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Inducing Oscillations in an Inertia Wheel Pendulum via Two-Relays Controller: Theory and Experiments

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***Abstract*— A tool for generating a self-excited oscillations for an inertia wheel pendulum by means of a variable structure controller is proposed. The original system is transformed into the normal form for exact linearization. The design procedure, based on Describing Function (DF) method, allows for finding the explicit expressions of the two-relays controller gain parameters in terms of the desired frequency and amplitude. Necessary condition for orbital asymptotic stability of the output of the exactly linearized system is derived. Performance issues of the system with self-excited oscillations are validated with experiments.**

I. Introduction

A. Overview

Traditionally, control systems are categorized into: regulators, which are supposed to maintain a certain process variable at a desired set point value, and servo systems that are supposed to track external inputs as precisely as possible. However, in practice there are some other control tasks that fall into neither of the above categories. One of those tasks is generating a *functional motion*: the motion having some properties important to the functionality of a certain system without involvement of set point tracking and specification of other properties of the motion.

In this paper, we consider the control of one of the simplest types of a functional motion: generation of a periodic motion in an inertial (reaction) wheel pendulum. Current representative works on periodic motions in an orbital stabilization of *non-minimum-phase underactuated* mechanical systems involve finding and using a reference model as a generator of limit cycles (e.g., [11], [13]), thus considering the problem of obtaining a periodic motion as a servo problem. Orbital stabilization of underactuated systems finds applications in the coordinated motion of biped robots [7], gymnastic robots, and others (see, e.g., [8], [12], [14], [15] and references therein).

The twisting algorithm [9] originally created as a SOSM controller -to ensure the finite-time convergence - is generalized, so that it can generate self-excited oscillations in the closed-loop system containing an underactuated plant. The required frequencies and amplitudes of periodic motions are produced without tracking of precomputed trajectories. It allows for generating a wider (than the original twisting algorithm with additional dynamics) range of frequencies and encompassing a variety of plant dynamics.

Below we will exploit the fact that the dynamics of the inertial wheel pendulum can be transformed, via normal form, into the exactly linearized one with internal locally stable dynamics [8]. The specific feature of the considered system is that the dynamics are of relative degree three.

B. Contributions

The contributions of the paper are:

- The two-relays control algorithm proposed in [3] is modified ensuring the intersection of the describing function plot with the Nyquist plot in any quadrant of the complex plane.
- For the exactly linearized system, the DF method allows for finding the explicit expressions of the two-relays controller gains in terms of the desired frequency and amplitude.
- Necessary condition for orbital asymptotic stability of periodic solution for the exactly linearized system is derived.
- The results are validated by experiments.

C. Organization of the paper

The paper is organized as follows: The equation of motion of an inertia wheel pendulum and the self-oscillation problem are introduced in Section II. The change of coordinates which transform the dynamics of the wheel pendulum into the exactly linearized one is explained. In Section III, the parameter design formulas for the two-relays controller to obtain desired frequency and amplitude is derived from DF method. Orbital stability analysis is proposed. Performance issues is illustrated by experiments in Section IV. Finally, Section V presents some conclusions.

II. Problem statement

Dynamics of an inertia wheel pendulum can be described as follows [1]:

$$\begin{bmatrix} J_1 & J_2 \\ J_2 & J_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h\sin q_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau \tag{1}$$

where $q_1 \in \mathbb{R}$ is the absolute angle of the pendulum, counted clockwise from the vertical downward position; $q_2 \in \mathbb{R}$ is the absolute angle of the disk; J_1, J_2, and h are

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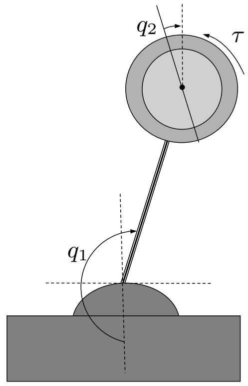

Fig. 1. Inertia wheel pendulum.

positive physical parameters, that depends on the geometric dimensions and the inertia-mass distribution; and τ is the controlled torque applied to the disk (see Fig. 1). It should be noted that the system (1) is nonlinear and underactuated. The purpose of the control is to produce a periodic motion of the underactuated link with desired frequency and amplitude.

In contrast to previous works in ([5], [6]) where DF method is used as chattering analysis for linear systems, now we are dealing with a nonlinear system; therefore linearization is required as a previous step for applying DF as a design method.

The inertia wheel pendulum has underactuation degree one and satisfies certain structural property noted in [8]. As a result, it is possible to make exact linearization thus achieving local stability of zero dynamics. Following [8], let us rewrite (1) in the modified normal form:

$$\dot{q}_1 = J_1^{-1}\zeta - J_1^{-1}J_2\dot{q}_2, \quad \dot{\zeta} = h\sin(q_1), \quad \ddot{q}_2 = v \qquad (2)$$

where ζ is the generalized momentum conjugate to q_1 and the control input

$$\tau = J_2\left(1 - J_1^{-1}J_2\right)v - J_1^{-1}J_2 h\sin(q_1) \qquad (3)$$

consists of the preliminary feedback needed to place the system in the normal form. The output that makes the internal dynamics locally stable is chosen as [8]

$$\eta = K\,p_1 + \zeta, \qquad p_1 = q_1 - \pi + J_1^{-1}\,J_2\,q_2 \qquad (4)$$

where $K \in \mathbb{R}$ is a constant. Direct calculation confirms relative degree three with respect to η:

$$\dddot{\eta} = R(q,\dot{q}) - H(q)v$$

where $R(q,\dot{q}) = J_1^{-1}h[K\cos(q_1)\dot{q}_1 + \cos(q_1)\dot{\zeta}] - h\sin(q_1)\dot{q}_1^2$ and $H(q) = J_1^{-1}J_2 h\cos(q_1)$. By selecting

$$v = H^{-1}(q)\left[-u + a_0\eta + a_1\dot{\eta} + a_2\ddot{\eta} + R(q,\dot{q})\right] \qquad (5)$$

where $H(q)$ is nonsingular around the equilibrium point $(q_1^\star, q_1^\star) = (\pi, 0)$, a_0, a_1, and a_2 are positive constants, the model (2) can be written in terms of the states $x = [x_1, x_2, x_3, x_4]^T = [\eta, \dot{\eta}, \ddot{\eta}, p_1]^T$ as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -a_0 & -a_1 & -a_2 & 0\\ J_1^{-1} & 0 & 0 & -KJ_1^{-1}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}}_{B}u \qquad (6)$$

$$y = \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}}_{C}x.$$

The system (6) is linear with generalized output y. Let us design a stabilizing controller for a periodic solution with desired frequency and amplitude for $y(t)$. The motivation for this is as follows. Suppose we have achieved this. Consider a periodic function $\eta_*(t) = y_*(t)$ that satisfies the three differential equations in (6) with appropriate u. They are decoupled from the last one given by $\dot{p}_1 = -K\,J_1^{-1}\,p_1 + J_1^{-1}\,y_*(t)$. It is obvious that p_1 exponentially converges to a periodic function $p_{1*}(t)$. Now, it is clear that for sufficiently small amplitudes of the oscillations, $q_1(t)$ exponentially converges to $q_{1*}(t) = \arcsin((\dot{y}_*(t) - K\,\dot{p}_{1*}(t))/h)$, which is periodic with the same frequency as $y_*(t)$ and an amplitude defined by the amplitude and the frequency of $y_*(t)$. Therefore, the presented change of coordinates and of the control input transforms the system into the one that has locally stable zero dynamics, while the remaining dynamics are forced to have a desired periodic motion.

III. The two-relays controller

The *analysis and design objectives* are to find the parameter values c_1 and c_2 of the two-relays controller

$$u = -c_1\,\mathrm{sign}(y) - c_2\,\mathrm{sign}(\dot{y}) \qquad (7)$$

where c_1 and c_2 are scalars designed such that the scalar-valued function output $y(t)$ has a periodic motion with the desired frequency Ω and amplitude A_1.

A. Describing function based design

Let firstly, the linearized plant (6) can be represented in the transfer function form as follows:

$$W(s) = C(sI - A)^{-1}B.$$

The Describing Function (DF) N of the variable structure controller (7) is the first harmonic of the periodic control signal divided by the amplitude of $y(t)$ [4]:

$$N = \frac{\omega}{\pi A_1}\int_0^{\frac{2\pi}{\omega}} u(t)\sin\omega t dt + j\frac{\omega}{\pi A_1}\int_0^{\frac{2\pi}{\omega}} u(t)\cos\omega t dt \qquad (8)$$

where A_1 is the amplitude of the input to the nonlinearity (of $y(t)$ in our case) and ω is the frequency of $y(t)$. However, the algorithm (7) can be analyzed as the parallel connection of two ideal relays where the input to the first relay is the output variable and the input to the second relay is the derivative of the output variable. For the first relay the DF is:

$$N_1 = \frac{4c_1}{\pi A_1},$$

and for the second relay it is:

$$N_2 = \frac{4c_2}{\pi A_2},$$

where A_2 is the amplitude of dy/dt. Also, take into account the relationship between y and dy/dt in the Laplace domain, which gives the relationship between the amplitudes A_1 and A_2: $A_2 = A_1\Omega$, where Ω is the frequency of the oscillation. Using the notation of the algorithm (7) we can rewrite this equation as follows:

$$N = N_1 + sN_2 = \frac{4c_1}{\pi A_1} + j\Omega\frac{4c_2}{\pi A_2} = \frac{4}{\pi A_1}(c_1 + jc_2), \quad (9)$$

where $s = j\Omega$. Let us note that the DF of the algorithm (7) depends on the amplitude value only. This suggests the technique of finding the parameters of the limit cycle - via the solution of the harmonic balance equation [4]:

$$W(j\Omega)N(a) = -1, \quad (10)$$

where a is the generic amplitude of the oscillation at the input of the nonlinearity and $W(j\omega)$ is the complex frequency response characteristic (Nyquist plot) of the plant. Using the notation of the algorithm (7) and replacing the generic amplitude with the amplitude of the oscillation of the input to the first relay this equation can be rewritten as follows:

$$W(j\Omega) = -\frac{1}{N(A_1)}, \quad (11)$$

where the function at the right-hand side is given by:

$$-\frac{1}{N(A_1)} = \pi A_1 \frac{-c_1 + jc_2}{4(c_1^2 + c_2^2)}. \quad (12)$$

Equation (10) is equivalent to the condition of the complex frequency response characteristic of the open-loop system intersecting the real axis in the point $(-1, j0)$. The function $-1/N$ is a straight line the slope of which depends on c_2/c_1 ratio. The point of intersection of this function and of the Nyquist plot $W(j\omega)$ provides the solution of the periodic problem. Now, we summarize the steps to tune c_1 and c_2:

1) Nyquist quadrant identification: Identify the quadrant in the Nyquist plot where the desired frequency is located, that is, Ω can belongs to any of the following sets (see Fig. 2):

$$Q_1 = \{\omega \in \mathbb{R} : \mathrm{Re}\{W(j\omega)\} > 0, \mathrm{Im}\{W(j\omega)\} \geq 0\}$$
$$Q_2 = \{\omega \in \mathbb{R} : \mathrm{Re}\{W(j\omega)\} \leq 0, \mathrm{Im}\{W(j\omega)\} \geq 0\}$$
$$Q_3 = \{\omega \in \mathbb{R} : \mathrm{Re}\{W(j\omega)\} \leq 0, \mathrm{Im}\{W(j\omega)\} < 0\}$$
$$Q_4 = \{\omega \in \mathbb{R} : \mathrm{Re}\{W(j\omega)\} > 0, \mathrm{Im}\{W(j\omega)\} < 0\}.$$

2) Gain parameters computation: The frequency of the oscillations depends only on the c_2/c_1 ratio, and it is possible to obtain the desired frequency Ω by tuning the $\xi = c_2/c_1$ ratio:

$$\xi = \frac{c_2}{c_1} = -\frac{\mathrm{Im}\{W(j\Omega)\}}{\mathrm{Re}\{W(j\Omega)\}}. \quad (13)$$

Since the amplitude of oscillations is given by

$$A_1 = \frac{4}{\pi}|W(j\Omega)|\sqrt{c_1^2 + c_2^2}, \quad (14)$$

then the c_1 and c_2 values can be computed as follows

$$c_1 = \begin{cases} \frac{\pi}{4}\frac{A_1}{|W(j\Omega)|}\left(\sqrt{1+\xi^2}\right)^{-1} & \text{if } \Omega \in Q_2 \cup Q_3 \\ -\frac{\pi}{4}\frac{A_1}{|W(j\Omega)|}\left(\sqrt{1+\xi^2}\right)^{-1} & \text{elsewhere} \end{cases} \quad (15)$$

$$c_2 = \xi \cdot c_1. \quad (16)$$

Remark 1: It is possible to obtain the formulas for computing the exact values of c_1 and c_2 using the Locus of Perturbed Relay Systems (LPRS) method (see details in [2]).

Remark 2: Necessary and sufficient condition for the existence of the desired periodic solution could be obtained via Poincaré sections.

B. Stability analysis

The approach for the stability analysis of the periodic motions is similar to the one proposed in [10]. We shall consider that the harmonic balance condition still holds for small perturbation of the amplitude and the frequency with respect to the steady values. In this case the oscillation can be described as a damped one. If the limit cycle is asymptotically stable then the damping parameter must be negative at a positive increment of the amplitude and positive at a negative increment of the amplitude then the perturbation will vanish.

Theorem 1: Suppose that for the values of the c_1 and c_2 given by (15) and (16) there exists an asymptotic orbitally stable solution to the system (6). Then

$$\mathrm{Re}\left.\frac{d\ln W}{ds}\right|_{s=j\Omega} + \frac{c_1 c_2}{\Omega(c_1^2 + c_2^2)} < 0. \quad (17)$$

Proof: The approach for the stability analysis of the periodic motions is similar to the one proposed in [10]. Writing the harmonic balance equation of the perturbed motion:

$$\{N_1(A_1+\Delta A_1)+[j\Omega+(\Delta\sigma+j\Delta\Omega)]N_2(A_2+\Delta A_2)\} \times W(\Delta\sigma+j(\Omega+\Delta\Omega))=-1 \quad (18)$$

where $A_2 = \Omega A_1$. The Laplace variable for the damped oscillation is $s = j\Omega + (\Delta\sigma + j\Delta\Omega)$. Taking the derivative of both sides of this equation with respect to ΔA_1:

$$\left.\frac{\partial N_1}{\partial \Delta A_1}\right|_{\Delta A_1=0} \cdot W(j\Omega) + \left.\frac{dW}{dS}\right|_{s=j\Omega}\left(\frac{d\Delta\sigma}{d\Delta A_1}+j\frac{d\Delta\Omega}{d\Delta A_1}\right)N_1(A_1) + \left.\frac{\partial N_2}{\partial \Delta A_1}\right|_{\Delta A_1=0}\cdot j\Omega W(j\Omega)+N_2(A_2)\Psi$$

where

$$\Psi=\left(\frac{d\Delta\sigma}{d\Delta A_1}+j\frac{d\Delta\Omega}{d\Delta A_1}\right)W(j\Omega) + j\Omega\left.\frac{dW}{ds}\right|_{s=j\Omega}\left(\frac{d\Delta\sigma}{d\Delta A_1}+j\frac{d\Delta\Omega}{d\Delta A_1}\right)=0$$

$$\frac{\partial N_1}{\partial \Delta A_1}=-\frac{4c_1}{\pi A_1^2}; \quad \text{and} \quad \frac{\partial N_2}{\partial \Delta A_1}=-\frac{4c_2}{\pi A_2^2}\Omega=-\frac{4c_2}{\pi\Omega A_1^2}.$$

Thus, the following equation is obtained

$$-\frac{4c_1}{\pi A_1^2}W(j\Omega)-j\frac{4c_2}{\pi A_1^2}W(j\Omega) = \left(\frac{d\Delta\sigma}{d\Delta A_1}+j\frac{d\Delta\sigma}{d\Delta A_1}\right) \times\left\{-N_1(A_1)\left.\frac{dW}{ds}\right|_{s=j\Omega} -N_2(A_2)\left[W(j\Omega)+j\Omega\left.\frac{dW}{ds}\right|_{s=j\Omega}\right]\right\}.$$

Express the quantity $\frac{d\Delta\sigma}{d\Delta A_1}+j\frac{d\Delta\Omega}{d\Delta A_1}$ from that equation

$$\frac{d\Delta\sigma}{d\Delta A_1}+j\frac{d\Delta\Omega}{d\Delta A_1} = \frac{W(j\Omega)[c_1+jc_2]}{A_1\left\{c_1\left.\frac{dW}{ds}\right|_{s=j\Omega}+c_2\left[\frac{1}{\Omega}W(j\Omega)+j\left.\frac{dW}{ds}\right|_{s=j\Omega}\right]\right\}} = \frac{1}{\underbrace{A_1\left\{\left.\frac{d\ln W}{ds}\right|_{s=j\Omega}+\frac{c_2}{\Omega}\frac{c_1-jc_2}{c_1^2+c_2^2}\right\}}_{\Lambda}}.$$

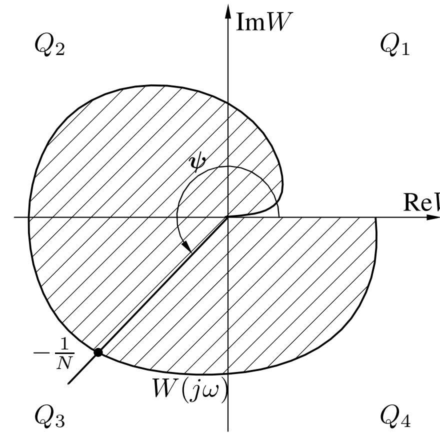

Fig. 2. Example of a Nyquist plot of the open-loop system $W(j\omega)$ with the two-relays controller.

Then for the inequality

$$\frac{d\Delta\sigma}{d\Delta A_1}<0 \quad (19)$$

to be true, the following inequality hold:

$$\mathrm{Re}\frac{1}{A_1\Lambda}<0 \quad \text{or} \quad \mathrm{Re}\Lambda<0. \quad (20)$$

Then for the real part of Λ, we can write:

$$\mathrm{Re}\left.\frac{d\ln W}{ds}\right|_{s=j\Omega}+\frac{c_1c_2}{\Omega(c_1^2+c_2^2)}<0. \quad (21)$$

Representing the transfer function in the exponential format and taking the derivative with respect to s leads to the following inequality:

$$\left.\frac{d\arg W}{d\omega}\right|_{\omega=\Omega}<-\frac{c_1c_2}{\Omega(c_1^2+c_2^2)} \quad (22)$$

or finally

$$\left.\frac{d\arg W}{d\ln\omega}\right|_{\omega=\Omega}<-\frac{\xi}{\xi^2+1}. \quad (23)$$

Therefore, the stability of the periodic motion is determined just by the slope of the phase characteristic of the plant, which must be steeper than a certain value for the oscillation to be asymptotically stable. ■

Remark 3: Sufficient conditions for the existence and stability of desired periodic solutions could be obtained by using Poincaré sections.

IV. Experimental results

A. Experimental setup

In this section, we present experimental results using the laboratory inertial wheel pendulum from Mechatronics Control Kit manufactured by Quanser Inc., depicted in Figure

3 where $J_1 = 4.572 \times 10^{-3}$, $J_2 = 2.495 \times 10^{-3}$, and $h = 0.3544$ (see [1]). It consists of a physical pendulum (q_1) with a motor/flywheel assembly attached to the free end of the-pendulum (q_2). The wheel is actuated by a 24-Volt, permanent magnet DC-motor and the coupling torque between the wheel and pendulum can be used to control the motion of the system. The pendulum angle is measured by an encoder. The experimental setup includes a PC equipped with a C6713 DSK Quanser interface/PWM amplifier board. The algorithm was implemented using C programming language. The sampling frequency for algorithm implementation was set to 400 Hz.

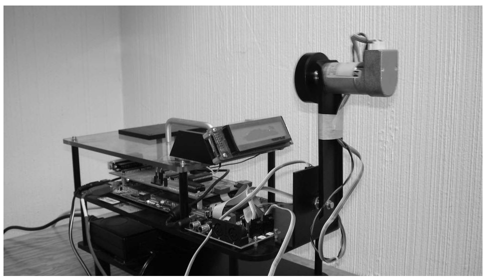

Fig. 3. Inertial wheel pendulum testbench.

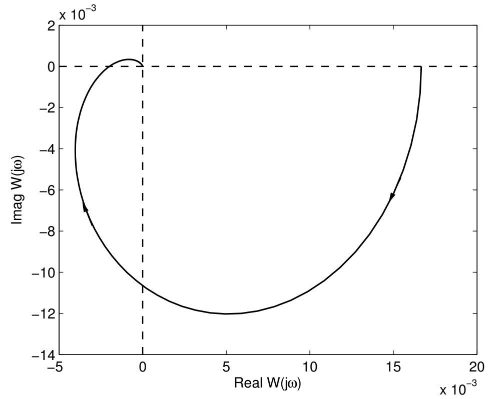

Fig. 4. Nyquist plot of the inertia wheel pendulum.

TABLE I

COMPUTED c_1 AND c_2 VALUES FOR SEVERAL DESIRED FREQUENCIES AND AMPLITUDES USING DF.

Ω	A_1	Quadrant	c_1	c_2	Ω^r	A_1^r
1.0	0.3	Q_3	-0.333	0.333	1.04	0.28
2.0	0.2	Q_2	0.0444	0.3110	1.79	0.04
2.5	0.1	Q_1	0.211	-0.741	2.2	0.1

B. Experimental validation

Experiments were carried out to achieve the orbital stabilization of the unactuated link (pendulum $y = \eta$ around the equilibrium point $q^\star = (\pi, 0)$. In the experiments, the inertia wheel pendulum was required to move from $[q_1(0), q_2(0)] = [3.1, 0.1]$ to the desired periodic motion. The initial velocity $\dot{q}(0) \in \mathbb{R}^2$ were set to zero for the experiments. The Nyquist plot of (1)-(7) evolving three quadrants is shown in Fig. 4. The parameters of the linearized systems are $K = 1 \times 10^{-4}$, $a_0 = 350$, $a_1 = 155$, and $a_2 = 22$.

Let us select $\Omega = 12$ [rad/s] and $A_1 = 0.005$ [rad] as desired frequency and amplitude. The exponential stability of the periodic solution is verified by using Theorem 1 where

$$\text{Re}\left\{\frac{d \arg W}{d \ln \omega}\right\} = -0.1070, \quad \frac{\xi}{\xi^2 + 1} = 0.0039,$$

thus satisfying inequality (23).

Since that we are interested in presenting the results in the original coordinates q_1 and q_2, let us show how to compute an approximation for the amplitude of oscillations for the pendulum itself. Since $\dot{p}_1 + J_1^{-1} K p_1 = J_1^{-1}\eta(t)$ we know that p_1 exponentially converge to a periodic steady-state function (the rate of convergence can be regulated by K).

Taking into account only the first harmonic and letting the steady-state value for $\eta(t)$ to be $\eta(t) \approx A_1 \sin(\Omega t)$ we can compute the approximate steady-state value for $p_1(t)$ in the form

$$p_1(t) \approx \frac{A_1}{\sqrt{J_1^2 \Omega^2 + K^2}} \sin\left(\Omega t + \arg\left\{\frac{1}{j J_1 \Omega + K}\right\}\right).$$

Now, using the equation $\dot{\eta}(t) \approx \Omega A_1 \cos(\Omega t)$ and the equation $h \sin(q_1) = \dot{\eta} - K \dot{p}_1$ one can conclude that q_1 exponentially converges to a periodic steady-state provided oscillations are small.

Finally, for q_1 close enough to π we have $\sin(q_1) \approx \pi - q_1$, and so

$$q_1(t) \approx \pi - \frac{\Omega A_1}{h} \cos(\Omega t) - \frac{\Omega A_1}{h\sqrt{J_1^2 \Omega^2 + K^2}} \sin\left(\Omega t + \arg\left\{\frac{1}{j J_1 \Omega + K}\right\}\right).$$

This expression gives us an estimate on the amplitude of achieved oscillations of the pendulum around π to be

$$A^r \approx \frac{\Omega A_1}{h} \sqrt{1 + \frac{1}{J_1^2 \Omega^2 + K^2}}.$$

Since $q_2 = J_1(p_1 - (\pi - q_1))/J_2$, the steady-state amplitude of q_2 can be estimated as well.

In Table I the measured frequencies and amplitudes denoted by Ω^r and A_1^r respectively are depicted; with the values of c_1 and c_2 computed from (15) and (16). In Figures 5 and 6, the oscillations for the output y and (q_1, q_2), where $\Omega = 12$ [rad/s] and $A_1 = 0.005$ [rad] are displayed. Figures 7 and 8 illustrates the phase portrait of $(q_1, \dot{q}_1)$ and (q_1, q_2), respectively; revealing the limit cycle behavior of the closed-loop system.

V. CONCLUSIONS

In this paper, the algorithm for periodic motion design for an inertial wheel pendulum is presented. Two steps were proposed to achieve the control objective which consists of

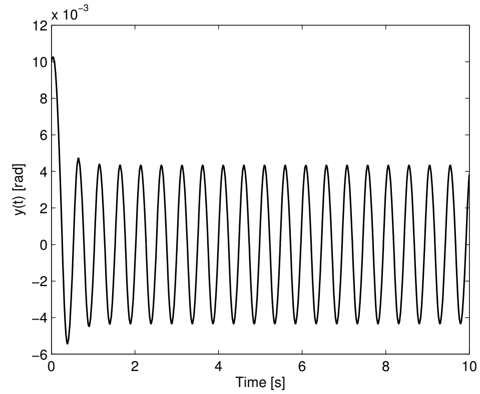

Fig. 5. Time evolution of the output $y(t)$.

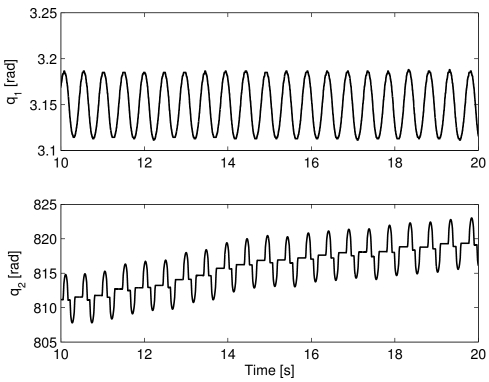

Fig. 6. Time evolution of the joint positions $q_1(t)$ and $q_2(t)$.

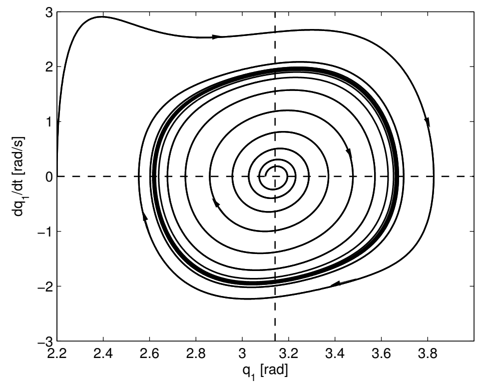

Fig. 7. Phase portrait of the unactuated joint (q_1 vs $\dot{q}_1$) for initial conditions inside and outside the limit cycle.

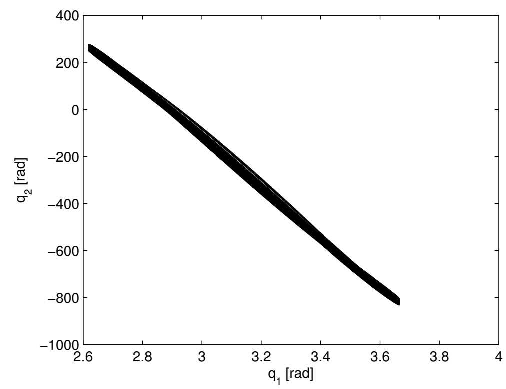

Fig. 8. Behaviour of the joint position trajectories q_1 vs q_2.

the exact linearization via normal form, where internal dynamics is locally stable, and the two-relays gains adjustment via the DF method. For the exactly linearized system, the DF method gives explicit expressions of the two-relays controller gains in terms of the desired frequency and amplitude of the output. Necessary condition for orbital asymptotic stability of the system with the linearized plant was derived. The proposed design procedure was validated by experiments.

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