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Exponential Stability and Static Output Feedback Stabilization of Singular Time-Delay Systems with Saturating Actuators

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Abstract— **This paper deals with the class of continuous-time singular linear systems with time-varying delays. The stability and stabilization problems of this class of systems are addressed. Delay-range-dependent sufficient conditions such that the system is regular, impulse free and α-stable are developed in the linear matrix inequality (LMI) setting and an estimate of the convergence rate of such stable systems is also presented. An iterative LMI (ILMI) algorithm to compute a static output feedback controller gains is proposed. A numerical example is employed to show the usefulness of the proposed results.**

Keywords: Singular time-delay systems, delay-dependent, stability, linear matrix inequality, stabilization.

I. INTRODUCTION

Singular time-delay systems arise in a variety of practical systems such as networks, circuits, power systems and so on [2]. Since singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study of such systems is much more complicated compared to standard state-space time-delay systems or singular systems. The existence and uniqueness of a solution to a given singular time-delay system is not always guaranteed and the system can also have undesired impulsive behavior.

Both delay-independent and delay-dependent stability conditions for singular time-delay systems have been derived using the time domain method, see [1], [2], [3], [16], [17], [18] and references therein. However, most of the delay-dependent results in the literature tackle only the case of constant time delay where two approaches were used to prove the stability of the system. The first approach consists of decomposing the system into fast and slow subsystems and the stability of the slow subsystem is proved using some Lyapunov functional. Then, the fast variables is expressed explicitly by an iterative equation in terms of the slow variables [1]. The second approach introduced by [2] and it consistes of constructing a Lyapunov-Krasovskii functional that corresponds directly to the descriptor form of the system. The extension of these approaches to time-varying delays has not been addressed yet. In [18], where time-varying delays are considered, the response of the fast variables has been bounded by an exponential term using a different approach. Using this approach, it is not possible to give an estimate of the convergence rate of the states of the system.

Recently, a free-weighting matrices method is proposed in [6] and [7] to study the delay-dependent stability for time-delay systems. The new method has been shown to be more effective in reducing conservatism entailed in previous results. In 2007, Zhu et al. adopted this technique for singular time-delay systems [3]. Also, delay-range-dependent concept was recently studied, where the delays are considered to vary in a range and thereby more applicable in practice [8].

Formally speaking, these conditions provide only the asymptotic stability of singular time-delay systems. In [13], the global exponential stability for a class of singular systems with multiple constant time delays is investigated and an estimate of the convergence rate of such systems is presented. One may ask if there exists a possibility to use the LMI approach for deriving exponential estimates for solutions of singular time-delay systems. In [18], exponential stability conditions in terms of LMIs are given but no estimate of the convergence rate is presented.

The problem of stabilizing linear systems with saturating controls has been widely studied because of its practical interest. However, only few works have dealt with stability analysis and the stabilization of singular linear systems in the presence of actuator saturation, see for example [10]. It is established in [10] that a singular linear system with actuator saturation is semi-globally asymptotically stabilizable by linear state feedback if its reduced system under actuator saturation is semi-globally asymptotically stabilizable by linear feedback. To the best of the authors' knowledge, the stabilization for singular time-delay systems in the presence of actuator saturation has not been fully addressed yet.

The static output feedback problem is probably the most important open question in control engineering. In contrast to the linear systems, there are only few papers solving the static output feedback problem for singular systems, see [9], [12]. In [12], the authors introduce an equality constraint in order to get an LMI sufficient conditions for admissibility of closed-loop systems. However, this equality constraint introduces conservatism. This approach has been generalized by [4] to singular time-delay systems. In [9], singular systems are assumed to have some characteristics in advance: regularity and absence of direct action of control inputs on the fast variables, which is not always the case.

This paper addresses two important problems that have not been fully investigated. First, delay-range-dependent exponential stability conditions for singular time-delay systems are established in terms of LMIs and an estimate of the

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convergence rate of the state is presented. The Lyapunov functional and some inequalities from [8] are adopted, with some modifications, in order to prove the exponential stability of the slow subsystem. The fast variables are expressed explicitly by an iterative method which can be seen as a generalization of the iterative expression in [1] for constant time delay. This means that many of the existing results for singular systems with constant time delay can be extended easily to the systems with time-varying delay. For instance, the results in [1], [3], [17].

Second, an iterative LMI algorithm is proposed to design a stabilizing static output feedback controller in the presence of actuator saturation. The objective of the control design is twofold. It consists in determining both a static output feedback control law to guarantee that the system is regular, impulse-free and exponentially stable with a predefined decaying rate for the closed-loop system, and a set of safe initial conditions for which the exponential stability of the saturated closed-loop system is guaranteed.

Notation: $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote, respectively, the maximal and minimal eigenvalue of matrix P. co $\{\cdot\}$ denotes a convex hull. $C_\tau = C([-\tau,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to the Euclidean vector norm whereas $\|\phi\|_c = sup_{-\tau\leq t\leq 0}\|\phi(t)\|$ stands for the norm of a function $\phi \in C_\tau$. C_τ^v is defined by $C_\tau^v = \{\phi \in C_\tau; \|\phi\|_c < v, v > 0\}$.

II. PROBLEM STATEMENT AND DEFINITIONS

Consider the following linear singular time-delay system:

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + B\mathrm{sat}(u(t)) \quad (1a)$$
$$y(t) = Cx(t) \quad (1b)$$
$$x(t) = \phi(t), \quad t \in [-d_2, 0] \quad (1c)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the saturating control input, $y(t) \in \mathbb{R}^q$ is the measurement, the matrix $E \in \mathbb{R}^{n\times n}$ may be singular, and we assume that $\mathrm{rank}(E) = r \leq n$, A, A_d, B and C are known real constant matrices, $\mathrm{sat}(u(t)) = [\mathrm{sat}(u_1(t)),...,\mathrm{sat}(u_m(t))]$, where $-\bar{u}_i \leq \mathrm{sat}(u_i(t)) \leq \bar{u}_i$, $\phi(t) \in C_\tau^v$ is a compatible vector valued continuous function and $d(t)$ is a time-varying continuous function that satisfies:

$$0 < d_1 \leq d(t) \leq d_2 \qquad \text{and} \qquad \dot{d}(t) \leq \mu < 1 \quad (2)$$

The following definitions will be used in this paper:

Definition 2.1:

i. System (1) with $u(t) = 0$ is said to be regular if the polynomial, $\det(sE - A)$ is not identically zero.
ii. System (1) with $u(t) = 0$ is said to be impulse-free if $\deg(\det(sE - A)) = \mathrm{rank}(E)$.
iii. System (1) with $u(t) = 0$ is said to be *exponentially stable* if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial conditions $\phi(t)$, the solution $x(t)$ to the singular time-delay system satisfies
$$\|x(t)\| \leq \gamma e^{-\sigma t}\|\phi\|_c$$
iv. System (1) is said to be exponentially admissible if it is regular, impulse-free and exponentially stable.

Lemma 2.1 ([5]): If system (1) with $u(t) = 0$ is regular and impulse free, then its solution exists and is impulse free and unique on $[0,\infty)$.

Lemma 2.2: Given a matrix D, let a positive-definite matrix S and a positive scalar $\eta \in (0,1)$ exist such that

$$D^\top SD - \eta^2 S < 0$$

then, the matrix D satisfies the bound

$$\left\|D^i\right\| \leq \chi e^{-\lambda i} \text{ with } \chi = \sqrt{\frac{\lambda_{max}(S)}{\lambda_{min}(S)}} \text{ and } \lambda = -\ln(\eta)$$

Consider the following static output feedback controller:

$$u(t) = Ky(t), \; K \in \mathbb{R}^{m\times q} \quad (3)$$

Applying this controller to system (1), we obtain the closed-loop system as follows:

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + B\mathrm{sat}(KCx(t)) \quad (4)$$

Generally, for a given stabilizing static output feedback K, it is not possible to determine exactly the region of attraction of the origin with respect to system (4). Hence, a domain of initial conditions, for which the exponential stability of system (4) is ensured, has to be determined.

III. MAIN RESULTS

The two problems to be tackled in this section can be summarized as follows:

- Find delay-range-dependent LMI conditions that guarantees the exponential admissibility of system (1) with $u(t) = 0$, with a predefined minimum decaying rate.
- Find a static output feedback law of the form (3) and a set of initial conditions such that the closed-loop system (4) is exponentially admissible with a predefined minimum decaying rate.

Now, we present the first result.

Theorem 3.1: Let $0 < d_1 < d_2$, $\mu < 1$ and $\alpha > 0$ be given scalars. System (1) with $u(t) = 0$ is exponentially admissible with $\sigma = \alpha$ if there exist a nonsingular matrix P, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 > 0$, $Z_2 > 0$, and matrices M_i, N_i and S_i, $i = 1,2$ such that the following LMI holds:

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & e^{\alpha d_1}M_1E & -e^{\alpha d_2}S_1E & c_2N_1 & cS_1 & cM_1 & \Pi_{18} \\ \star & \Pi_{22} & e^{\alpha d_1}M_2E & -e^{\alpha d_2}S_2E & c_2N_2 & cS_2 & cM_2 & A_d^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -c_2Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix}$$
$$< 0 \quad (5)$$

with the following constraint:

$$E^\top P = P^\top E \geq 0 \quad (6)$$

where

$$\Pi_{11} = P^\top A + A^\top P + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top + 2\alpha E^\top P$$
$$\Pi_{12} = P^\top A_d + (N_2 E)^\top - N_1 E + S_1 E - M_1 E$$
$$\Pi_{22} = -(1-\mu)e^{-2\alpha d_2} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top - M_2 E - (M_2 E)^\top,\ d_{12} = d_2 - d_1,\ U = d_2 Z_1 + d_{12} Z_2$$
$$\Pi_{18} = A^\top U, \quad c = \frac{e^{2\alpha d_2} - e^{2\alpha d_1}}{2\alpha} \quad c_2 = \frac{e^{2\alpha d_2} - 1}{2\alpha}.$$

Proof: Choose two nonsingular matrices R, L such that $\bar{E} = REL$, $\bar{A} = RAL$ and $\bar{P} = R^{-\top} PL$ are written as

$$\bar{E} = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix},\ \bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ \bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Then, the following relation can be shown: $A_{22}^\top P_{22} + P_{22}^\top A_{22} < 0$, which implies that A_{22} is nonsingular and the system is regular and impulse-free [5]. Now, there exist other two matrices R, L such that (see [5])

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{A} = RAL = \begin{bmatrix} \widehat{A} & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix} \tag{7}$$

Now, let $\bar{A}_d = RA_d L$ and $\bar{Q}_i = L^\top Q_i L$,

$$\bar{A}_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \qquad \bar{Q}_{ki} = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i21} & Q_{i22} \end{bmatrix}$$

Then, the following relations can be shown

$$A_{d22}^\top Q_{322} A_{d22} - e^{-2\alpha d_2} Q_{322} < 0, \quad \rho(e^{\alpha d_2} A_{d22}) < 1 \tag{8}$$

So there exist constants $\beta > 1$ and $\gamma \in (0,1)$ such that

$$\|e^{i\alpha d_2} A_{d22}^i\| \le \beta \gamma^i,\ i = 1, 2, \cdots. \tag{9}$$

Let $\zeta(t) = L^{-1} x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. Then, system (1) becomes equivalent to the following one

$$\dot{\zeta}_1(t) = A_1 \zeta_1(t) + A_{d11} \zeta_1(t - d(t)) + A_{d12} \zeta_2(t - d(t)), \tag{10}$$
$$0 = \zeta_2(t) + A_{d21} \zeta_1(t - d(t)) + A_{d22} \zeta_2(t - d(t)). \tag{11}$$

Now, choose the Lyapunov functional as follows:

$$\begin{aligned} V(t) &= \zeta^\top(t) \bar{E}^\top \bar{P} \zeta(t) + \sum_{i=1}^{2} \int_{t-d_i}^{t} \zeta^\top(s) e^{2\alpha(s-t)} \bar{Q}_i \zeta(s) ds \\ &+ \int_{t-d(t)}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_3 \zeta(s) ds \\ &+ \int_{-d_2}^{0} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_1 \bar{E} \dot{\zeta}(s) ds d\theta \\ &+ \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_2 \bar{E} \dot{\zeta}(s) ds d\theta \end{aligned}$$

Then, the following estimation can be obtained.

$$\|\zeta_1(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c e^{-\alpha t} \tag{12}$$

where λ_1 and λ_2 are positive integers. In order to prove the exponential stability of the fast subsystem, define

$$t_i = t_{i-1} - d(t_{i-1}),\ i = 1, 2, \ldots$$
$$t_0 = t$$

From (11), we get

$$\begin{aligned} \zeta_2(t) &= -A_{d21} \zeta_1(t_1) - A_{d22} \zeta_2(t_1) \\ &= -A_{d21} \zeta_1(t_1) - A_{d22}[-A_{d21} \zeta_1(t_2) - A_{d22} \zeta_2(t_2)] \\ &\text{and so on} \end{aligned}$$

Note that $t_i < t_{i-1}$, therefore, there exists a positive finite integer $k(t)$ such that

$$\zeta_2(t) = (-A_{d22})^{k(t)} \zeta_2(t_{k(t)}) - \sum_{i=0}^{k(t)-1} (-A_{d22})^i A_{d21} \zeta_1(t_{i+1}) \tag{13}$$

and $t_{k(t)} \in (-d_2, 0]$. Therefore, from (8), (9), (12), Lemma 2.2 and noting that

$$k(t) d_2 \ge t, \qquad t_i = t - \sum_{j=0}^{i-1} d(t_j) \ge t - i d_2$$

we get

$$\begin{aligned} \|\zeta_2(t)\| &\le \|A_{d22}{}^{k(t)}\| \|\phi\|_c + \|A_{d21}\| \sum_{i=0}^{k(t)-1} \left\|A_{d22}{}^i\right\| \|\zeta_1(t_{i+1})\| \\ &\le \chi e^{-\alpha d_2 k(t)} \|\phi\|_c \\ &+ \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c \sum_{i=0}^{k(t)-1} \left\|A_{d22}{}^i\right\| e^{-\alpha(t-(i+1)d_2)} \\ &\le \left[\chi + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha d_2} \sum_{i=0}^{k(t)-1} \|A_{d22}\|^i e^{i\alpha d_2}\right] \|\phi\|_c e^{-\alpha t} \\ &\le \left[\chi + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha d_2} M\right] \|\phi\|_c e^{-\alpha t} \end{aligned}$$

where

$$M = \frac{\beta}{1-\gamma} \quad \chi = \sqrt{\frac{\lambda_{max}(Q_{322})}{\lambda_{min}(Q_{322})}},$$

Thus, the singular time-delay system is exponentially stable with a minimum decaying rate $= \alpha$. Therefore, by Definition (2.1), the system is exponentially admissible. ■

Remark 3.1: Eq. (13) can be seen as a generalization of the iterative equation in [1] for systems with constant time delay. Therefore, the results in [1], [3] and [17] can be extended easily to the case of time-varying delay.

Remark 3.2: Strict LMI conditions are more desirable than non-strict ones from numerical point of view. Considering this, Eqs. (5) and (6) can be combined into a single strict LMI. Let $P > 0$ and $S \in R^{n\times(n-r)}$ be any matrix with full column rank and satisfies $E^\top S = 0$. Changing P to $PE + SQ$ in (5) yields the strict LMI.

Now, the stabilization problem of system (1) via static output feedback controller will be addressed. The technique introduced in [14] will be adopted in order to write the saturated nonlinear system (4) in a linear polytopic form. Let us write the saturation term as

$$\text{sat}(KCx(t)) = D(\rho(x)) KCx(t), \quad D(\rho(x)) \in \mathbb{R}^{m\times m}$$

where $D(\rho(x))$ is a diagonal matrix for which the diagonal elements $\rho_i(x)$ are defined for $i = 1,...,m$ as

$$\rho_i(x) = \begin{cases} -\frac{\overline{u}_i}{(KC)_i x} & \text{if } (KC)_i x \leq -\overline{u}_i \\ 1 & \text{if } -\overline{u}_i < (KC)_i x < \overline{u}_i \\ \frac{\overline{u}_i}{(KC)_i x} & \text{if } (KC)_i x \geq \overline{u}_i \end{cases}$$

Using this, system (4) can be written as follows:

$$E\dot{x}(t) = (A + BD(\rho(x))KC)x(t) + A_d x(t - d(t)) \qquad (14)$$

Let $0 < \underline{\rho}_i \leq 1$ be a lower bound to $\rho_i(x)$, and define a vector $\underline{\rho} = [\underline{\rho}_1,...,\underline{\rho}_m]$. The vector $\underline{\rho}$ is associated to the following region in the state space:

$$S(K, \overline{u}^{\rho}) = \{x \in \mathbb{R}^n \mid -\overline{u}^{\rho} \leq KCx \leq \overline{u}^{\rho}\}$$

where every component of the vector $\overline{u}^{\rho}$ is defined by $\overline{u}_i/\underline{\rho}_i$. This vector can be viewed as a specification on the saturation tolerance. Define now matrices A_j, $j = 1,...,2^m$, as follows:

$$A_j = A + BD(\gamma_j)KC$$

where $D(\gamma_j)$ is a diagonal matrix of positive scalars $\gamma_{j(i)}$ for $i = 1,...,m$, which arbitrarily take the value one or $\underline{\rho}_i$. Note that the matrices A_j are the vertices of a convex polytope of matrices. If $x(t) \in S(K, \overline{u}^{\rho})$, it follows that $(A + BD(\rho(x))KC) \in co\{A_1,...,A_{2^m}\}$. We conclude that if $x(t) \in S(K, \overline{u}^{\rho})$, then $E\dot{x}(t)$ can be determined from the following polytopic model:

$$E\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j x(t) + A_d x(t - d(t)) \qquad (15)$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$ and $\lambda_{j,t} \geq 0$. Then, we have the following result.

Theorem 3.2: Let $0 < d_1 < d_2$, $\alpha > 0$, a vector $\underline{\rho}$ and $\mu < 1$ be given. If there exist symmetric and positive-definite matrices P, Q_1, Q_2, Q_3, Z_1 and Z_2, matrices M_i, N_i and S_i, $i = 1,2$, matrices K and Q and a positive scalar κ such that

$$\begin{bmatrix} \Pi_{j11} & \Pi_{12} & e^{\alpha d_1} M_1 E & -e^{\alpha d_2} S_1 E & c_2 N_1 & cS_1 & cM_1 & \Pi_{j18} \\ \star & \Pi_{22} & e^{\alpha d_1} M_2 E & -e^{\alpha d_2} S_2 E & c_2 N_2 & cS_2 & cM_2 & A_d^{\top} U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -c_2 Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1 + Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix}$$

$$< 0, \ j = 1,...,2^m \qquad (16)$$

$$\begin{bmatrix} E^{\top}(PE + SQ) & \underline{\rho}_i (KC)_i^{\top} \\ \underline{\rho}_i (KC)_i & \kappa \overline{u}_i^2 \end{bmatrix} \geq 0, \quad i = 1,...,m \qquad (17)$$

where

$$\begin{aligned} \Pi_{j11} &= (PE + SQ)^{\top} A + A^{\top}(PE + SQ) + \sum_{i=1}^{3} Q_i + N_1 E \\ &\quad + (N_1 E)^{\top} + (PE + SQ)^{\top} BD(\gamma_j)KC \\ &\quad + \Big((PE + SQ)^{\top} BD(\gamma_j)KC\Big)^{\top} + 2\alpha E^{\top}(PE + SQ) \\ \Pi_{12} &= (PE + SQ)^{\top} A_d + (N_2 E)^{\top} - N_1 E + S_1 E - M_1 E \\ \Pi_{22} &= -(1 - \mu)e^{-2\alpha d_2} Q_3 + S_2 E + (S_2 E)^{\top} - N_2 E \\ &\quad -(N_2 E)^{\top} - M_2 E - (M_2 E)^{\top}, \ d_{12} = d_2 - d_1 \\ U &= d_2 Z_1 + d_{12} Z_2, \ \Pi_{j18} = A^{\top} U + (BD(\gamma_j)KC)^{\top} U \\ c &= \frac{e^{2\alpha d_2} - e^{2\alpha d_1}}{2\alpha}, \ c_2 = \frac{e^{2\alpha d_2} - 1}{2\alpha}. \end{aligned}$$

where $S \in \mathbb{R}^{n \times (n-r)}$ is any matrix with full column rank and satisfies $E^{\top} S = 0$, then there exists a static output feedback controller (3) such that the closed-loop system (4) is locally exponentially admissible with $\sigma = \alpha$ for any compatible initial condition satisfying:

$$\Omega(\nu_1, \nu_2) = \{\phi \in C_{d_2}^{v} \quad : \quad \frac{\|\phi\|_c^2}{\nu_1} + \frac{\|\dot{\phi}\|_c^2}{\nu_2} \leq 1\} \qquad (18)$$

where

$$\nu_1 = \frac{\kappa^{-1}}{\chi_1}, \qquad \nu_2 = \frac{\kappa^{-1}}{\chi_2}$$

$$\begin{aligned} \chi_1 &= \lambda_{max}(E^{\top} PE) + \sum_{i=1}^{2} \lambda_{max}(Q_i) \frac{1 - e^{-2\alpha d_i}}{2\alpha} + \lambda_{max}(Q_3) \times \\ &\quad \frac{1 - e^{-2\alpha d_2}}{2\alpha}, \ \chi_2 = \lambda_{max}(Z_1)\lambda_{max}(E^{\top} E) \frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2} \\ &\quad + \lambda_{max}(Z_2)\lambda_{max}(E^{\top} E) \frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2} \end{aligned}$$

Proof: Assume that $x(t) \in S(K, \overline{u}^{\rho})$, $\forall t > 0$ (will be proved later). Therefore, $E\dot{x}(t)$ can be determined from the polytopic system (15). Applying Remark 3.2 to (5)-(6) in Theorem 3.1 yields a single matrix inequality. Then, if we apply this matrix inequality 2^m times to the parameters A_j with $j = 1, \ldots, 2^m$, A_d, E, d_1, d_2 and μ, we will have (16). Then, the following relations can be shown

$$A_{j22}^{\top} P_{22} + P_{22}^{\top} A_{j22} < 0, \ j = 1, \ldots 2^m$$

Using the fact that $\lambda_{j,t} \geq 0$,

$$\lambda_{j,t} A_{j22}^{\top} P_{22} + P_{22}^{\top} \lambda_{j,t} A_{j22} \leq 0, \ j = 1, \ldots 2^m, \ \forall t \in (0, \infty)$$

adding these inequalities together and noting that $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$, gives

$$\left[\sum_{j=1}^{2^m} \lambda_{j,t} A_{j22}\right]^{\top} P_{22} + P_{22}^{\top} \sum_{j=1}^{2^m} \lambda_{j,t} A_{j22} < 0$$

$$\sum_{j=1}^{2^m} \lambda_{j,t} A_{j22} \text{ is nonsingular} \quad \forall t \in (0, \infty)$$

which implies that system (15) is regular and impulse-free. Now, proceeding similar to the proof of Theorem 3.1, the system can be shown to be exponentially stable. Now, by virtue of condition (17), the ellipsoid defined by $\Gamma = \{x \in \mathbb{R}^n : x^{\top} E^{\top}(PE + SQ)\, x \leq \kappa^{-1}\}$ is included in the set $S(K, \overline{u}^{\rho})$ [14]. Suppose now that the initial condition $\phi(t)$ satisfies (18), and conditions (16)-(17) hold. Then, from the definition of $V(t)$, it follows that $x^{\top}(0)E^{\top}(PE + SQ)x(0) \leq V(0) \leq \chi_1 \|\phi\|_c^2 + \chi_2 \|\dot{\phi}\|_c^2 \leq \kappa^{-1}$ and, one has $x(0) \in \Gamma \subset S$. Now, since $\dot{V}(0) < 0$, we conclude that $x^\top(t)E^\top(PE+SQ)x(t) \leq V(t) \leq V(0) \leq \chi_1\|\phi\|_c^2 + \chi_2\|\dot{\phi}\|_c^2 \leq \kappa^{-1}$, which means that $x(t) \in S(K, \overline{u}^\rho)$, $\forall t > 0$. This completes the proof. ■

It is obvious that (16) is a BMI, and consequently its solution is very difficult. Thus, an ILMI approach similar to [15] and [11] will be proposed. This algorithm has the same disadvantages as those in [15] and [11]. The following is the proposed algorithm and the explanation is given later.

- Step 1. OP1.

$$\min_{P_0>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_{p0}>0,M_p,N_p,S_p,p=1,2,\kappa} \beta_0$$
$$s.t. (21)-(22)$$
$$K = 0 \text{ and } X_0 = E.$$

  Set $i = 1$, $X_1 = E$, $Z_{11} = Z_{10}$ and $Z_{21} = Z_{20}$.
- Step 2. OP2.

$$\min_{P_i>0,Q,Q_1>0,Q_2>0,Q_3>0,M_p,N_p,S_p,p=1,2,K,\kappa} \beta_i$$
$$s.t. (21)-(22)$$

  Let β_i^* and K^* be the solution of OP2. If $\beta_i^* \leq -\alpha$, where α is a prescribed decay rate, then K^* is a stabilizing static output feedback gain, go to step 5, otherwise, go to step 3.
- Step 3. OP3.

$$\min_{P_i>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_{pi}>0,M_p,N_p,S_p,p=1,2,\kappa} \mathrm{tr}(E^\top T_i)$$
$$s.t. (21)-(22)$$
$$\beta_i = \beta_i^* \text{ and } K = K^*.$$

  If $\|X_iB - T_i^{*}B\| < \varepsilon$, go to step 4, else set $i = i+1$, $X_i = T_{i-1}^*$, $Z_{1i} = Z_{1(i-1)}^*$ and $Z_{2i} = Z_{2(i-1)}^*$, then go to step 2.
- Step 4. The system may not be stabilizable via static output feedback. Stop.
- Step 5. OP4.

$$\min_{P>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_p>0,M_p,N_p,S_p,p=1,2,K,\kappa} r$$
$$s.t. (21)-(22) \quad \beta_i = \alpha$$
$$\delta_1\mathbb{I} \geq E^\top PE \qquad \delta_2\mathbb{I} \geq Q_1 \qquad \delta_3\mathbb{I} \geq Q_2 \qquad (19)$$
$$\delta_4\mathbb{I} \geq Q_3 \qquad \delta_5\mathbb{I} \geq Z_1 \qquad \delta_6\mathbb{I} \geq Z_2 \qquad (20)$$

  where $r = w_1\left(\delta_1 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_2 + \frac{1-e^{-2\alpha d_2}}{2\alpha}\delta_3 + \frac{1-e^{-2\alpha d_2}}{2\alpha}\delta_4\right)$ $+w_2\left(\lambda_{max}(E^\top E)\frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2}\delta_5 + \lambda_{max}(E^\top E)\times \frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2}\delta_6\right) + w_3\kappa$, and w_1, w_2 and w_3 are weighting factors. We solve this problem iteratively in two steps as follows:
  a) Fix K, and solve for $P > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_p > 0$, M_p, N_p, S_p, $p = 1,2$, and κ.
  b) Fix Z_1 and Z_2, and solve for $P > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, M_p, N_p, S_p, $p = 1,2$, K and κ. Set $X = T$.
  The set (18) is calculated from the matrices that solve OP4.

$$\begin{bmatrix} \Pi_{11} & \Pi_j & \Pi_{12} & c_3M_1E & c_4S_1E & c_2N_1 & cS_1 & cM_1 & \Pi_{j18} \\ \star & -\mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \Pi_{22} & c_3M_2E & c_4S_2E & c_2N_2 & cS_2 & cM_2 & A_d^\top U \\ \star & \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c_2Z_{1i} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -c(Z_{1i}+Z_{2i}) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -cZ_{2i} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix}$$
$$< 0, \; j = 1, ..., 2^m \qquad (21)$$

$$\begin{bmatrix} E^\top T_i & \underline{\rho}_r(KC)_r{}^\top \\ \underline{\rho}_r(KC)_r & \kappa\overline{u}_r{}^2 \end{bmatrix} \geq 0, \quad r = 1, ..., m \qquad (22)$$

where

$$\Pi_{11} = T_i{}^\top A + A^\top T_i + \sum_{i=1}^{3} Q_i + N_1E + (N_1E)^\top - X_iBB^\top T_i - (X_iBB^\top T_i)^\top + X_iBB^\top X_i - 2\beta_iE^\top T_i$$
$$\Pi_{j18} = A^\top U + (BD(\gamma_j)KC)^\top U, \; c_2 = \frac{e^{-2\beta_i d_2} - 1}{-2\beta_i}$$
$$\Pi_{22} = -(1-\mu)e^{2\beta d(\beta)}Q_3 + S_2E + (S_2E)^\top - N_2E - (N_2E)^\top - M_2E - (M_2E)^\top, T_i = (P_iE + SQ)$$
$$c_3 = e^{-\beta_i d_1}, \; c_4 = -e^{-\beta_i d_2}, \; d(\beta) = \begin{cases} d_1 & \text{if } \beta > 0 \\ d_2 & \text{if } \beta < 0 \end{cases}$$
$$c = \frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i}, \quad \Pi_j = (B^\top T_i + D(\gamma_j)KC)^\top$$

and the other variables as defined previously.

Remark 3.3: The core of this algorithm is in OP2 and OP3. As shown in [15], OP2 guarantees the progressive reduction of β_i while OP3 guarantees the convergence of the algorithm. Yet, in [15], only X needs to be fixed in order to get LMIs, while in our case, we have also to fix either Z_1 and Z_2 or K to get LMIs. Thus, we will fix Z_1 and Z_2 in OP2, and K in OP3. This way of solving this problem will not affect the convergence of the algorithm. It is worth noting that although this ILMI algorithm is convergent, we may not achieve the solution because β may not always converge to its minimum. For more details on the numerical properties of the algorithm, we refer the reader to [15].

Remark 3.4: OP4 is used in order to enlarge the set of initial conditions (18). The satisfaction of (19)-(20) means that $\chi_1 \leq \delta_1 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_2 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_3 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_4$ and $\chi_2 \leq \lambda_{max}(E^\top E)\frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2}\delta_5 + \lambda_{max}(E^\top E)\frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2}\delta_6$. Therefore, because $\nu_i = \frac{\kappa^{-1}}{\chi_i}$, if we minimize the criterion as defined in OP4, then greater the bounds on $\|\phi\|_c^2$ and $\|\dot{\phi}\|_c^2$ tend to be. In other words, by using OP4, we orient the solutions of LMIs (16)-(17) in a sense to obtain the set $\Omega(\nu_1, \nu_2)$ as large as possible. For more discussion, we refer the reader to [14].

IV. EXAMPLES

Consider the singular time-delay system described by:

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},A=\begin{bmatrix}0&0&1\\1&0&0\\1&0&-1\end{bmatrix},C=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}^{\top}$$

$$A_d=\begin{bmatrix}0&0&0.3\\0&0.4&0\\0.2&0.3&0\end{bmatrix},\quad B=\begin{bmatrix}1&-2\\0.1&0.3\\0.1&-0.3\end{bmatrix}$$

The exponential stabilizability of this system will be investigated using Theorem 3.2 and the iterative algorithm. Letting $d_1=0.2$, $d_2=0.6$, $\mu=0.5$, $\bar{u}=7$ and $\alpha=0.3$, the ILMI algorithm gives after 14 iterations

$$K=\begin{bmatrix}-1.4186&-1.2682\\1.3943&0.8652\end{bmatrix},v_1=14.8960,v_2=82.6586$$

Figure 1 gives the simulation results for the closed-loop system when $d(t)=|0.4+0.15\sin(3t)|$ and the initial function is $\phi(t)=[5\ \ 12\ \ 9.6]^{\top},t\in[-0.5,0]$. Changing the control amplitude saturation level, Table I presents the functional dependence of v_1 and v_2 on the level of control saturation $\bar{u}$. For various α, the values v_1 and v_2 for which we guarantee the exponential admissibility of the saturated system are listed in Table II.

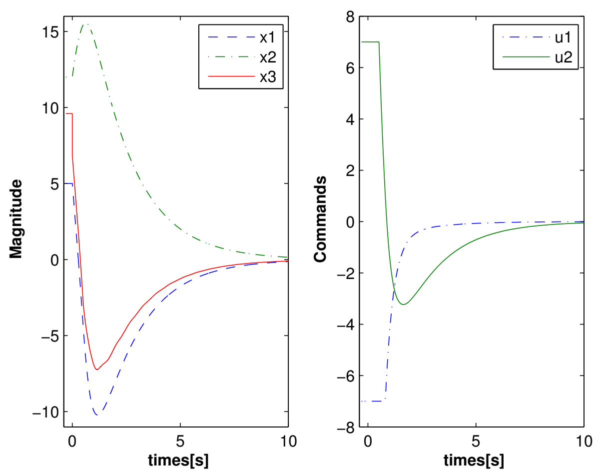

Fig. 1. Simulation results

TABLE I

COMPUTATION RESULTS OF EXAMPLE 2 WITH $\alpha=0.3$

$\bar{u}$	1	3	5	7	9	11	13	15
v_1	0.3	2.8	7.8	14.9	24.8	35.8	49.9	65.5
v_2	1.1	11.6	44.5	82.7	156.9	262	387.1	394.9

V. CONCLUSION

TABLE II

COMPUTATION RESULTS OF EXAMPLE 2 WITH $\bar{u}=15$

α	0.001	0.2	0.4	0.6	0.8
v_1	192.1172	97.0467	48.7601	25.8165	14.0812
v_2	967.1209	509.6311	268.5460	165.2845	90.6967
Iterations	11	13	14	15	16

This paper has dealt with the stability and the stabilization of the class of singular time-delay systems. A delay-range-dependent exponential stability conditions have been developed for singular time-delay systems. Also, a delay-range-dependent static output feedback controller with input saturation has been designed for singular time-delay systems and an ILMI algorithm has been proposed to compute the controller gains. The effectiveness of the results has been illustrated through an example.

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