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Characterization of Backward Reachable Set and Positive Invariant Set in Polytopes

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Abstract— **The paper studies reachability problems of autonomous affine systems in polytopes. Our goal is to find in a given polytope both the largest positive invariant set and backward reachable sets (or attraction domains) of facets. Special attention is paid to the stable invariant affine subspace. After presenting several useful properties of these sets, a partition procedure is given so that the polytope is divided into a positive invariant set and several backward reachable sets.**

I. INTRODUCTION

Recently, a particular class of hybrid systems—*piecewise affine* (or *piecewise linear*) *hybrid systems* (PWAHS), first introduced by Sontag [15], [16] in the 1980's, have been studied quite extensively. A piecewise affine hybrid system consists of a partition of the state space into regions (discrete modes) and a collection of affine dynamics with each valid in a corresponding region. For a piecewise affine hybrid system, as soon as the continuous state reaches the boundary of a region, a discrete event occurs, transferring the system to a new discrete mode. Also, the continuous state is restarted and continues to evolve by the new governing dynamics on the new discrete mode. Since many physical systems can in a first approximation be described by piecewise affine hybrid systems and their computational complexity issues seem relative simple, piecewise affine hybrid systems have gained considerable research attention [2], [3], [4], [12], [11], [13], [6].

We focus on reachability problems of piecewise affine hybrid systems but only on one discrete mode in the paper. That is, we restrict our attention to affine systems defined on n-dimensional polytopes. The reachability problem of affine systems on polytopes is composed of the following two subproblems. One subproblem is to determine attraction domain of each facet of the polytope since leaving through different facets corresponds to different transitions in discrete modes, which may result in a totally different behavior for the whole system. The other subproblem is to find the largest positive invariant set in the polytope since if a trajectory enters a positive invariant set, it will never leave it and no further discrete transition can occur. Therefore, these two subproblems are of great importance for reachability analysis. Our work draws inspiration heavily from [8], in which the reachability problem of affine systems on polytopes in the plane is studied. In the paper, we extend their work to n-dimension. For n-dimensional polytope, we start by showing that the largest positive invariant set lies in the stable eigenspace of the system, which reduce the complexity of determining the largest positive invariant set in the polytope by looking at only a lower dimensional carrying affine space. After introducing exit sets on the facets of the polytope, we prove that both the largest positive invariant set and attraction domain of every exit set are open when considered in their carrying affine subspace. In this way, a procedure is then proposed to partition the polytope and determine the largest positive invariant set in the polytope and the attraction domains of facets. The division is based on numerical computation of dividing hypersurfaces that are convex combinations of trajectories starting from a finite number of points.

In the literature, there are some related work on finding the largest positive invariant set in polytopes. Most of them are based on Nagumo Theorem and Lyapunov level sets, which can only give an approximation of the largest positive invariant set, see e.g. [5] for detailed discussion. However, the result in the paper gives an explicit way to find more accurately the largest positive invariant set in a polytope while the computation complexity is relatively simple and acceptable. It thus also contributes to the field of set invariance study for independent interest.

The rest of the paper is organized as follows. In section II we formally state the problem after introducing some notations and definitions. Characterization properties of positive invariant set and attraction domains and a computational procedure to determine them are presented in section III. In section IV two simple examples are given to illustrate our results. We conclude in section V with final remarks and directions for future work.

II. PRELIMINARIES

In this section, we provide some background materials and then formulate the problem we study.

A. Terminologies and Notations

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Denote $\mathrm{Re}(x)$ the real part of a complex number x.

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Consider a set of m points $V = \{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$. The *linear combination* of all points in V is denoted by

$$\text{span}(V) \triangleq \sum_{i=1}^{m} a_i v_i, \text{ where } a_i \in \mathbb{R}.$$

The *convex hull* of V is the *convex combination* of all points in V, i.e.,

$$\text{cov}(V) \triangleq \sum_{i=1}^{m} \alpha_i v_i, \text{ where } \alpha_i \in [0,1] \text{ and } \sum_{i=1}^{m} \alpha_i = 1.$$

The *affine hull* of V is the *affine combination* of all points in V, i.e.,

$$\text{aff}(V) \triangleq \sum_{i=1}^{m} \alpha_i v_i, \text{ where } \alpha_i \in \mathbb{R} \text{ and } \sum_{i=1}^{m} \alpha_i = 1.$$

Let $\mathcal{S}$ be an m-dimensional set in $\mathbb{R}^n$. we use int$(\mathcal{S})$ and $\partial\mathcal{S}$ to denote the relative interior and relative boundary of $\mathcal{S}$, respectively. Here the relative topology is used. When $m = n$, these notions are in the normal sense. Denote $\bar{\mathcal{S}}$ the closure of $\mathcal{S}$. Finally, we say that a collection of sets $\mathcal{S}_1, \ldots, \mathcal{S}_n$ is a partition of $\mathcal{S}$ if they are disjoint and their union is $\mathcal{S}$.

Let $\mathcal{P}$ be an n-dimensional polytope in $\mathbb{R}^n$. It can be written as the intersection of d half spaces where d is the least number required. That is,

$$\mathcal{P} = \bigcap_{i=1}^{d} \{x \in \mathbb{R}^n | n_i \cdot x \leq \gamma_i\},$$

where n_i is a unit normal vector and γ_i is a constant in $\mathbb{R}$. The set $\{x \in \mathbb{R}^n | n_i \cdot x = \gamma_i\}$ is called its *supporting hyperplanes*. A *facet* of polytope $\mathcal{P}$ is the intersection of $\mathcal{P}$ with one of its supporting hyperplanes, which is of $(n-1)$-dimension. That is,

$$\mathcal{F}_i = \{x \in \mathcal{P} | n_i \cdot x = \gamma_i\}, \ i = 1, \ldots, d.$$

B. Problem Formulation

We now formulate the reachability problem addressed in this paper.

Consider an affine system with its state restricted in an n-dimensional polytope $\mathcal{P}$,

$$\dot{x} = Ax + a, \ x(0) = x_0, \ x \in \mathcal{P}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $a \in \mathbb{R}^n$. That is, the above governing dynamics remains valid as long as the state x lies in $\mathcal{P}$. But as soon as the state reaches the boundary, a discrete event occurs and the governing dynamics might change to another. Because the occurrence of discrete event depends on the facet through which the state leaves the polytope, we want to determine a partition of $\mathcal{P}$ so that we know which set of states reaches which facet and which set of states remains in $\mathcal{P}$ forever.

Let $x(t, x_0)$ be the solution trajectory of (1) starting at x_0. A point $\bar{x}$ satisfying $A\bar{x} + a = 0$ is called an *equilibrium point* of system (1). It is clear that if A is nonsingular, the equilibrium point of the system is unique and is given by $\bar{x} = -A^{-1}a$. Otherwise if A is singular and additionally rank$[A]$ = rank$[A \mid a]$, the equilibrium points form an affine subspace of dimension $n -$ rank$[A]$. In the paper, we assume that the affine system has a unique equilibrium point (namely, A is nonsingular) and the equilibrium point is not on the boundary of $\mathcal{P}$. The assumptions here are to make us focus on the systematic analysis instead of putting too much efforts on some complicated arguments for trivial cases, see [7] for similar assumptions and discussion.

Next we introduce a few definitions and then the problem.

Definition 1: Let $\mathcal{F}_i$ be a facet of $\mathcal{P}$ with its normal vector n_i. We define the *identifier function on* $\mathcal{F}_i$ as

$$g_i(x) \triangleq n_i \cdot (Ax + a), \ x \in \mathcal{F}_i. \tag{2}$$

Remark 1: From the definition, it can be easily verified that for any point x_1 in $\mathcal{F}_i$, if $g_i(x_1) > 0$ then the solution from x_1 leaves the polytope immediately. On the other hand, for any point $x_2 \in \mathcal{F}_j$, if $g_j(x_2) < 0$ then the local backward solution from x_2 (namely, $x(t, x_2), \ t \in (-\epsilon, \ 0)$) can not entirely lie in $\text{int}(\mathcal{P})$. Roughly speaking, it implies that no trajectory inside $\mathcal{P}$ can reach x_2. Taking this fact into account, we know that if a point x_3 lies in an intersection of several facets and if one identifier function is less than zero at this point, then no trajectory inside $\mathcal{P}$ can reach x_3. (See Fig. 1 for example, where $g_j(x_3) < 0$.)

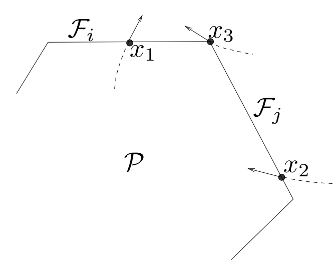

Fig. 1. Illustration for Remark 1. (The real line with arrow represents the vector field at that point.)

Definition 2: The *attraction domain* of a facet $\mathcal{F}_i$ is defined as a set of interior points of $\mathcal{P}$ from which the solution trajectories reach $\mathcal{F}_i$ in the smallest time, and then leave the polytope immediately, i.e.

$$\mathcal{A}(\mathcal{F}_i) \triangleq \{x_0 \in \text{int}(\mathcal{P}) | \exists T > 0 \text{ such that } x(t, x_0) \in \text{int}(\mathcal{P}) \text{ for } t \in [0, T), \ x(T, x_0) \in \mathcal{F}_i, \text{ and } g_i(x(T, x_0)) > 0\}.$$

Definition 3: Define

$$\mathcal{O} \triangleq \{x_0 \in \text{int}(\mathcal{P}) | x(t, x_0) \in \text{int}(\mathcal{P}) \text{ for } t \in [0, \infty)\},$$

the set of all points from which the solutions remain in the interior of the polytope $\mathcal{P}$ forever. It is called *the largest positive invariant set in* $\text{int}(\mathcal{P})$.

Problem 1: Consider affine system (1) on $\mathcal{P}$, the reachability problem is to determine

(1) the corresponding attraction domain of every facet, namely, $\mathcal{A}(\mathcal{F}_i)$ for $i = 1, \ldots, d$;

(2) the largest positive invariant set $\mathcal{O}$ in $\text{int}(\mathcal{P})$.

To solve the problem, we further introduce several notions of exit set and attraction domain of exit set. The definitions are drawn from [8] with some modification.

Definition 4: We say a point $x \in \partial\mathcal{P}$ *satisfies exit condition* if $g_i(x) > 0$ holds for all the facets $\mathcal{F}_i$ that x belongs to.

As an example, in Fig. 1 the point x_1 satisfies exit condition while x_2 and x_3 do not.

Definition 5: A *total exit set* $\mathcal{U}_{tot}$ contains those points in $\partial\mathcal{P}$ that satisfy exit condition.

We divide the total exit set $\mathcal{U}_{tot}$ into a collection of K disjoint sets $\mathcal{U}_1, \ldots, \mathcal{U}_K$ so that each $\mathcal{U}_i$ is connected and we call each $\mathcal{U}_i$ an *exit set*. Notice that every facet $\mathcal{F}_i$ is partitioned into at most two subsets by the identify function on $\mathcal{F}_i$. One of them (which, if exists, is of $n-1$ dimension and convex) belongs to $\mathcal{U}_{tot}$ while the other is not. So each exit set $\mathcal{U}_i$ may consist of just a subset of one facet (see for example Fig. 2) or several subsets from different facets, which are connected through the intersection of facets (see for example Fig. 3).

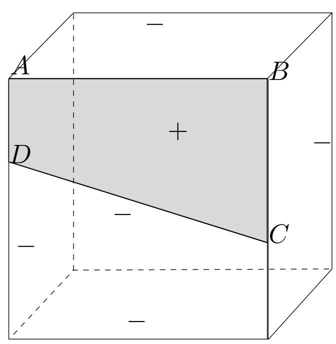

Fig. 2. The sign of identifier function on each facet is marked with $+/-$. In this case, an exit set $\mathcal{U}_i$ is the shaded set excluding the relative boundary $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$.

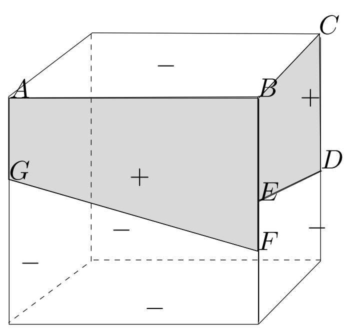

Fig. 3. According to Remark 1, it is clear that the points whose neighborhood (in the topological sense) has both positive and negative sign do not belong to exit sets. So in this case, an exit set $\mathcal{U}_i$ consists of two pieces from two facets, which is the shaded set excluding the points on $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EF}, \overline{FG}, \overline{GA}$.

Following the method adopted in [8], we will start by computing the attraction domain of each exit set $\mathcal{U}_i$ and then the attraction domain of each facet $\mathcal{F}_i$. We define the attraction domain of an exit set in a similar way.

Definition 6: The *attraction domain* of an exit set $\mathcal{U}_i$ is defined as

$$\mathcal{A}_i \triangleq \{x_0 \in \mathrm{int}(\mathcal{P}) | \exists T > 0 \text{ such that } x(t, x_0) \in \mathrm{int}(\mathcal{P}) \text{ for } t \in [0, T), \text{ and } x(T, x_0) \in \mathcal{U}_i\}.$$

Moreover, we define $\mathcal{D}$ the set of points in $\mathrm{int}(\mathcal{P})$, from which the trajectories reach $\partial\mathcal{P}$ in finite time and on that occasion the vector field is tangent to at least one facet of $\mathcal{P}$. In a mathematical form,

$$\mathcal{D} \triangleq \{x_0 \in \mathrm{int}(\mathcal{P}) | \exists T > 0 \text{ such that } x(t, x_0) \in \mathrm{int}(\mathcal{P}) \text{ for } t \in [0, T),\ x(T, x_0) \in \partial\mathcal{P} \text{ and } \exists i : g_i(x(T, x_0)) = 0\}.$$

III. Main Results

In this section, we first present results on characterizing properties of backward reachable set and positive invariant set in $\mathcal{P}$. Next, a computational procedure is given to determine them.

A. Characterization

Firstly, we provide several known results on the reachability problem, namely, a partition of the polytope $\mathcal{P}$ in terms of reachability and a nonexistence condition of positive invariant set.

Lemma 1: [8] For affine system (1) on $\mathcal{P}$, the collection of sets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_K, \mathcal{O}$, and $\mathcal{D}$ is a partition of $\mathrm{int}(\mathcal{P})$.

The lemma can be deduced from the definitions directly.

Lemma 2: [14] For affine system (1) on $\mathcal{P}$, if the equilibrium point $\bar{x} \notin \mathcal{P}$, then for each $x_0 \in \mathcal{P}$, the trajectory starting at x_0 leaves $\mathcal{P}$ in finite time.

The above lemma means that there is no positive invariant set in $\mathcal{P}$ when the equilibrium point is not in it.

Secondly, we present some properties of the largest positive invariant set when it exists, i.e., the equilibrium point $\bar{x}$ is inside $\mathcal{P}$. It will be shown that the largest positive invariant set lies in the stable eigenspace.

Let $\lambda_1, \ldots, \lambda_m$ be m $(m \leq n)$ distinct eigenvalues of A. Let μ_i and ν_i be the algebraic and geometric multiplicity of λ_i, respectively. Hence, $\sum_{i=1}^{m} \mu_i = n$. First, if $\mathrm{Re}(\lambda_i) < 0$, then let $\mathcal{V}_i$ be the set of eigenvectors (including generalized eigenvectors) of λ_i. Therefore, $\mathcal{V}_i$ has μ_i vectors if λ_i is real and it has $2\mu_i$ vectors if λ_i is complex. Second, if $\mathrm{Re}(\lambda_i) = 0$, then let $\mathcal{V}_i$ be the set of eigenvectors (excluding the generalized eigenvectors) of λ_i. Hence, $\mathcal{V}_i$ has ν_i vectors for real eigenvalue and $2\nu_i$ for complex eigenvalue. Third, if $\mathrm{Re}(\lambda_i) > 0$, let $\mathcal{V}_i = \emptyset$. It should be pointed out that $\mathcal{V}_i = \mathcal{V}_{i+1}$ if λ_i and λ_{i+1} are two conjugate complex eigenvalues.

Now we define

$$\mathcal{V}_{\mathcal{Q}} = \bigcup_{i=1}^{m} \mathcal{V}_i, \tag{3}$$

which forms the bases of the stable eigenspace. In addition, we let

$$\mathcal{Q} = (\bar{x} + \mathrm{span}(\mathcal{V}_{\mathcal{Q}})) \cap \mathrm{int}(\mathcal{P}). \tag{4}$$

Then we have the following result.

Theorem 1: For affine system (1) on $\mathcal{P}$, if $\bar{x} \in \mathrm{int}(\mathcal{P})$, then

(a) $\mathcal{O} \subseteq \mathcal{Q}$;
(b) $\text{aff}(\mathcal{O}) = \text{aff}(\mathcal{Q})$;
(c) $\mathcal{O}$ is convex.

Proof: (a) Let $z = x - \bar{x}$, then $\dot{z} = Az$. It can be checked that $\text{span}(\mathcal{V}_{\mathcal{Q}})$ is the largest Lyapunov stable subspace ([9][1]). Therefore, if $z(0) \notin \text{span}(\mathcal{V}_{\mathcal{Q}})$ then $z(t)$ goes to infinity. That is equivalent to say, when $x(0) \notin \text{aff}(\mathcal{Q})$, the trajectory $x(t)$ goes to infinity and of course leaves the polytope $\mathcal{P}$. Then from the definition of the positive invariant set, it follows that $\mathcal{O} \subset \text{aff}(\mathcal{Q})$. Moreover, since $\mathcal{O} \subseteq \text{int}(\mathcal{P})$, we get $\mathcal{O} \subseteq \mathcal{Q}$.

(b) On the one hand, we have $\text{aff}(\mathcal{O}) \subseteq \text{aff}(\mathcal{Q})$ from (a). On the other hand, as $\bar{x}$ is in the interior of $\mathcal{P}$, we can select an $\epsilon > 0$ so that the ϵ-ball centered at $\bar{x}$, $\mathcal{B}(\bar{x}, \epsilon)$, is entirely in the interior of $\mathcal{P}$, too. Recall that $\text{aff}(\mathcal{Q})$ is a Lyapunov stable affine subspace, so there exists a $\delta > 0$ such that the trajectories starting from any point in $\mathcal{B}(\bar{x}, \delta) \cap \text{aff}(\mathcal{Q})$ remain in $\mathcal{B}(\bar{x}, \epsilon) \cap \text{aff}(\mathcal{Q})$ and therefore in $\text{int}(\mathcal{P})$. Hence, $\mathcal{B}(\bar{x}, \delta) \cap \text{aff}(\mathcal{Q}) \subseteq \text{aff}(\mathcal{O})$. Furthermore, since $\mathcal{B}(\bar{x}, \delta)$ is full-dimensional, it follows that $\text{aff}(\mathcal{O}) \supseteq \text{aff}(\mathcal{Q})$.

(c) Consider any two points $x_1, x_2 \in \mathcal{O}$. Then we know $x(t, x_1)$ and $x(t, x_2)$, $t \geq 0$, are entirely in $\text{int}(\mathcal{P})$. Let x_3 be any convex combination of x_1 and x_2, i.e.

$$x_3 = \alpha x_1 + (1 - \alpha) x_2, \alpha \in [0, 1].$$

It can be easily deduced that the trajectory

$$x(t, x_3) = \alpha x(t, x_1) + (1 - \alpha) x(t, x_2).$$

Combining the fact that $\mathcal{P}$ is convex and the fact that $x(t, x_1)$, $x(t, x_2)$ are in $\text{int}(\mathcal{P})$, it follows that $x(t, x_3)$ is also in $\text{int}(\mathcal{P})$ for all $t \geq 0$, which means by definition that $x_3 \in \mathcal{O}$. So $\mathcal{O}$ is convex. ■

Remark 2: Generally, $\mathcal{O} \neq \mathcal{Q}$. An example showing that $\mathcal{O}$ is a strict subset of $\mathcal{Q}$ is given in Section IV. Some examples showing $\mathcal{O} = \mathcal{Q}$ can be found in [8]. In the trivial case, $\mathcal{O} = \mathcal{Q} = \{\bar{x}\}$ when $\mathcal{V}_{\mathcal{Q}}$ is empty.

Next, some properties of attraction domains are investigated.

Theorem 2: Consider affine system (1) on $\mathcal{P}$ with exit sets $\mathcal{U}_1, \ldots, \mathcal{U}_K$. Then for $i = 1, \ldots, K$,

(a) $\mathcal{A}_i$ is open;
(b) $\text{aff}(\mathcal{A}_i) = \mathbb{R}^n$;
(c) $\mathcal{A}_i$ is connected.

Proof: (a) Let $x_0 \in \mathcal{A}_i$. By the definition of $\mathcal{A}_i$, there exists $T > 0$ such that $x(t, x_0) \in \text{int}(\mathcal{P})$ for $t \in [0, T)$ and $x(T, x_0) \in \mathcal{U}_i$. Since $\mathcal{U}_i$ is relatively open in the topological sense from its definition and solutions of the system depend continuously on the initial values [10], there exists a neighborhood of x_0 such that all solution trajectories with initial states in the neighborhood leave the polytope $\mathcal{P}$ in finite time through the exit set $\mathcal{U}_i$. Hence the neighborhood of x_0 is also contained in $\mathcal{A}_i$, which means that $\mathcal{A}_i$ is open.

(b) Since the neighborhood of x_0 is of full dimension and is contained in $\mathcal{A}_i$, we obtain $\text{aff}(\mathcal{A}_i) = \mathbb{R}^n$.

(c) First, consider the case that $\mathcal{U}_i$ lies just in one facet, say $\mathcal{F}_j$, (see Fig. 2 for an example). For this case, suppose by contradiction that $\mathcal{A}_i$ is not connected. Then it can be decomposed into a collection of subsets $\mathcal{A}_i^1, \mathcal{A}_i^2, \ldots$ such that each subset $\mathcal{A}_i^j$ is a connected set but no pair is connected. Now select any two points x_1, x_2 in $\mathcal{A}_i^1$ and $\mathcal{A}_i^2$, respectively. Then by the definition of attraction domain, there exist $T_1, T_2 > 0$ such that $x(T_1, x_1) \in \mathcal{U}_i$, $x(T_2, x_2) \in \mathcal{U}_i$, $x(t, x_1) \in \mathcal{A}_i$ for all $t \in [0, T_1)$, and $x(t, x_2) \in \mathcal{A}_i$ for all $t \in [0, T_2)$. Moreover, since $\mathcal{A}_i^1$ is a connected set and $x_1 \in \mathcal{A}_i^1$, we obtain that $x(t, x_1) \in \mathcal{A}_i^1$ for all $t \in [0, T_1)$. For the same reason, we get $x(t, x_2) \in \mathcal{A}_i^2$ for all $t \in [0, T_2)$. Hence, we can select two points, say x_1' and x_2', on the trajectories $x(t, x_1), t \in [0, T_1)$ and $x(t, x_2), t \in [0, T_2)$, respectively, such that the trajectories starting from x_1' and x_2' reach $\mathcal{U}_i$ at the same time instant T. That is, $x(T, x_1') \in \mathcal{U}_i$ and $x(T, x_2') \in \mathcal{U}_i$. On the other hand, since no pair from the collection of sets $\mathcal{A}_i^1, \mathcal{A}_i^2, \ldots$ is connected, it follows that there must be a point $x_3' = \alpha x_1' + (1 - \alpha) x_2'$ for some $\alpha \in (0, 1)$ such that $x_3' \notin \mathcal{A}_i$. Hence, one obtains that $x(T, x_3')$ cannot be in $\mathcal{U}_i$ by the definition of $\mathcal{A}_i$. However, by the convex argument and the fact that $\mathcal{U}_i$ is convex, $x(T, x_3') = \alpha x(T, x_1') + (1 - \alpha) x(T, x_2')$ is in $\mathcal{U}_i$, a contradiction.

Second, consider the case that $\mathcal{U}_i$ lies in several facets, say $\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_l}$, (see Fig. 3 for an example). For this case, it is clear that $\mathcal{A}_i$ can be written as

$$\mathcal{A}_i = \mathcal{A}\left(\mathcal{F}_{i_1}\right) \cup \ldots \cup \mathcal{A}\left(\mathcal{F}_{i_l}\right).$$

By the same argument as above, it can be shown that each $\mathcal{A}\left(\mathcal{F}_{i_j}\right)$ is a connected set. Notice that facets $\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_l}$ are connected through intersection points. Say for example, the facets $\mathcal{F}_{i_1}$ and $\mathcal{F}_{i_2}$ share common intersection points. Thus $\mathcal{A}\left(\mathcal{F}_{i_1}\right)$ and $\mathcal{A}\left(\mathcal{F}_{i_2}\right)$ must have common points that can reach the intersection of $\mathcal{F}_{i_1}$ and $\mathcal{F}_{i_2}$. Hence, the sets $\mathcal{A}\left(\mathcal{F}_{i_1}\right)$ and $\mathcal{A}\left(\mathcal{F}_{i_2}\right)$ are connected. Repeating the argument, it then follows that $\mathcal{A}\left(\mathcal{F}_{i_1}\right), \ldots, \mathcal{A}\left(\mathcal{F}_{i_l}\right)$ are connected. That is, $\mathcal{A}_i$ is a connected set. ■

Remark 3: The attraction domain is connected as we showed, but in most cases it is not convex, see Example 2.

Finally, since the largest positive invariant set $\mathcal{O}$ lies entirely in the set $\mathcal{Q}$ as we proved in Theorem 1, we are going to partition the set $\mathcal{Q}$ and investigate the properties of the partition in order to get $\mathcal{O}$.

Let

$$\mathcal{D}^{\mathcal{Q}} \triangleq \mathcal{D} \cap \mathcal{Q} \text{ and } \mathcal{A}_i^{\mathcal{Q}} \triangleq \mathcal{A}_i \cap \mathcal{Q} \; (i = 1, \ldots, K).$$

Then we have the following result.

Lemma 3: For affine system (1) on $\mathcal{P}$, the collection of sets $\mathcal{A}_1^{\mathcal{Q}}, \mathcal{A}_2^{\mathcal{Q}}, \ldots, \mathcal{A}_K^{\mathcal{Q}}, \mathcal{O}$ and $\mathcal{D}^{\mathcal{Q}}$ is a partition of $\mathcal{Q}$.

The lemma can be deduced directly from Lemma 1 and the fact that $\text{aff}(\mathcal{Q})$ is invariant.

Theorem 3: Consider affine system (1) on $\mathcal{P}$. Suppose $\mathcal{Q}$ is not empty. Then

(a) $\text{aff}(\mathcal{A}_i^{\mathcal{Q}}) = \text{aff}(\mathcal{Q})$ and $\mathcal{A}_i^{\mathcal{Q}}$ is open in $\text{aff}(\mathcal{Q})$;
(b) if in addition $\bar{x} \in \text{int}(\mathcal{P})$, then $\mathcal{O}$ is open in $\text{aff}(\mathcal{Q})$.

The proof is similar to the one for Theorem 2, so it is omitted.

Now we are able to extend a result from 2-dimension (in [8]) to higher dimension, which is presented as follows.

Theorem 4: For affine system (1) on $\mathcal{P}$, the following holds:

(a) $v \in \partial\mathcal{A}_i^{\mathcal{Q}} \cap \mathrm{int}(\mathcal{P})$ implies that $v \in \mathcal{D}^{\mathcal{Q}}$,

(b) $v \in \partial\mathcal{O} \cap \mathrm{int}(\mathcal{P})$ implies that $v \in \mathcal{D}^{\mathcal{Q}}$.

Proof: By Lemma 3, the collection of sets $\mathcal{O}$, $\mathcal{A}_i^{\mathcal{Q}}$, $\mathcal{D}^{\mathcal{Q}}$ is a partition of $\mathcal{Q}$. Moreover, both $\mathcal{O}$ and $\mathcal{A}_i^{\mathcal{Q}}$ are connected and open in aff($\mathcal{Q}$). Hence, all boundaries among $\mathcal{A}_1^{\mathcal{Q}}, \ldots, \mathcal{A}_K^{\mathcal{Q}}$ and $\mathcal{O}$ consist of trajectories belonging to the set $\mathcal{D}^{\mathcal{Q}}$. ■

The above theorem states that $\mathcal{D}^{\mathcal{Q}}$ is the hypersurface dividing $\mathcal{A}_1^{\mathcal{Q}}, \ldots, \mathcal{A}_K^{\mathcal{Q}}$, and $\mathcal{O}$. Therefore, in order to derive the explicit description of the largest positive invariant set $\mathcal{O}$, it is important to obtain $\mathcal{D}^{\mathcal{Q}}$. In the next subsection, we will give a result for the computation of $\mathcal{D}$.

B. Computation

Let

$$\mathcal{C}_i \triangleq \{x \in \mathcal{F}_i | g_i(x) = 0\}, \tag{5}$$

the set of points in the facet $\mathcal{F}_i$ with vector fields tangent to the facet. Clearly, the set $\mathcal{C}_i$ is a (lower dimension) polytope, so we denote $\mathrm{vert}(\mathcal{C}_i)$ the set of vertices of $\mathcal{C}_i$. In addition, if $W = \{v_1, \ldots, v_l\}$ is a collection of finite numbers of points, we denote $x(t, W)$ the collection of trajectories starting from $v_1, \ldots, v_l$, i.e.

$$x(t, W) \triangleq \{x(t, v) | v \in W\}. \tag{6}$$

Finally, the notation $\mathrm{cov}\,(x(t, W))$ is used to represent the convex combination of the points $x(t, v_1), \ldots, x(t, v_l)$ at time instant t, which should be pointed out that it is not a convex combination of these trajectories.

The following theorem gives a computation for $\mathcal{D}$.

Theorem 5: For affine system (1) on $\mathcal{P}$, the set

$$\mathcal{D} = \bigcup_{i=1}^{d} \mathcal{D}_i, \tag{7}$$

where

$$\mathcal{D}_i = \left(\bigcup_{t \in (-\infty, 0)} \mathrm{cov}\,(x\,(t, \mathrm{vert}(\mathcal{C}_i))) \right) \bigcap \mathrm{int}(\mathcal{P}). \tag{8}$$

Proof: ($\Leftarrow$) Let $x_0 \in \mathcal{D}_i$ for some i. According to (8) we know $x_0 \in \mathrm{cov}\,(x\,(-T, \mathrm{vert}(\mathcal{C}_i)))$ for some $T > 0$, which is equivalent to say that $x(T, x_0) \in \mathcal{C}_i$ by convexity argument. Thus, it follows from the definition of $\mathcal{D}$ that $x_0 \in \mathcal{D}$.

($\Rightarrow$) Let $x_0 \in \mathcal{D}$. Then by the definition of $\mathcal{D}$, there exists $T > 0$ such that $x_1 \triangleq x(T, x_0) \in \mathcal{C}_i$ for some i. Notice that x_1 can be written as a convex combination of points in $\mathrm{vert}(\mathcal{C}_i)$ as $\mathcal{C}_i$ is convex. So $x_0 \in \mathrm{cov}\,(x\,(-T, \mathrm{vert}(\mathcal{C}_i)))$ and thus $x_0 \in \mathcal{D}_i$. ■

Finally, a procedure is given to determine the largest positive invariant set $\mathcal{O}$ and attraction domains $\mathcal{A}(\mathcal{F}_i)$.

Procedure 1:

1) Compute $\mathcal{D}$ (Theorem 5).
2) Compute the equilibrium point $\bar{x}$.
   - If $\bar{x} \notin \mathcal{P}$, then $\mathcal{O} = \emptyset$ (Lemma 2).
   - If $\bar{x} \in \mathrm{int}(\mathcal{P})$ and $\mathcal{V}_{\mathcal{Q}} = \emptyset$, then $\mathcal{O} = \{\bar{x}\}$ (Remark 2).
   - If $\bar{x} \in \mathrm{int}(\mathcal{P})$ and $\mathcal{V}_{\mathcal{Q}} \neq \emptyset$, then compute $\mathcal{Q}$ and the partition of $\mathcal{Q}$ (Lemma 3). The set that contains $\bar{x}$ is $\mathcal{O}$.
3) Compute $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_K$, and the partition of $\mathcal{P}$ (Lemma 1). The set that is connected to $\mathcal{U}_i$ is the attraction domain $\mathcal{A}_i$.
   - If $\mathcal{U}_i$ lies just in one facet, say $\mathcal{F}_j$, then $\mathcal{A}(\mathcal{F}_j) = \mathcal{A}_i$.
   - If $\mathcal{U}_i$ lies in more than one facet, say $\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_l}$, then for every adjacent facets $\mathcal{F}_{i_j}$ and $\mathcal{F}_{i_k}$, compute

$$\mathcal{H}_{jk} = \bigcup_{t \in (-\infty, 0)} \mathrm{cov}\left(x\left(t, \mathrm{vert}(\mathcal{F}_{i_j} \cap \mathcal{F}_{i_k})\right)\right).$$

   The hypersurfaces $\mathcal{H}_{jk}$ divide the attraction domain $\mathcal{A}_i$ into $\mathcal{A}(\mathcal{F}_{i_1}), \ldots, \mathcal{A}(\mathcal{F}_{i_l})$.

Remark 4: There is no common point for any pair of attraction domains of exit sets ($\mathcal{A}_i, i = 1, \ldots, K$), but there might be common points for some pair of attraction domains of facets ($\mathcal{A}(\mathcal{F}_1), \ldots, \mathcal{A}(\mathcal{F}_d)$). As we can see, the common points reach the intersection of facets.

IV. Illustrative Examples

In this section, two examples are given for illustration. One shows the largest positive invariant set and the other shows the attraction domain of a facet.

Example 1: (The largest positive invariant set). Consider the affine system

$$\dot{x} = \begin{bmatrix} 0.3980 & -0.2921 & -0.1312 \\ 0.9652 & 0.0567 & 0.8763 \\ -0.4724 & 0.5916 & 0.6590 \end{bmatrix} x + \begin{bmatrix} 0.0253 \\ -1.8982 \\ -0.7782 \end{bmatrix}$$

on a polytope $\mathcal{P}$, where the polytope $\mathcal{P}$ is the cube

$$\{x \in \mathbb{R}^3 : -2 \leq x_1 \leq 2, -2 \leq x_2 \leq 2, -2 \leq x_3 \leq 2\}.$$

The system matrix in the above system has three eigenvalues:

$$\lambda_{12} = -0.0040 \pm 0.1811i, \quad \lambda_3 = 0.1218.$$

The equilibrium point of the system is $\bar{x} = [1, 1, 1]^{\mathrm{T}}$, which is inside the polytope. By Theorem 1, the largest positive invariant set $\mathcal{O}$ belongs to the plane aff($\mathcal{Q}$)

$$\{x \in \mathbb{R}^3 : 0.0082x_1 + 0.0685x_2 + 0.1273x_3 = 0.2040\}.$$

In Fig. 4, the quadrangle $ABCD$ is the set $\mathcal{Q}$. The set $\mathcal{Q}$ is then divided by $\mathcal{D}^{\mathcal{Q}}$ (two spiral curves in Fig. 4) into three parts, and the part that contains the equilibrium point $\bar{x}$ is the largest positive invariant set in $\mathcal{P}$.

Example 2: (Attraction domain of facet $\mathcal{F}_1$). Consider the affine system

$$\dot{x} = \begin{bmatrix} -0.3727 & 0.8380 & 0.5220 \\ -0.1990 & 0.3455 & 0.2966 \\ -0.4231 & -0.2945 & -0.9401 \end{bmatrix} x + \begin{bmatrix} -0.9873 \\ -0.4431 \\ 1.6577 \end{bmatrix}$$

on the same polytope $\mathcal{P}$ as in Example 1.

The system matrix in this example has three eigenvalues

$$\lambda_1 = -0.4827, \;\; \lambda_{23} = -0.2423 \pm 0.2813i.$$

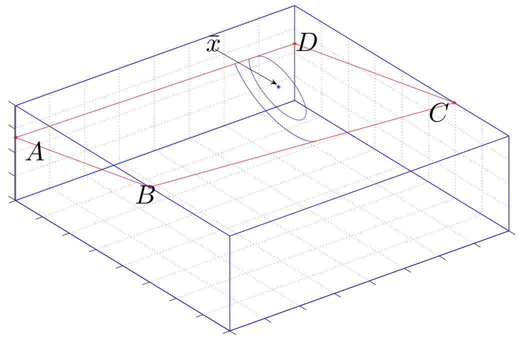

Fig. 4. Example 1: the largest positive invariant set $\mathcal{O}$

So aff($\mathcal{Q}$) is $\mathbb{R}^3$.

The facet $\mathcal{F}_1$ is the front surface of the cube shown in Fig. 5. The set $\mathcal{C}_1$ (see eq. (5) for its definition) determined by the identifier function on the facet $\mathcal{F}_1$ is the straight line on $\mathcal{F}_1$ in the figure, which divides $\mathcal{F}_1$ into two parts. Applying the procedure, we obtain the attraction domain of $\mathcal{F}_1$ as shown in Fig. 5 where its boundary is the shaded surface and the bottom surface of the cube.

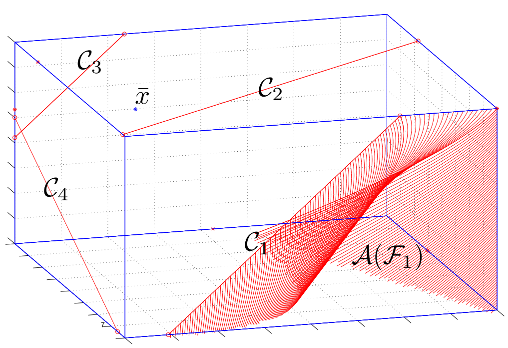

Fig. 5. Example 2: the attraction domain of $\mathcal{F}_1$

V. Conclusion

In the paper, we first observe that for an autonomous affine system, the largest positive invariant set lies in the stable invariant affine subspace. Then a hypersurface is computed to partition the polytope into attraction domains and positive invariant sets. These sets are determined thereafter. As a result, the reachability problem has been solved. In this work, the most numerical computation burden is in computing the dividing hypersurface, which requires to calculate the solution trajectories from a finite number of points and then the convex combination of these trajectory points at every time instant.

Two special cases have not been considered in the paper: the case with a singular system matrix and the case with the equilibrium point on the boundary of the polytope. But it is possible to extend these results to these two special cases. In addition, the attraction domain of each facet is not convex in general, which leads to difficulty in solving the reachability problem for piecewise affine systems. Hence, some "good properties" such as convexity of the attraction domain may be required in order to use the reachability results of an affine system on one polytope to solve the reachability problem of piecewise affine systems. However, no general condition is given so far.

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