8: Consider a positive switched system (1), commuting among 3 subsystems of size 3. If the system is reachable, then $\mathbf{R}_3 = \mathbb{R}^3_+$.

Proof: Assume w.l.o.g. that (2) holds for suitable $\alpha_i \geq 0$ and $\beta_i > 0$, and consider any positive vector $\mathbf{x}_f \in \mathbb{R}^3_+$. If $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| = 1$, namely $\mathbf{x}_f$ is a monomial vector, then it can be reached by using the single subsystem (A_i, b_i), with $\{i\} = \overline{\mathrm{ZP}}(\mathbf{x}_f)$. Consequently, $\mathbf{x}_f \in \mathbf{R}_1$.

If $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| = 3$, namely $\mathbf{x}_f$ is a strictly positive vector, then it can be reached by using the three subsystems $(A_i, b_i), i \in \langle 3 \rangle$, as shown in Proposition 4. This ensures that $\mathbf{x}_f \in \mathbf{R}_3$.

So, we want to show that all vectors $\mathbf{x}_f$, with $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| = 2$, belong to $\mathbf{R}_2$. Set $\mathcal{S} := \overline{\mathrm{ZP}}(\mathbf{x}_f)$. Two cases possibly occur: either (a) there exists an index $i \in \langle 3 \rangle$ such that $i \in \mathcal{I}_\mathcal{S}$ and $i \in \mathcal{S}$, or (b) for every index $i \in \langle 3 \rangle$ such that $i \in \mathcal{I}_\mathcal{S}$, it follows that $i \notin \mathcal{S}$. In the first situation, as in the proof of Proposition 5, by resorting to Lemma 2, we can claim that, since $i \in \mathcal{I}_\mathcal{S}$, then $\mathbf{x}_f = e^{A_i\tau}\mathbf{w}$, for some $\tau > 0$ and some positive vector $\mathbf{w}$ with $\overline{\mathrm{ZP}}(\mathbf{w}) = \mathcal{S}$. On the other hand, as $i \in \mathcal{S}$, then one can also write $\mathbf{x}_f = e^{A_i\tau}\mathbf{w}' + \mathbf{v}_i$, for suitable $\mathbf{w}'$ and $\mathbf{v}_i$, with $\overline{\mathrm{ZP}}(\mathbf{w}') = \mathcal{S} \setminus \{i\}$ and $\overline{\mathrm{ZP}}(\mathbf{v}_i) = \{i\}$. Since $\mathbf{w}'$ is a monomial vector, and hence belongs to $\mathbf{R}_1$, this implies that $\mathbf{x}_f$ belongs to $\mathbf{R}_2$.

Suppose, now, that we are in case (b) and hence $\mathcal{I}_\mathcal{S} \cap \mathcal{S} = \emptyset$. This implies that every index $i \in \mathcal{I}_\mathcal{S}$ must belong to $\langle 3 \rangle \setminus \mathcal{S}$ which is a set consisting of a single index, say d. In other words, in this case it must be $\mathcal{I}_\mathcal{S} = \{d\}$. As the system is reachable, then $\mathbf{x}_f$ can be expressed (see, also, the proof of Lemma 3) as

$$\mathbf{x}_f = e^{A_d\tau}\mathbf{v} + \mathbf{v}_d,$$

where $\tau > 0$, $\mathbf{v}_d$ is a state reachable by the subsystem (A_d, b_d), and $\mathbf{v}$ is a reachable state. By applying the same reasoning we resorted to within the proof of Proposition 6, we can claim that two cases may occur:

(b1) $\mathbf{v}_d \neq 0$;
(b2) $\mathbf{v}_d = 0$.

Case (b1): if $\mathbf{v}_d \neq 0$, then it would be $\{d\} = \overline{\mathrm{ZP}}(\mathbf{v}_d) \subset \mathcal{S}$, thus contradicting the assumption corresponding to case (b) (as it would be $d \in \mathcal{I}_\mathcal{S} \cap \mathcal{S}$). So, we must be in case (b2): $\mathbf{v}_d = 0$. If $\overline{\mathrm{ZP}}(\mathbf{v}) \subsetneq \mathcal{S}$, then $\mathbf{v}$ is a monomial vector and hence it belongs to $\mathbf{R}_1$. This ensures that $\mathbf{x}_f$ belongs to $\mathbf{R}_2$. So, the only hypothetical case remaining is the case when $\mathbf{v}_d = 0$ and $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$. If so, by proceeding as in the proof of Proposition 6, we can claim that vector $\mathbf{v}$, in turn, could be expressed as $\mathbf{v} = e^{A_j\tau_j}\mathbf{w} + \mathbf{w}_j$, where $j \in \langle 3 \rangle, j \neq d$, $\tau_j > 0$, $\mathbf{w}_j$ is a state reachable by the subsystem (A_j, b_j), and $\mathbf{w}$ is a reachable state. By Lemma 3, however, this would imply $j \in \mathcal{I}_\mathcal{S}$, but since $j \neq d$ this is not possible. ■

V. CONCLUSIONS

In this paper we addressed the reachability problem for the class of single-input positive switched systems. We have shown that the sequence of the k-switching reachability sets, $\{\mathbf{R}_k, k \in \mathbb{N}\}$, is a non-decreasing one and if at some stage it does not increase, namely $\mathbf{R}_k = \mathbf{R}_{k+1}$ for some $k \in \mathbb{N}$, then it cannot further increase, namely all states reachable in finite time belong to $\mathbf{R}_k$. A necessary and sufficient condition for this to happen has also been provided.

By restricting our attention to the class of n-dimensional single-input positive switched systems that commute among n subsystems, we have provided sufficient conditions that ensure that every vector $\mathbf{x}_f > 0$ with $|\overline{\mathrm{ZP}}(\mathbf{x}_f)| = k$ belongs to $\mathbf{R}_k$ and it can be reached by resorting to a switching sequence σ taking values only in $\overline{\mathrm{ZP}}(\mathbf{x}_f)$. As a consequence, under these conditions, $\mathbf{R}_n = \mathbb{R}^n_+$.

Finally, we have proved that this result is always true when restricting our attention to 2-dimensional and 3-dimensional reachable systems belonging to this class.

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