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Sliding-Mode Observers for Uncertain Systems

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***Abstract*— Sliding-mode observer design is considered for linear systems with unknown inputs when the so-called observer matching condition is not satisfied. To circumvent the restriction imposed by the observer matching condition, the method of utilizing auxiliary outputs generated by high-order sliding-mode exact differentiators in the sliding-mode observer design has been proposed in the literature. In this paper, an alternative approach is proposed to use high-gain approximate differentiators of simpler architecture instead of high-order sliding-mode exact differentiators. The capability of reconstructing the unknown inputs using the proposed high-gain approximate differentiator based sliding-mode observer is also discussed and then illustrated with a numerical example.**

I. Introduction

Unknown input observer (UIO) has been developed to estimate the states of the system with inputs that are unknown or partially known. Linear UIO architectures that have been developed for linear system are presented in [1]–[5]. UIO architectures for non-linear systems with unknown inputs have been reported in [6], [7]. Motivated by the design of sliding-mode controllers, first-order sliding mode based UIOs have been discussed in [5], [8]–[10]. The main advantage of sliding-mode observers over their linear counterparts is that while in sliding, they are insensitive to the unknown inputs, and, moreover, they can be used to reconstruct unknown inputs which could be a combination of system disturbances, faults or non-linearities. The reconstruction of unknown inputs has found impressive applications in fault-detection and isolation [4], [9], [10].

The necessary and sufficient conditions for the existence of most of the unknown input observers proposed thus far are that the observer matching condition is satisfied and the invariant zeros of the system involving unknown input are in the open left half complex plane. However, the observer matching condition seriously restrict the applicability of sliding mode observers. Recently, high-order sliding mode based unknown input observers [11]–[14] have been developed to overcome this restrictive condition. In [13], a change of coordinates is performed using a constructive algorithm to transform the system into a quasi-block triangular observable form. Then a step-by-step second order sliding-mode observer is constructed for the transformed system. In [14], auxiliary outputs are defined so that the conventional unknown input sliding-mode observer proposed in [9] can be constructed for systems that do not satisfy the observer matching condition.

In this paper, we design the sliding-mode observer presented in [8] for systems that do not satisfy the observer matching condition. We adopt the idea of auxiliary outputs used in [14], but propose an alternative approach for the generation of auxiliary outputs. We use high-gain observers rather than high-order sliding-mode observers to obtain the estimates of auxiliary outputs. The high-gain observer is often referred to as approximate differentiator [15]. The proposed high-gain approximate differentiator based sliding-mode observer can achieve good state estimation performance. The advantage of our developed technique is that the overall observer architecture is simpler than the high-order sliding-mode exact differentiator based sliding-mode observer proposed in [14]. We also discuss the capability of the proposed high-gain approximate differentiator based sliding-mode observer in the unknown input reconstruction, which is then illustrated with a numerical example.

II. System Description and Problem Statement

We consider the following class of linear time-invariant systems with unknown inputs

$$\left.\begin{array}{l}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2 \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x},\end{array}\right\} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^p$, $\boldsymbol{u}_1 \in \mathbb{R}^{m_1}$ and $\boldsymbol{u}_2 \in \mathbb{R}^{m_2}$ are the state, output, known and unknown input vectors, and $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$, $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$ and $\boldsymbol{C} \in \mathbb{R}^{p\times n}$ are known constant matrices. For the above system, we assume that

1) $\boldsymbol{B}_2$ and $\boldsymbol{C}$ are of full rank, that is, $\operatorname{rank}\boldsymbol{B}_2 = m_2$ and $\operatorname{rank}\boldsymbol{C} = p$, and $m_2 \le p$;
2) there is $\rho > 0$ such that $\|\boldsymbol{u}_2(t)\| \le \rho$ for all t, where $\|\cdot\|$ denotes the standard Euclidean norm;
3) the invariant zeros of the system model given by the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ are in the open left-hand complex plane, or equivalently,

$$\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{O}\end{bmatrix} = n + m_2. \tag{2}$$

for all s such that $\Re(s) \ge 0$.

It follows from [8] that, if the so-called observer matching condition [13] is also satisfied for the system modeled by (1), that is,

$$\operatorname{rank}\boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = m_2, \tag{3}$$

we can construct the Walcott-Żak sliding-mode observer,

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{L}\left(\boldsymbol{y} - \hat{\boldsymbol{y}}\right) - \boldsymbol{B}_2\boldsymbol{E}(\boldsymbol{y}, \hat{\boldsymbol{y}}, \eta) \tag{4}$$

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with $\hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}}$ and

$$\boldsymbol{E}(\boldsymbol{y}, \hat{\boldsymbol{y}}, \eta) = \begin{cases} \eta \frac{\boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})}{\|\boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})\|} & \text{if } \boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y}) \neq \boldsymbol{0} \\ \boldsymbol{0} & \text{if } \boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y}) = \boldsymbol{0}, \end{cases} \tag{5}$$

where η is a positive design parameter, $\boldsymbol{L} \in \mathbb{R}^{n\times p}$ and $\boldsymbol{F} \in \mathbb{R}^{m_2 \times p}$ are matrices such that

$$(\boldsymbol{A} - \boldsymbol{LC})^\top \boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{LC}) = -2\boldsymbol{Q} < 0$$

and $\boldsymbol{FC} = \boldsymbol{B}_2^\top \boldsymbol{P}$ for some symmetric positive definite $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$. The design procedures for the matrices $\boldsymbol{L}_i$, $\boldsymbol{F}_i$ and $\boldsymbol{P}_i^o$ is given in [5].

However, many physical systems that can be modeled by (1) do not satisfy the observer matching condition (3). The observer matching condition (3) is sometimes too restrictive in practical applications.

III. High-Gain Approximate Differentiator

In this section, we propose a high-gain approximate differentiator based sliding-mode observer for the systems that do not satisfy the observer matching condition.

A. Auxiliary Output Signals

We first define as in [14] the auxiliary outputs that are then used to construct the sliding-mode observer. Let $\boldsymbol{c}_i$ be the i-th row of the output matrix $\boldsymbol{C}$. Recall that the relative degree of the i-th output y_i with respect to the unknown input $\boldsymbol{u}_2$ is defined to be the smallest positive integer r_i such that

$$\begin{aligned} \boldsymbol{c}_i\boldsymbol{A}^k\boldsymbol{B}_2 &= \boldsymbol{0}, \quad k = 0, \dots, r_i - 2 \\ \boldsymbol{c}_i\boldsymbol{A}^{r_i-1}\boldsymbol{B}_2 &\neq \boldsymbol{0}. \end{aligned}$$

We can choose integers γ_i ($1 \le \gamma_i \le r_i$) such that

$$\boldsymbol{C}_a = \begin{bmatrix} \boldsymbol{c}_1 \\ \vdots \\ \boldsymbol{c}_1\boldsymbol{A}^{\gamma_1-1} \\ \vdots \\ \boldsymbol{c}_p \\ \vdots \\ \boldsymbol{c}_p\boldsymbol{A}^{\gamma_p-1} \end{bmatrix}$$

is of full rank with $\operatorname{rank}(\boldsymbol{C}_a\boldsymbol{B}_2) = \operatorname{rank}\boldsymbol{B}_2$. It is proved in [14] that the system zeros of the system model given by the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C}_a)$ are in the open left-hand complex plane if the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ satisfies (2). Thus, we can construct the sliding-mode observer of the form (4) for the following system model

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2 \\ \boldsymbol{y}_a &= \boldsymbol{C}_a\boldsymbol{x}, \end{aligned}\right\}$$

if the output $\boldsymbol{y}_a = \boldsymbol{C}_a\boldsymbol{x}$ is available. However, some components of the vector $\boldsymbol{y}_a$ are not measurable and, therefore, additional observers are needed to estimate them.

B. High-Gain Observer Construction

In [14], high-order sliding-mode observers have been employed to obtain the auxiliary outputs in $\boldsymbol{y}_a$. We propose to use high-gain observers to estimate the auxiliary outputs instead. The reason behind this is because they have simpler architectures than high-order sliding-mode observers.

To proceed, we let $y_{ij} = \boldsymbol{c}_i\boldsymbol{A}^{j-1}\boldsymbol{x}$, $i = 1, \dots, p$ and $j = 1, \dots, \gamma_i$. Thus, we have $\boldsymbol{y}_a = [\boldsymbol{y}_{a1}^\top \ \cdots \boldsymbol{y}_{ap}^\top]^\top$, where $\boldsymbol{y}_{ai} = [y_{i1} \ \cdots \ y_{i\gamma_i}]^\top$. If $\gamma_i > 1$, the dynamics of $\boldsymbol{y}_{ai}$, $i = 1, \dots, p$, are given by

$$\left.\begin{aligned} \dot{\boldsymbol{y}}_{ai} &= \bar{\boldsymbol{A}}_i\boldsymbol{y}_{ai} + \bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}, \boldsymbol{u}_2) + \bar{\boldsymbol{b}}_{i2}\boldsymbol{u}_1 \\ y_{i1} &= \bar{\boldsymbol{c}}_i\boldsymbol{y}_{ai}, \end{aligned}\right\} \tag{6}$$

where the pair $(\bar{\boldsymbol{A}}_i, \bar{\boldsymbol{b}}_{i1})$ is in canonical controllable form representing the chain of γ_i integrators,

$$f_i(\boldsymbol{x}, \boldsymbol{u}_2) = \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i}\boldsymbol{x} + \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i-1}\boldsymbol{B}_1\boldsymbol{u}_2, \tag{7}$$

$\bar{\boldsymbol{b}}_{i2} = [\boldsymbol{c}_i\boldsymbol{B}_1 \ \cdots \ \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i-1}\boldsymbol{B}_1]^\top$ and $\bar{\boldsymbol{c}}_i = [1 \ 0 \ \cdots \ 0]$. We assume, as in [14], that $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are bounded and $|y_{ij}| \le d_{ij}$, which implies that $\boldsymbol{u}_1$ is bounded. If $\gamma_i > 1$, we construct the following high-gain observers,

$$\left.\begin{aligned} \dot{\hat{y}}_{i1} &= \hat{y}_{i2} + \tfrac{\alpha_{i1}}{\epsilon}(y_{i1} - \hat{y}_{i1}) + \boldsymbol{c}_i\boldsymbol{B}_1\boldsymbol{u}_1 \\ &\ \vdots \\ \dot{\hat{y}}_{i(\gamma_i-1)} &= \hat{y}_{i\gamma_i} + \tfrac{\alpha_{i(\gamma_i-1)}}{\epsilon^{\gamma_i-1}}(y_{i1} - \hat{y}_{i1}) + \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i-2}\boldsymbol{B}_1\boldsymbol{u}_1 \\ \dot{\hat{y}}_{i\gamma_i} &= \tfrac{\alpha_{i\gamma_i}}{\epsilon^{\gamma_i}}(y_{i1} - \hat{y}_{i1}) + \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i-1}\boldsymbol{B}_1\boldsymbol{u}_1, \end{aligned}\right\} \tag{8}$$

where $\epsilon \in (0,1)$ is a design parameter and α_{ij}, $j = 1, \dots, \gamma_i$, are selected so that the roots of the equation, $s^{\gamma_i} + \alpha_{i1}s^{\gamma_i-1} + \cdots + \alpha_{i(\gamma_i-1)}s + \alpha_{i\gamma_i} = 0$, have negative real parts. Let $\boldsymbol{y}_{hi} = [\hat{y}_{i1} \ \cdots \ \hat{y}_{i\gamma_i}]^\top$ and $\boldsymbol{l}_i = [\alpha_{i1}/\epsilon \ \cdots \ \alpha_{i\gamma_i}/\epsilon^{\gamma_i}]^\top$. We can rewrite (8) as

$$\dot{\boldsymbol{y}}_{hi} = \bar{\boldsymbol{A}}_i\boldsymbol{y}_{hi} + \boldsymbol{l}_i\bar{\boldsymbol{c}}_i(\boldsymbol{y}_{ai} - \boldsymbol{y}_{hi}) + \bar{\boldsymbol{b}}_{i2}\boldsymbol{u}_1. \tag{9}$$

If $\gamma_i = 1$, we do not need to construct the above high-gain observer (9) because of the availability of y_{i1}. In such a case, we have $\boldsymbol{y}_{hi} = \boldsymbol{y}_{ai} = y_{i1}$. To proceed, let $\boldsymbol{\zeta}_i = 0$ if $\gamma_i = 1$ and let $\boldsymbol{\zeta}_i = [\zeta_{i1} \ \cdots \ \zeta_{i\gamma_i}]^\top$ if $\gamma_i > 1$, where

$$\zeta_{ij} = \frac{y_{ij} - \hat{y}_{ij}}{\epsilon^{\gamma_i - j}}, \quad j = 1, \dots, \gamma_i. \tag{10}$$

It follows from (6) and (9) that if $\gamma_i > 1$, we have

$$\epsilon\dot{\boldsymbol{\zeta}}_i = \bar{\boldsymbol{A}}_{ci}\boldsymbol{\zeta}_i + \epsilon\bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}, \boldsymbol{u}_2), \tag{11}$$

where $\bar{\boldsymbol{A}}_{ci} = \epsilon\boldsymbol{D}_i^{-1}(\bar{\boldsymbol{A}}_i - \boldsymbol{l}_i\bar{\boldsymbol{c}}_i)\boldsymbol{D}_i$ is a Hurwitz matrix independent of ϵ. Applying the method in [16], we can prove the following proposition.

Proposition: For the high-gain observer (9), there exists a finite time $T_i(\epsilon)$ such that $\|\boldsymbol{\zeta}_i(t)\| \le \beta_i\epsilon$ for some positive constant β_i and $t \ge t_0 + T_i(\epsilon)$. Moreover, $T_i(\epsilon)$ approaches zero when ϵ approaches to zero, that is, $\lim_{\epsilon\to 0^+} T_i(\epsilon) = 0$.

It follows from (10) that $\boldsymbol{y}_{ai} - \boldsymbol{y}_{hi} = \boldsymbol{D}_i\boldsymbol{\zeta}_i$, where $\boldsymbol{D}_i = \operatorname{diag}[\epsilon^{\gamma_i-1} \ \epsilon^{\gamma_i-2} \ \cdots \ 1]$. Let $\boldsymbol{y}_h = [\boldsymbol{y}_{h1}^\top \ \cdots \ \boldsymbol{y}_{hp}^\top]^\top$, $\boldsymbol{D} = \operatorname{diag}[\boldsymbol{D}_1 \ \cdots \ \boldsymbol{D}_p]$ and $\boldsymbol{\zeta} = [\boldsymbol{\zeta}_1^\top \ \cdots \ \boldsymbol{\zeta}_p^\top]^\top$. We have

$$\boldsymbol{y}_a - \boldsymbol{y}_h = \boldsymbol{D\zeta}. \tag{12}$$

Note that the induced Euclidean norm of $\boldsymbol{D}$ is 1, that is, $\|\boldsymbol{D}\| = 1$. Let $\beta_i = 0$ and $T_i(\epsilon) = 0$ if $\gamma_i = 1$. Thus, it follows from the proposition that $\|\boldsymbol{\zeta}\| \le \beta\epsilon$, where $\beta = (\sum_{i=1}^{p} \beta_i^2)^{\frac{1}{2}}$, after a finite time $T(\epsilon) = \max_{1\le i\le p} T_i(\epsilon)$, and $\lim_{\epsilon\to 0} T(\epsilon) = 0$.

IV. State Estimation Performance Analysis

In order to eliminate the peaking phenomena that accompanies the operation of the above high-gain observer [17], we introduce the saturation of the signal $\boldsymbol{y}_h$ such that $\boldsymbol{y}_h^s = [\boldsymbol{y}_{h1}^{s\,\top} \ \cdots \ \boldsymbol{y}_{hp}^{s\,\top}]^\top$, where $\boldsymbol{y}_{hi}^s = \boldsymbol{y}_{ai} = y_{i1}$ if $\gamma_i = 1$ and

$$\boldsymbol{y}_{hi}^s = \left[\begin{array}{ccc} S_{i1}\,\mathrm{sat}\left(\dfrac{\hat{y}_{i1}}{S_{i1}}\right) & \cdots & S_{i\gamma_i}\,\mathrm{sat}\left(\dfrac{\hat{y}_{i\gamma_i}}{S_{i\gamma_i}}\right) \end{array} \right]^\top$$

with $S_{ij} > d_{ij}$ if $\gamma_i > 1$. Then we construct the following sliding-mode observer,

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{L}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right) - \boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta), \quad (13)$$

where $\hat{\boldsymbol{y}}_a = \boldsymbol{C}_a\hat{\boldsymbol{x}}$ and

$$\boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta) = \begin{cases} \eta\dfrac{\boldsymbol{F}_a(\hat{\boldsymbol{y}}_a - \boldsymbol{y}_h^s)}{\|\boldsymbol{F}_a(\hat{\boldsymbol{y}}_a - \boldsymbol{y}_h^s)\|} & \text{if } \boldsymbol{F}_a\left(\hat{\boldsymbol{y}}_a - \boldsymbol{y}_h^s\right) \ne \boldsymbol{0} \\ \boldsymbol{0} & \text{if } \boldsymbol{F}_a\left(\hat{\boldsymbol{y}}_a - \boldsymbol{y}_h^s\right) = \boldsymbol{0}. \end{cases}$$

where $\boldsymbol{L}_a \in \mathbb{R}^{n\times\gamma}$ and $\boldsymbol{F}_a \in \mathbb{R}^{m_2\times\gamma}$ are matrices such that

$$\left(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a\right)^\top \boldsymbol{P}_a + \boldsymbol{P}_a\left(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a\right) = -2\boldsymbol{Q}_a < 0$$

and

$$\boldsymbol{F}_a\boldsymbol{C}_a = \boldsymbol{B}_2^\top\boldsymbol{P}_a \quad (14)$$

for some symmetric positive definite $\boldsymbol{P}_a \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Q}_a \in \mathbb{R}^{n\times n}$. It follows from (1) and (13) that

$$\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{L}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right) - \boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta). \quad (15)$$

In the following, we analyze the performance of the proposed high-gain approximate differentiator based sliding-mode observer given by (13).

Theorem 1: For the dynamical system (1) and the associated sliding-mode observer (13) with high-gain approximate differentiators (9), there exists a constant $\epsilon^* \in (0, 1)$ such that if $\epsilon \in (0,\ \epsilon^*)$ and $\eta \ge \rho$, then the state estimation error $\boldsymbol{e}(t)$ is uniformly ultimately bounded. Specifically, after a finite time $T_f(\epsilon)$, we have $\|\boldsymbol{e}(t)\| \le \kappa(\epsilon)$, where

$$\kappa(\epsilon) = \frac{\kappa_1\epsilon + \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_1\epsilon}}{2\mu_a}\sqrt{\frac{2}{\lambda_{\min}(\boldsymbol{P}_a)}}$$

for positive constants μ_a, κ_1 and κ_2.

Proof: It follows from the proposition that $\|\boldsymbol{\zeta}(t)\| \le \beta\epsilon$ for $t \ge t_0 + T(\epsilon)$. Then, it follows from (12) that $\|\boldsymbol{y}_a(t) - \boldsymbol{y}_h(t)\| \le \beta\epsilon$ for $t \ge t_0 + T(\epsilon)$. There exists a constant $\bar{\epsilon}$ such that if $\|\boldsymbol{y}_a(t) - \boldsymbol{y}_h(t)\| \le \beta\bar{\epsilon}$, then $\boldsymbol{y}_h(t)$ is not saturated, that is, $\boldsymbol{y}_h^s(t) = \boldsymbol{y}_h(t)$. Thus, we can choose $\epsilon^* = \min\{\bar{\epsilon}, 1\}$ such that if $\epsilon \in (0,\ \epsilon^*)$, then $\|\boldsymbol{\zeta}(t)\| \le \beta\epsilon$ and $\boldsymbol{y}_h^s(t) = \boldsymbol{y}_h(t)$ after a finite time $T(\epsilon)$.

For $t_0 \le t \le t_0 + T(\epsilon)$, it is guaranteed that the observer state vector $\hat{\boldsymbol{x}}(t)$ in (13) is bounded because $\boldsymbol{u}_1$, $\boldsymbol{y}_h^s$ and $\boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta)$ are bounded and $\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a$ is Hurwitz. Thus, $\boldsymbol{e}(t)$ is bounded for $t_0 \le t \le t_0 + T(\epsilon)$. For $t \ge t_0 + T(\epsilon)$, because $\boldsymbol{y}_h^s(t) = \boldsymbol{y}_h(t)$ and $\boldsymbol{y}_h = \boldsymbol{y}_a - \boldsymbol{D}\boldsymbol{\zeta}$, the dynamics of the state estimation error (15) become

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{A}\boldsymbol{e} + \boldsymbol{L}_a\left(\boldsymbol{y}_h - \hat{\boldsymbol{y}}_a\right) - \boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta) \\ &= \left(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a\right)\boldsymbol{e} - \boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} - \boldsymbol{B}_2\boldsymbol{u}_2 \\ &\quad - \boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta). \end{aligned} \quad (16)$$

Consider the Lyapunov function candidate $V = \frac{1}{2}\boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{e}$ for $t \ge t_0 + T(\epsilon)$. Evaluating the time derivative of V on the solutions of (16), we obtain

$$\begin{aligned}\dot{V} &= \boldsymbol{e}^\top\boldsymbol{P}\left(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a\right)\boldsymbol{e} - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} \\ &\quad - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta) \\ &= -\boldsymbol{e}^\top\boldsymbol{Q}_a\boldsymbol{e} - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} \\ &\quad - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e})^\top\boldsymbol{u}_2 - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e})^\top\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta) \\ &= -\boldsymbol{e}^\top\boldsymbol{Q}_a\boldsymbol{e} - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{u}_2 \\ &\quad - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta) \\ &\quad + (\boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{u}_2 + (\boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta). \end{aligned}$$

If $\boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\boldsymbol{\zeta}) = \boldsymbol{0}$, then

$$\begin{aligned}&-(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{u}_2 \\ &\qquad - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e})^\top\boldsymbol{E}_a = 0. \end{aligned} \quad (17)$$

On the other hand, if $\boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\boldsymbol{\zeta}) \ne \boldsymbol{0}$, then

$$\begin{aligned}&-(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{u}_2 - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e})^\top\boldsymbol{E}_a \\ &= -(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{u}_2 \\ &\quad - \eta(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\frac{\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta}}{\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta}\|} \\ &\le -(\eta - \rho)\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta}\| \le 0. \end{aligned} \quad (18)$$

It follows from (17) and (18) that in both cases we have

$$\begin{aligned}\dot{V} &\le -\boldsymbol{e}^\top\boldsymbol{Q}_a\boldsymbol{e} - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} \\ &\quad + (\boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{u}_2 + (\boldsymbol{F}_a\boldsymbol{D}\boldsymbol{\zeta})^\top\boldsymbol{E}_a(\boldsymbol{y}_h, \hat{\boldsymbol{y}}_a, \eta) \\ &\le -\lambda_{\min}(\boldsymbol{Q}_a)\|\boldsymbol{e}\|^2 + \beta\epsilon\|\boldsymbol{P}_a\boldsymbol{L}_a\|\|\boldsymbol{e}\| \\ &\quad + (\eta + \rho)\beta\epsilon\|\boldsymbol{F}_a\| \\ &= -2\mu_a V + \kappa_1\epsilon\sqrt{V} + \kappa_2\epsilon, \end{aligned} \quad (19)$$

where $\kappa_1 = \sqrt{2}\beta\|\boldsymbol{P}_a\boldsymbol{L}_a\|/\sqrt{\lambda_{\max(\boldsymbol{P}_a)}}$ and $\kappa_2 = (\eta + \rho)\beta\|\boldsymbol{F}_a\|$. It follows from (19) that

$$\begin{aligned}\dot{V} &\le -\mu_a V - \mu_a V + \kappa_1\epsilon\sqrt{V} + \kappa_2\epsilon \\ &= -\mu_a V - \left(\sqrt{V} - R_-\right)\left(\sqrt{V} - R_+\right), \end{aligned} \quad (20)$$

where

$$R_- = \frac{\kappa_1\epsilon - \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_2\epsilon}}{2\mu_a} < 0$$

and

$$R_+ = \frac{\kappa_1\epsilon + \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_1\epsilon}}{2\mu_a} > 0.$$

We conclude from (20) that $\dot{V} < 0$ when $\|\boldsymbol{e}\| > R_+$. Thus, the state estimation error $\boldsymbol{e}$ is uniformly ultimately bounded with respect to any closed ball of radius greater than R_+.

Hence, as long as $\sqrt{V} > R_+$, that is, $V > R_+^2$, we have $(\sqrt{V} - R_-)(\sqrt{V} - R_+) < 0$. Therefore, if $V(t_0 + T(\epsilon)) = V(\boldsymbol{e}(t_0 + T(\epsilon))) > R_+^2$ and $V(t) > R_+^2$ for $t \geq t_0 + T_f(\epsilon)$, then $\dot{V} \leq -\mu_a V$, which implies that $V(t) \leq \exp\left(-\mu_a\left(t - t_0 - T(\epsilon)\right)\right) V\left(t_0 + T(\epsilon)\right)$. Thus, we can find a finite time $T_f(\epsilon)$ such that $V(t) \leq R_+^2$ for $t \geq t_0 + T(\epsilon)$, where $T_f(\epsilon)$ is the solution to the equation $V(t_0 + T(\epsilon))\exp(-\mu_a(T_f(\epsilon) - T(\epsilon))) = R_+^2$ as

$$T_f(\epsilon) = T(\epsilon) + \frac{1}{\mu_a}\ln\left(\frac{V(t_0 + T(\epsilon))}{R_+^2}\right).$$

On the other hand, if $V(t_0 + T(\epsilon)) \leq R_+^2$, then $V(t) \leq R_+^2$ for $t \geq t_0 + T(\epsilon)$. In such a case, we can choose $T_f(\epsilon) = T(\epsilon)$. Therefore, there exists a finite time $T_f(\epsilon)$ such that $V(t) \leq R_+^2$ for $t \geq t_0 + T_f(\epsilon)$, which implies that $\|\boldsymbol{e}(t)\| \leq \kappa(\epsilon)$. The proof of the theorem is complete. ∎

Remark: It follows from Theorem 1 that the state estimation error enters the closed ball $\{\boldsymbol{e} : \|\boldsymbol{e}\| \leq \kappa(\epsilon)\}$ after a finite time $T_f(\epsilon)$. It is easy to verify that

$$\lim_{\epsilon\to 0^+} T_f(\epsilon) = \begin{cases} \infty & \text{if } V(t_0) \neq 0 \\ 0 & \text{if } V(t_0) = 0, \end{cases}$$

because $\lim_{\epsilon\to 0^+} T(\epsilon) = 0$ and $\lim_{\epsilon\to 0^+} R^+ = 0$. Moreover, the radius of the above closed ball can be adjusted by the design parameter ϵ and because $\lim_{\epsilon\to 0^+}\kappa(\epsilon) = 0$, the state estimation error e converges to the origin as ϵ goes to zero.

Theorem 2: For sufficiently large η, the sliding surface, $\{(\boldsymbol{e}, \boldsymbol{\zeta}) : \boldsymbol{\sigma} = \boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\boldsymbol{\zeta}) = \boldsymbol{0}\}$ is invariant in the $(\boldsymbol{e}, \boldsymbol{\zeta})$-space and is reached in finite time.

Proof: Let $\boldsymbol{\zeta} = [\boldsymbol{\zeta}_1^\top \;\cdots\; \boldsymbol{\zeta}_p^\top]^\top$. Using (11) for $\gamma_i > 1$ and the fact that $\boldsymbol{\zeta}_i = 0$ if $\gamma_i = 1$, we obtain

$$\epsilon\dot{\boldsymbol{\zeta}} = \bar{\boldsymbol{A}}_c\boldsymbol{\zeta} + \epsilon\bar{\boldsymbol{B}}_1\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2), \tag{21}$$

where $\bar{\boldsymbol{A}}_c = \text{diag}[\bar{\boldsymbol{A}}_{c1} \;\cdots\; \bar{\boldsymbol{A}}_{cp}]$, $\bar{\boldsymbol{B}}_1 = \text{diag}[\bar{\boldsymbol{b}}_{11} \;\cdots\; \bar{\boldsymbol{b}}_{p1}]$ with $\bar{\boldsymbol{A}}_{ci} = \boldsymbol{O}$ and $\bar{\boldsymbol{b}}_{i1} = \boldsymbol{0}$ if $\gamma_i = 1$ and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2) = [f_1(\boldsymbol{x}, \boldsymbol{u}_2) \;\cdots\; f_p(\boldsymbol{x}, \boldsymbol{u}_2)]^\top$. Because $\boldsymbol{x}$ and $\boldsymbol{u}_2$ are bounded, we have $\|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2)\| \leq \beta_1$ for some $\beta_1 > 0$. For $t \geq t_0 + T_f(\epsilon)$, it follows from (16) and (21) that

$$\begin{aligned}
\boldsymbol{\sigma}^\top\dot{\boldsymbol{\sigma}} &= \boldsymbol{\sigma}^\top\left(\boldsymbol{F}_a\boldsymbol{C}_a\dot{\boldsymbol{e}} + \boldsymbol{F}_a\boldsymbol{D}\dot{\boldsymbol{\zeta}}\right) \\
&= \boldsymbol{\sigma}^\top\Big(\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)\boldsymbol{e} - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} \\
&\qquad - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{E}_a \\
&\qquad + \frac{1}{\epsilon}\boldsymbol{F}_a\boldsymbol{D}\bar{\boldsymbol{A}}_c\boldsymbol{\zeta} + \boldsymbol{F}_a\boldsymbol{D}\bar{\boldsymbol{B}}_1\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2)\Big) \\
&\leq \kappa(\epsilon)\|\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)\|\|\boldsymbol{\sigma}\| \\
&\quad + \beta\epsilon\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\|\|\boldsymbol{\sigma}\| + \beta_1\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{B}}_1\|\|\boldsymbol{\sigma}\| \\
&\quad + \lambda_{\max}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\|\boldsymbol{u}_2\|\|\boldsymbol{\sigma}\| \\
&\quad - \eta\lambda_{\min}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\|\boldsymbol{\sigma}\| + \beta\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{A}}_c\|\|\boldsymbol{\sigma}\| \\
&= -\left(\eta - \frac{\kappa_3 + \kappa_4 + \kappa_5 + \kappa_6 + \kappa_7}{\kappa_8}\right)\kappa_8\|\boldsymbol{\sigma}\|,
\end{aligned} \tag{22}$$

where

$$\kappa_3 = \kappa(\epsilon)\|\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)\|, \quad \kappa_4 = \beta\epsilon\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\|,$$
$$\kappa_5 = \rho\lambda_{\max}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2), \quad \kappa_6 = \beta\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{A}}_c\|,$$
$$\kappa_7 = \|\boldsymbol{F}_a\|\|\bar{\boldsymbol{B}}_1\|\|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_2)\|, \quad \kappa_8 = \lambda_{\min}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2).$$

It follows from (22) that if we choose η such that

$$\eta \geq \frac{\kappa_3 + \kappa_4 + \kappa_5 + \kappa_6 + \kappa_7}{\kappa_8} + \varepsilon,$$

where ε is a small positive constant, then

$$\boldsymbol{\sigma}^\top\dot{\boldsymbol{\sigma}} \leq -\varepsilon\|\boldsymbol{\sigma}\|, \tag{23}$$

which implies the above hyperplane is invariant. Let T_s denote the time the sliding surface is reached. Using the same arguments as in [9, p. 53], we obtain

$$T_s \leq t_0 + T_f(R) + \frac{\|\boldsymbol{\sigma}(t_0 + T_f(R))\|}{\varepsilon}.$$

Thus, the proof of the theorem is complete. ∎

V. UNKNOWN INPUT RECONSTRUCTION

It follows from Theorem 2 that the manifold $\{(\boldsymbol{e}, \boldsymbol{\zeta}) : \boldsymbol{\sigma} = \boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\boldsymbol{\zeta}) = \boldsymbol{0}\}$ is invariant and is reached after a finite time. Therefore, we have

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} = \boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)\boldsymbol{e} - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta} - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{u}_2 \\
- \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta) + \boldsymbol{F}_a\boldsymbol{D}\dot{\boldsymbol{\zeta}} = \boldsymbol{0}.
\end{aligned} \tag{24}$$

Substituting (14) into (24) and performing simple manipulations, we obtain

$$\begin{aligned}
\boldsymbol{u}_2 = \left(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2\right)^{-1}\Big(\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\dot{\boldsymbol{\zeta}} \\
\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\boldsymbol{D}\boldsymbol{\zeta}\Big) - \boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta)..
\end{aligned} \tag{25}$$

By the proposition, we have $\|\boldsymbol{\zeta}(t)\| \leq \beta\epsilon$ for $t \geq t_0 + T(\epsilon)$. By Theorem 1, we have $\|\boldsymbol{e}(t)\| \leq \kappa(\epsilon)$ for $t \geq t_0 + T_f(\epsilon)$, where $\lim_{\epsilon\to 0^+}\kappa(\epsilon) = 0$. Therefore, for sufficiently small ϵ, $\|\boldsymbol{\zeta}(t)\|$ and $\|\boldsymbol{e}(t)\|$ becomes negligible after a finite time. If, in addition, $\|\dot{\boldsymbol{\zeta}}(t)\|$ becomes negligible for sufficiently small ϵ, it follows from (25) that after a finite time,

$$\boldsymbol{u}_2 \approx -\boldsymbol{E}_a(\boldsymbol{y}_h^s, \hat{\boldsymbol{y}}_a, \eta). \tag{26}$$

That is, we can use the proposed architecture to estimate the unknown input $\boldsymbol{u}_2$ for sufficiently small ϵ.

Recall that $\boldsymbol{\zeta} = [\boldsymbol{\zeta}_1^\top \;\cdots\; \boldsymbol{\zeta}_p^\top]^\top$ and $\boldsymbol{\zeta}_i = \dot{\boldsymbol{\zeta}}_i = 0$ if $\gamma_i = 1$. Thus, in order to obtain (26), it remains to show that if $\gamma_i > 1$, then $\|\dot{\boldsymbol{\zeta}}_i(t)\|$ becomes negligible for sufficiently small ϵ. We first rewrite (11) as

$$\dot{\boldsymbol{\zeta}}_i(t) = \frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}\boldsymbol{\zeta}_i(t) + \boldsymbol{v}_i(t), \tag{27}$$

where $\boldsymbol{v}_i(t) = \bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}(t), \boldsymbol{u}_2(t))$. Because $\boldsymbol{x}(t)$ and $\boldsymbol{u}_2(t)$ are bounded, it follows from (7) that $f_i(\boldsymbol{x}(t), \boldsymbol{u}_2(t))$ is bounded. Thus, $\boldsymbol{v}_i(t)$ is bounded. To proceed, we define two notions regarding the function $\boldsymbol{v}_i(t)$.

Definition 1: A function $\boldsymbol{v}_i(t)$ is left-continuous if $\lim_{\epsilon\to 0^+}\boldsymbol{v}_i(t - \epsilon) = \boldsymbol{v}_i(t)$ for all t.

Definition 2: A function $\boldsymbol{v}_i(t)$ defined on $S \subset \mathbb{R}$ is weakly uniformly continuous if for every $\nu > 0$, there exists a $\delta > 0$ such that for each interval $\Omega \subset S$ with length less than δ, $\|\boldsymbol{v}_i(s) - \boldsymbol{v}_i(t)\| < \nu$ for $s, t \in \Omega$.

In the following, we use $S_1 + S_2$, where $S_1, S_2 \subset \mathbb{R}$, to denote the set $\{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$. If S_1 or S_2 is empty, then $S_1 + S_2$ is defined to be empty.

Let J denote the set of points at which $\boldsymbol{v}_i(t)$ is discontinuous and let $\tau > t_0 > 0$. It can be shown, as in [18], that if $\boldsymbol{v}_i(t)$ is left-continuous, then $\lim_{\epsilon \to 0^+} \dot{\boldsymbol{\zeta}}_i(t) = \mathbf{0}$ for each $t > t_0 \geq 0$. Moreover, if $\boldsymbol{v}_i(t)$ is also weakly uniformly continuous on $[\tau,\ \infty)\backslash J$, then the convergence of $\dot{\boldsymbol{\zeta}}_i(t)$ to $\mathbf{0}$ as $\epsilon \to 0^+$ is uniform on $[\tau,\ \infty)\backslash(J + (0,\ \xi))$ for each $\xi > 0$. In particular, if $\boldsymbol{v}_i(t)$ is uniformly continuous, then the convergence is uniform on $[\tau,\ \infty)$. The detailed proof of this result can be found in [18], where a more general case regarding $\boldsymbol{v}_i(t)$ is also considered.

VI. Numerical Example

In this section, we illustrate the effectiveness of our proposed high-gain approximate differentiator based sliding-mode observer with a numerical example. Our simulations demonstrate that its performance is quite similar to that of the high-order sliding-mode exact differentiator based sliding-mode observer. Due to lack of space, we only show simulations with the high-gain approximate differentiator based sliding-mode observer.

We consider a linear time invariant system modeled by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -5 & -10 & -10 & -5 \end{bmatrix},$$

$$\boldsymbol{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We do not consider $\boldsymbol{B}_1$, because we set $\boldsymbol{u}_1 = \mathbf{0}$ for simplicity. The initial condition is selected to be $\boldsymbol{x}(0) = [0.5\ 0.5\ 0.5\ -0.5\ -0.5]^\top$. The unknown input $\boldsymbol{u}_2$ consists of a square wave of amplitude 1 and frequency 1Hz, and a sawtooth signal of amplitude 2 and frequency 1Hz.

It is easy to check that for this system $\operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) \neq \operatorname{rank} \boldsymbol{B}_2$ because $\boldsymbol{c}_1\boldsymbol{B}_2 = \mathbf{0}$. Thus, we choose $\gamma_1 = r_1 = 3$ such that

$$\boldsymbol{C}_a = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_1\boldsymbol{A} \\ \boldsymbol{c}_1\boldsymbol{A}^2 \\ \boldsymbol{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is of full rank with $\operatorname{rank}(\boldsymbol{C}_a\boldsymbol{B}_2) = \operatorname{rank} \boldsymbol{B}_2$. We employ a high-gain observer to estimate the auxiliary outputs $y_{12} = \boldsymbol{c}_1\boldsymbol{A}\boldsymbol{x}$ and $y_{13} = \boldsymbol{c}_1\boldsymbol{A}^2\boldsymbol{x}$. The design parameters of the high-gain observer are selected to be $\alpha_{11} = 3$, $\alpha_{12} = 3$, $\alpha_{13} = 1$ and $\epsilon = 0.001$. The estimated and true values of the auxiliary outputs are shown in Fig. 1.

Now we use the estimates of the auxiliary outputs to construct the sliding-mode observer described by (13). Following the algorithm given in [5], we use $\kappa = 2.0659$ and

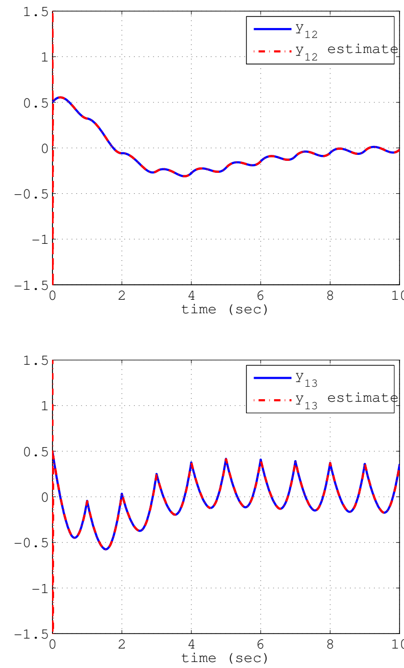

Fig. 1. True and estimated auxiliary outputs.

$\eta = 50$ to obtain

$$\boldsymbol{L}_a = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2.0659 & 0 \\ 0 & 0 & 0 & 2.0659 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{F}_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}^\top.$$

We set the initial states of the sliding-mode observer to be zero, that is, $\hat{\boldsymbol{x}}(0) = \mathbf{0}$, and select $S_{11} = S_{12} = S_{13} = 1.5$. In Fig. 2, we show the state estimation performance. The unknown inputs reconstruction is illustrated in Fig. 3.

VII. Conclusions

A novel sliding-mode observer has been proposed for systems with unknown inputs, where the observer matching condition is not satisfied. High-gain approximate differentiators were employed to estimate auxiliary outputs that are then used by the sliding-mode observer to estimate the states and reconstruct the unknown inputs. The proposed observer has simple architecture and performs comparably to the high-order sliding-mode exact differentiator based sliding-mode observer in [14].

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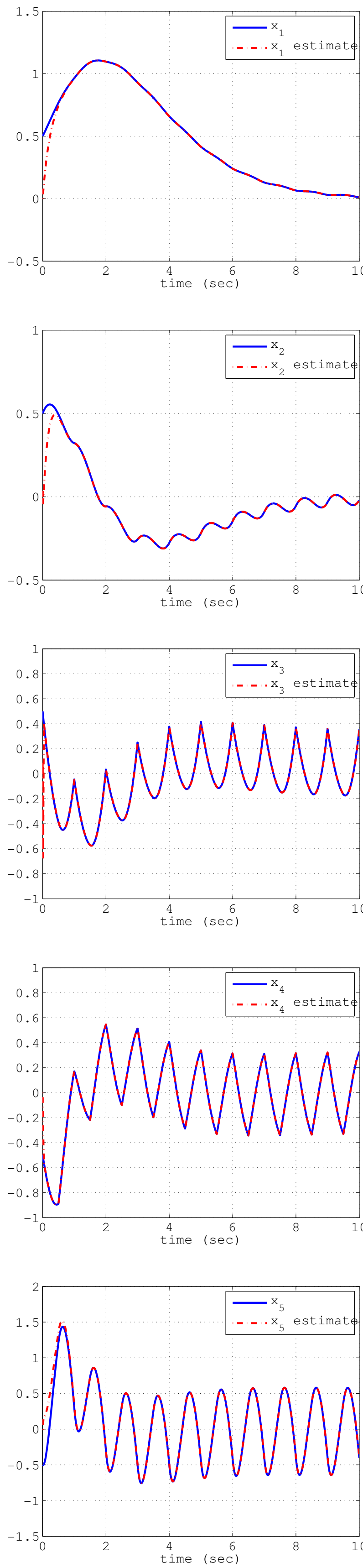

Fig. 2. True and estimated system states.

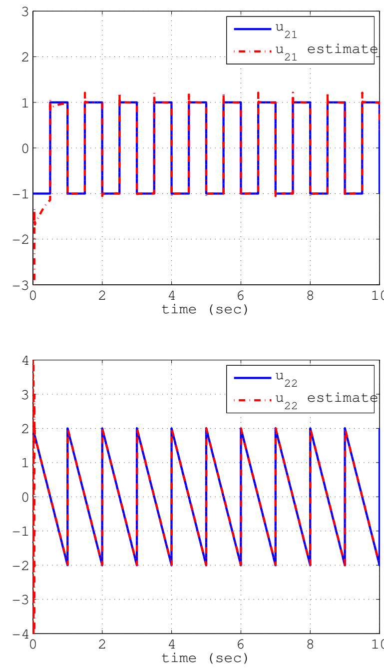

Fig. 3. Unknown input reconstruction.

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