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Impulsive Control of a Mechanical Oscillator with Friction

Yury Orlov, Raul Santiesteban, and Luis T. Aguilar

***Abstract*— Undesired dynamics (stick-slip motions, limit cycles, etc.) that appear in friction systems result in steady-state errors, limiting achievable performance. To enhance the performance a novel control approach counteracting friction effects is developed. The approach is based on an impulsive actuation while the underlying system is getting stuck due to the presence of dry friction. A spring-mass system serves as a simple test bed. Asymptotic stabilization is achieved not only for the nominal model but also for its perturbed version provided that external disturbances affecting the system are of sufficiently small magnitude.**

I. INTRODUCTION

Friction is a highly nonlinear phenomenon, producing undesired stick-slip motions, thereby resulting in steady state errors. To reduce friction effects without resorting to high gain control loops suitable modeling is required for friction compensation [2]. Alternatively [5], [6], a second-order sliding mode control algorithm, robust to discrepancies of friction modeling, can be utilized.

In this paper, we develop a novel control approach counteracting friction effects. The approach is based on an impulsive actuation while the underlying system is getting stuck due to the presence of dry friction. A spring-mass system serves as a simple test bed. Stabilization of this system is under study. In contrast to the afore-mentioned friction compensation methods and sliding mode approach, static position feedback is constructed provided that the information is available on whether the system is in steady state. Robustness against small parameter variations and external disturbances is additionally provided.

II. PROBLEM STATEMENT

The spring-mass system (Figure 1), affected by Coulomb friction, is governed by the differential equation

$$m\ddot{x} = -kx - \alpha\,\mathrm{sign}(\dot{x}) + w + u \tag{1}$$

where x is the displacement, $\dot{x}$ is the velocity, $m > 0$ is the mass of the system, $\alpha > 0$ is the Coulomb friction level, $k > 0$ is the stiffness coefficient, u denotes the control input, and w stands for external disturbances. As equation (1) appears with discontinuous right-hand side, its meaning is defined in the sense of Filippov [1].

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Y. Orlov and R. Santiesteban are with CICESE Research Center, Electronics and Telecommunication Department, P.O. BOX 434944, San Ysidro, CA, 92143-4944, (e-mail: yorlov{rsanties}@cicese.mx)

L.T. Aguilar is with Centro de Investigación y Desarrollo de Tecnología Digital, Instituto Politécnico Nacional, Avenida del Parque 1310 Mesa de Otay, Tijuana 22510 México (e-mail: luis.aguilar@ieee.org).

The Coulomb friction model has been chosen for treatment. Although augmenting with viscous friction is important from practical standpoint, we preferred not to do so as such an extension would be rather technical. Instead, we preferred to facilitate exposition and to focus on essential features of the general treatment.

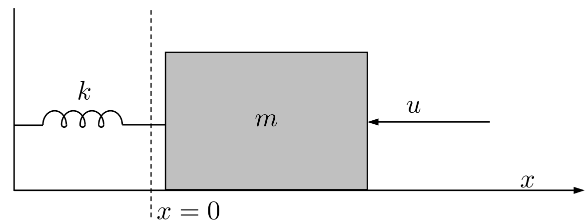

Fig. 1. Spring-mass system.

In order to account for model discrepancies, an unknown external disturbance $w(t)$ has been introduced into modeling. Throughout, the amplitude of the disturbance is assumed to be smaller than half the Coulomb level α, i.e.,

$$|w(t)| \le \alpha_0 < \frac{1}{2}\alpha \tag{2}$$

for all t and some constant $\alpha_0 > 0$. Assumption (2) is made for a technical reason that becomes clear as far as the controller derivation goes.

Being represented in the state space form, the above system (1) is modified to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\left[-kx_1 - \alpha\,\mathrm{sign}(x_2) + w + u\right] \end{aligned} \tag{3}$$

where $x_1 = x$ and $x_2 = \dot{x}$.

Since the unforced system (3) is dissipative and its dissipation is lower bounded by the positive constant $\alpha - \alpha_0$, system (3) under $u = 0$ is getting stuck in finite time at the disturbance-dependent zone (see Figure 2)

$$S_w \subset S = \{(x_1, x_2) : |x_1| \le \frac{\alpha + \alpha_0}{k},\ x_2 = 0\} \subset \mathbb{R}^2. \tag{4}$$

Assuming that the system is enforced by an impulsive

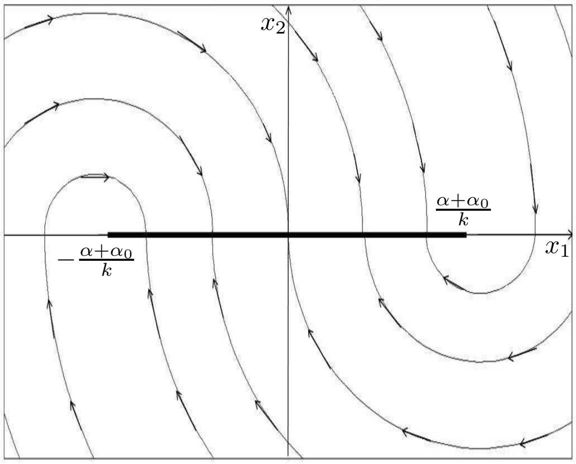

Fig. 2. Phase trajectories of the unforced spring-mass system (3) .

actuator

$$u = \sum_{i=1}^{\infty} v(x_1)\delta(t - t_i) \tag{5}$$

utilizing a continuous position feedback $v(x_1)$ and being applied at some state-dependent time instants $t_i(x_1, x_2),\ i = 1, 2, \ldots$, the closed-loop system appears to exhibit discrete-continuous dynamics

$$\dot{x}_1 = x_2,\ \dot{x}_2 = \frac{1}{m}\left(-kx_1 - \alpha\,\mathrm{sign}(x_2) + w\right),\ t \neq t_i, \tag{6}$$

$$x_1(t_i+) = x_1(t_i-),\ x_2(t_i+) = x_2(t_i-) + v(x_1(t_i)),$$
$$i = 1, 2, \ldots. \tag{7}$$

It is clear that dealing with the position feedback $v(x_1)$ yields the above restitution rule (7) because the impulsive input (5) results in the corresponding instantaneous change of the velocity while the position dynamics and, hence, the position feedback $v(x_1(t))$ remain continuous in time. It is worth noticing that applying a state feedback $v(x_1, x_2)$ would impose a certain nonlinear restitution rule, caused by an ill-posed product of the Dirac function $\delta(t - \tau)$, localized at a time instant τ, and the function $v(x_1(t), x_2(t))$, discontinuous at $t = \tau$ (see, e.g., [4] for details on the nonlinear impulse response).

Our objective is to design an impulsive controller (5) such that the closed-loop system (7) is asymptotically stable around the origin, regardless of whichever external disturbance (2) affects the system. The current position $x_1(t)$ is assumed to be available for measurement whereas the only available information on the velocity $x_2(t)$ is the knowledge of whether it is nullified or it is not.

III. Controller Design

Let the impulsive controller (5) be specified with

$$v(x_1) = -\sqrt{\frac{2\alpha|x_1| - kx_1^2}{m}}\,\mathrm{sign}(x_1) \tag{8}$$

and let it be applied to system (3) at the time instants $t_i,\ i = 1, 2, \ldots$ such that

$$|x_1(t_i)| \leq \frac{\alpha + \alpha_0}{k},\ x_2(t_i-) = 0. \tag{9}$$

The dynamics of the closed-loop system (5)–(9) is then as follows. Once the underlying system (3) hits the stuck zone (4), it is enforced by the impulsive controller (5) that changes the velocity of the system instantaneously (see Figure 3). The controller amplitude (8) has been pre-specified in such a manner to bring the underlying system (3) from the stuck zone (4) to the phase trajectory that while being disturbance-free arrives at the origin without oscillations. It is shown that the asymptotic stabilization is thus achieved not only for the disturbance-free system (3) with $w = 0$ but also for its perturbed version provided that external disturbances affecting the system meet the norm upper bound (2).

The following result is in order.

Theorem 1: Let the friction oscillator (3) be driven by the impulsive controller (5), specified with (8), (9). Then the closed-loop system (5)–(9) is globally asymptotically stable provided that the norm upper bound (2) holds for the external disturbance affecting the system.

Proof: It has been mentioned that while being unforced, system (3) possesses a globally finite time stable invariant manifold S_w, localized according to (4) within the estimated stuck zone S. Once the closed-loop system (5)–(9) attains S, say at $x_1(t_1) = \xi_1$ such that $|\xi_1| \leq \frac{\alpha+\alpha_0}{k}$, the impulsive controller (5), (8), (9) is applied at the time instant t_1. The restitution rule (7) is then specified as

$$x_1(t_1) = \xi_1,$$
$$x_2(t_1+) = -\sqrt{\frac{2\alpha|\xi_1| - k(\xi_1)^2}{m}}\,\mathrm{sign}(\xi_1). \tag{10}$$

If the closed-loop system is disturbance-free, the state trajectory, re-initialized with (10), would mono-directionally arrive at the origin in finite-time (see Figure 3). However, if an admissible disturbance (2) affects the closed-loop system, the state trajectory would hit the stuck zone at $x_1(t_2) = \xi_2 \neq 0$ provided that $x_2(t_2-) = 0$.

Our goal is to demonstrate that, even in the worst-disturbance case where $w = \alpha_0$ or $w = -\alpha_0$, the following inequality holds:

$$|\xi_2| \leq \frac{2\alpha_0}{\alpha}|\xi_1|. \tag{11}$$

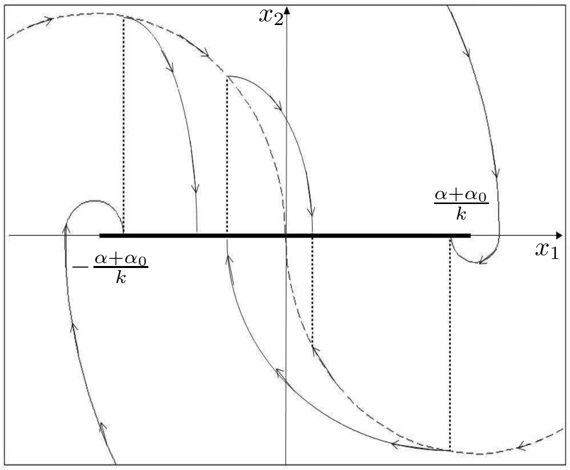

Fig. 3. Phase portrait of the impulsive closed-loop system (5)–(9): dotted lines are for jumps of the velocity, solid lines are for the perturbed trajectories, and dashed line is for the unperturbed trajectory.

Then by iteration on i, similar relations

$$|\xi_{i+1}| \leq \frac{2\alpha_0}{\alpha}|\xi_i|, \ i = 1, 2, \ldots. \tag{12}$$

could be obtained for the state positions $\xi_i = x_1(t_i)$ at the time instants $t_i, \ i = 1, 2, \ldots$ of reaching the stuck zone (4). Since $q = \frac{2\alpha_0}{\alpha} < 1$ by assumption, it would follow that

$$\lim_{i\to\infty} |x_1(t_i)| = \lim_{i\to\infty} q^{i-1}|\xi_1| = 0, \ \lim_{i\to\infty} |x_2(t_i+)| = 0 \tag{13}$$

where the relations $x_2(t_i+) = -\sqrt{\frac{2\alpha|\xi_i|-k(\xi_i)^2}{m}}\,\mathrm{sign}(\xi_i), \ i = 1, 2, \ldots,$ similar to (10), have been taken into account.

For the purpose of validating (11), let us compute the value of ξ_2 as a function of ξ_1, assuming, for certainty, that $\xi_1 < 0$, and performing similar computation, otherwise.

In the case where $w = \alpha_0$ the system dynamics between successive impacts are governed by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= \frac{1}{m}(-kx_1 - \alpha + \alpha_0), \ t \in (t_1, t_2). \end{aligned} \tag{14}$$

Initialized with (10), the solution to the perturbed system (14) is given by

$$\frac{1}{m}(-k\frac{x_1^2}{2} - \alpha x_1 + \alpha_0 x_1) = \frac{\alpha_0\xi_1}{m} + \frac{x_2^2}{2}. \tag{15}$$

Being confined to the time instant t_2 when $x_2(t_2-) = 0$, the above relation yields

$$kx_1^2(t_2) + 2(\alpha - \alpha_0)x_1(t_2) + 2\alpha_0\xi_1 = 0. \tag{16}$$

Setting $\xi_2 = x_1(t_2)$ and taking into account that the case where $\xi_1 < 0$ is under study, it follows that

$$\xi_2 = -\frac{\alpha - \alpha_0}{k} + \sqrt{\frac{(\alpha_0 - \alpha)^2}{k^2} + \frac{2\alpha_0|\xi_1|}{k}}. \tag{17}$$

Substituting (17) into (11) for ξ_2, we arrive at the inequality

$$-\frac{\alpha - \alpha_0}{k} + \sqrt{\frac{(\alpha_0 - \alpha)^2}{k^2} + \frac{2\alpha_0|\xi_1|}{k}} \leq \frac{2\alpha_0}{\alpha}|\xi_1| \tag{18}$$

to be verified. To validate (18) it suffices to represent it in the form

$$\frac{(\alpha_0 - \alpha)^2}{k^2} + \frac{2\alpha_0|\xi_1|}{k} \leq \left[\frac{2\alpha_0}{\alpha}|\xi_1| + \frac{\alpha - \alpha_0}{k}\right]^2, \tag{19}$$

and to observe that (19) is equivalent to the inequality

$$\frac{2\alpha_0|\xi_1|}{k} \leq \frac{4(\alpha_0)^2}{\alpha^2}|\xi_1|^2 + \frac{4\alpha_0(\alpha - \alpha_0)}{\alpha k}|\xi_1| \tag{20}$$

whose validation is reduced to the obvious inequality

$$1 \leq \frac{2(\alpha - \alpha_0)}{\alpha}, \tag{21}$$

resulted from (2). Thus, inequality (11) is verified in the case of $w = \alpha_0$.

It remains to verify inequality (11) in the case of $w = -1$ with the system dynamics between successive impacts governed by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= \frac{1}{m}(-kx_1 - \alpha - \alpha_0), \ t \in (t_1, t_2). \end{aligned} \tag{22}$$

The solution of the above system, initialized with (10), is given by

$$\frac{1}{m}(-k\frac{x_1^2}{2} - \alpha x_1 - \alpha_0 x_1) = -\frac{\alpha_0\xi_1}{m} + \frac{x_2^2}{2}. \tag{23}$$

Specified at the next impact time instant t_2 when $x_2(t_2-) = 0$, relation (23) yields

$$kx_1^2(t_2) + 2(\alpha + \alpha_0)x_1(t_2) - 2\alpha_0\xi_1 = 0. \tag{24}$$

Setting $\xi_2 = x_1(t_2)$, it follows that

$$\xi_2 = -\frac{\alpha_0 + \alpha}{k} + \sqrt{\frac{(\alpha_0+\alpha)^2}{k^2} - \frac{2\alpha_0|\xi_1|}{k}} \qquad (25)$$

provided that the case where $\xi_1 < 0$ is under study.

Substituting (25) into (11) for ξ_2, we arrive at the inequality

$$\frac{\alpha_0 + \alpha}{k} - \sqrt{\frac{(\alpha_0+\alpha)^2}{k^2} - \frac{2\alpha_0|\xi_1|}{k}} \le \frac{2\alpha_0}{\alpha}|\xi_1| \qquad (26)$$

to be verified. To validate (26) it suffices to represent it in the form

$$\left[\frac{\alpha + \alpha_0}{k} - \frac{2\alpha_0}{\alpha}|\xi_1|\right]^2 \le \frac{(\alpha_0+\alpha)^2}{k^2} - \frac{2\alpha_0|\xi_1|}{k}, \qquad (27)$$

and to observe that (27) is equivalent to the inequality

$$\frac{2\alpha_0|\xi_1|}{k} + \frac{4(\alpha_0)^2}{\alpha^2}|\xi_1|^2 \le \frac{4\alpha_0(\alpha+\alpha_0)}{\alpha k}|\xi_1|. \qquad (28)$$

Since ξ_1 is within the estimated stuck zone (4), i.e., $|\xi_1| \le \frac{\alpha+\alpha_0}{k}$, the validation of (28) is reduced to the inequality

$$1 + \frac{2\alpha_0(\alpha+\alpha_0)}{\alpha^2} \le \frac{2(\alpha+\alpha_0)}{\alpha}, \qquad (29)$$

straightforwardly resulted from (2). Thus, inequality (11) is verified in the case of $w = -\alpha_0$, too. This completes the proof of the theorem because as pointed out, inequality (11) leads to the global asymptotic stability of the the closed-loop system (5)–(9). ∎

IV. Numerical Results

Performance issues and robustness properties of the proposed impulsive controller are additionally tested in numerical experiments. In the simulations, performed with MATLAB, the dimensionless spring-mass model (1) is studied with the parameters $m = 1$, $k = 1$, and $\alpha = 1$. The initial position and the initial velocity are set to $x_1(0) = 3.5$ and $x_2(0) = -4$, respectively.

The impulsive controller (5), (8), (9) is first applied to the disturbance-free system. In order to test the controller robustness a harmonic external disturbance $w = 0.7 \sin t$ is then applied to the closed-loop system. Good performance and desired robustness properties of the controller are concluded from Figures 5 and 6.

Alternative application of impulsive control is in the throttle system (Figure 4) which is governed by the differential equation (see [3])

$$m\ddot{x} = -k(x - x_0) - \alpha\,\mathrm{sign}(\dot{x}) + u + w \qquad (30)$$

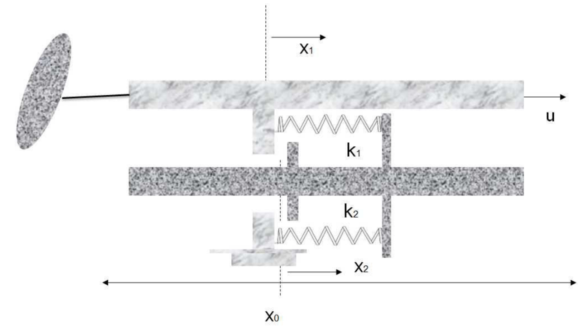

Fig. 4. Throttle mechanism (translational linear equivalent).

where

$$k = \begin{cases} k_1, & \text{if } x \le x_0 \\ k_2, & \text{otherwise.} \end{cases} \qquad (31)$$

In the simulations, the parameters $x_0 = 0$, $m = 1$, $k_1 = 1$, $k_2 = 3$, and $\alpha = 1$ were selected. The initial position and the initial velocity are set to $x_1(0) = -2$ and $x_2(0) = -2$. Simulation results are provided in Figures 7 and 8 for the unperturbed and perturbed cases, respectively.

V. Conclusions

A novel approach, counteracting friction affects, is developed for and tested on a spring-mass system. The approach is based on an impulsive actuation while the underlying system is getting stuck due to the presence of dry friction. Global asymptotic stability of the closed-loop system and its robustness against external disturbances of sufficiently small magnitude are carried out in the theoretical study. The effectiveness of the proposed approach is illustrated in numerical simulations. The approach is expected to enhance control of mechanical manipulators with relatively strong Coulomb friction forces.

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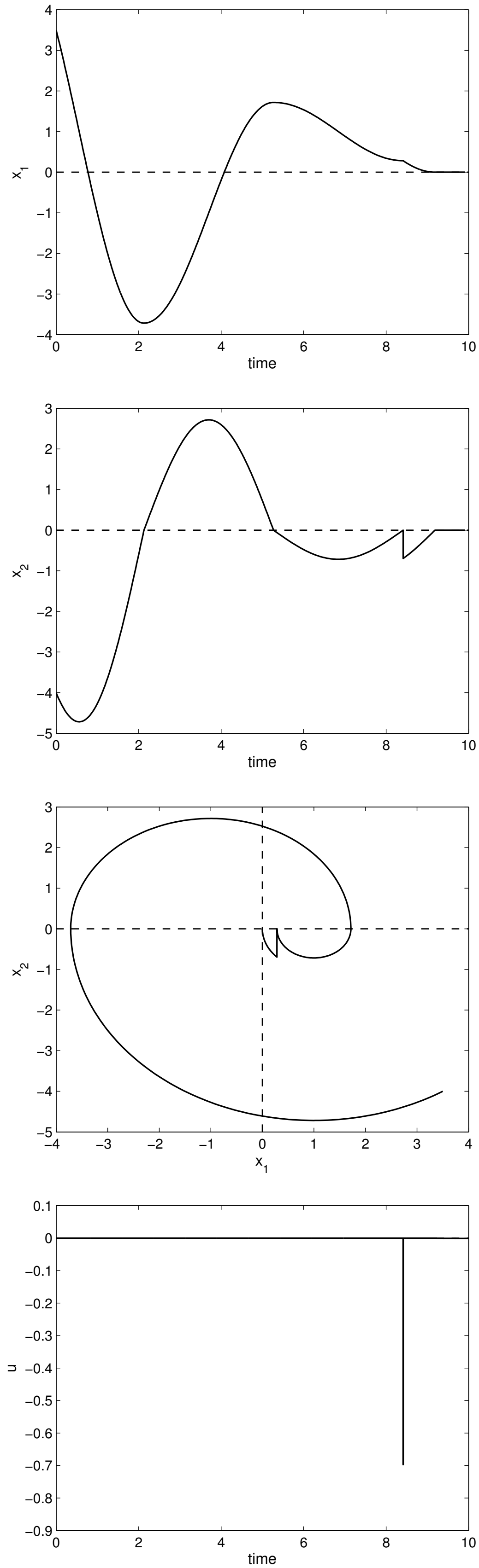

Fig. 5. Impulsive stabilization of the spring-mass system with Coulomb friction: the no disturbance case.

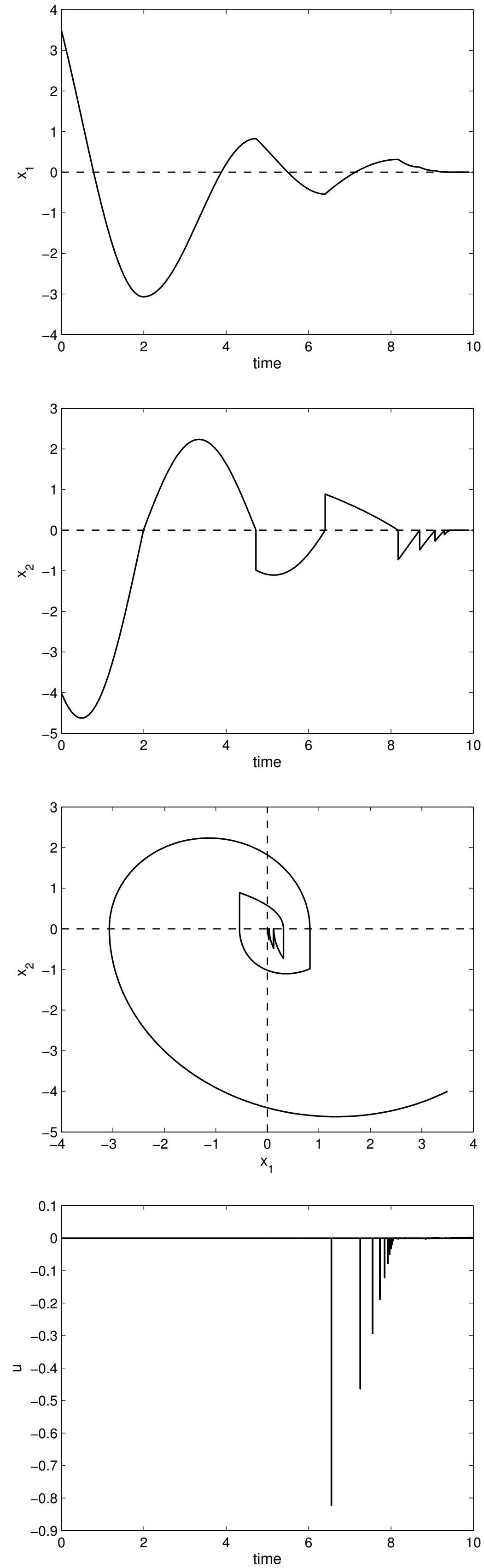

Fig. 6. Impulsive stabilization of the spring-mass system with Coulomb friction: the disturbance case.

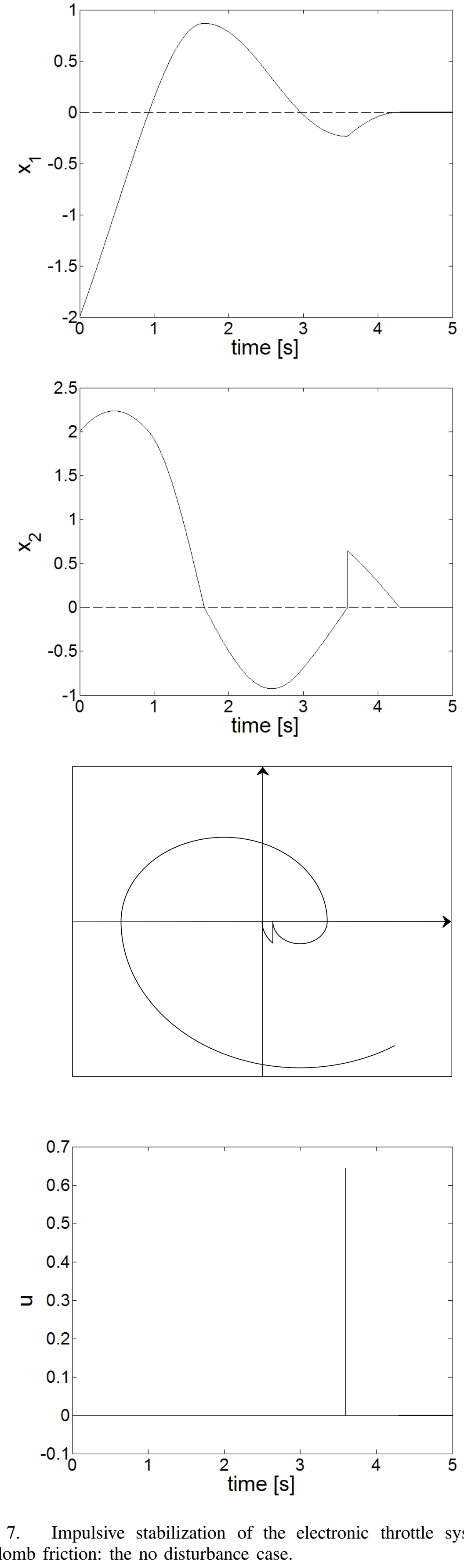

Fig. 7. Impulsive stabilization of the electronic throttle system with Coulomb friction: the no disturbance case.

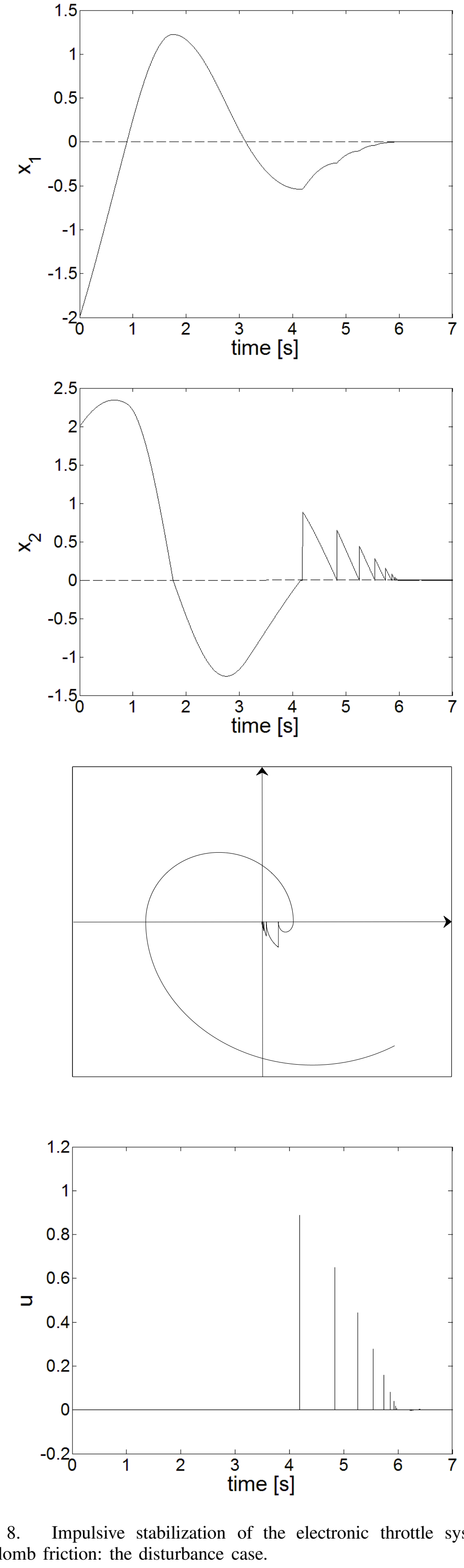

Fig. 8. Impulsive stabilization of the electronic throttle system with Coulomb friction: the disturbance case.