2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Linear Matrix Inequality Based Dynamic Output Feedback Sliding Mode Control for Uncertain Plants

José Manuel ANDRADE—DA SILVA, Christopher EDWARDS and Sarah K. SPURGEON

***Abstract*— The development of a synthesis methodology to design compensator-based sliding mode output feedback controllers using linear matrix inequalities is considered in this paper. The class of dynamical system dealt with belongs to the class of uncertain plants with matched, and mismatched uncertainties in polytopic form. The efficacy of the proposed design methodology is demonstrated by a numerical design and computer simulations.**

I. INTRODUCTION

Variable Structure Control (VSC) is a robust nonlinear control method consisting of a switched control law and a decision function. *Sliding Mode Control* (SMC) is a type of VSC that rejects completely, when appropriately designed, a class of uncertainty known as *matched uncertainty*. The dynamics in the sliding mode will, however, be subject to variations from any *mismatched uncertanty* in the system. It is important to highlight that most of the early developments in SMC theory assumed that the state variables of the plant can be physically measured. This is an unrealistic assumption as in many practical engineering applications some states do not have physical meaning and hence they cannot be measured. This problem can be addressed by using either an *observer-based controller* or an *output feedback controller*. The former approach requires additional dynamics. Its main drawback is that, unless properly designed, the observer undermines the robustness properties of the state feedback control. The latter uses only measured plant output signals. These kinds of controllers can be static or dynamic in nature. Static output feedback control is the simplest approach since no further dynamics are needed. However, it may not be applied in some particular cases, *e.g.* if the *Kimura-Davison* conditions are not satisfied. In such situations, an appropriately dimensioned dynamic compensator is required in order to introduce extra dynamics and to increase the degrees of design freedom. This approach belongs to the class of dynamic output feedback controllers.

Most of the papers referenced in the technical literature have considered the class of Linear Time Invariant (LTI) systems either without uncertainties or with matched uncertainties. Only a few papers have been devoted to the class of systems with mismatched uncertainties. For instance, sliding mode static output feedback (SMSOF) control systems based on *Linear Matrix Inequalities* (LMIs) [2] have been developed by Choi [4] and Xiang *et al.* [8], whilst a dynamic output feedback variable structure controller has recently been proposed by Park *et al.* in [7].

Choi in [4] presented a static output feedback variable structure control synthesis approach based on LMIs. The proposed scheme corresponds to a high gain control law which may not be desirable in practical applications. As stated by Choi, this disadvantage can be overcome by means of a trade-off between complexity and control effort using a dynamic variable structure output feedback control law [4]. Xiang *et al.* in [8] developed an iterative LMI algorithm which neither requires a change of coordinates nor solves a static output feedback problem. Nevertheless, since the proposed control law is high gain then it is necessary to solve an optimization problem in order to design a sliding surface. The LMIs involved in the synthesis methodology are relatively complex. Furthermore, since the algorithm is iterative, its convergence depends on the initial conditions defined by the designer. The dynamic output feedback VSC proposed by Park *et al.* in [7] considers mismatched norm-bounded time-varying uncertainty and is of the same order as the plant. The design methodology consists of an iterative algorithm developed using the so-called cone complementary linearization algorithm for bi-convex problems.

In this paper, a compensator-based sliding mode output feedback controller is proposed for plants with matched and mismatched uncertainties. This controller belongs to the class of sliding mode dynamic output feedback controllers. The design methodology is based on an LMI framework. The proposed approach represents an extension to the work in [6] where only matched uncertainties were considered. In particular the existence and reaching problems are formulated from a polytopic perspective. The switching surface design problem for the augmented system, *i.e.* the uncertain plant and dynamic compensator, is recast in terms of LMIs as a static output feedback problem with mismatched uncertainties. In order to design the linear component, a polytopic LMI based approach is developed in which the mismatched uncertainties are considered whilst the synthesis of the nonlinear part takes into account matched uncertainties, disturbances and/or nonlinearities.

This paper is structured as follows: Section 2 defines the class of systems to be considered and presents the problem formulation. The proposed design approach for the Sliding Mode Dynamic Output Feedback Controller (SMDOFC) for

José Manuel ANDRADE—DA SILVA* and Christopher EDWARDS are with Control and Instrumentation Research Group, Department of Engineering, University of Leicester, Leicester, LE1 7RH, U.K., Tel.: +44 (0) 116 252 2567, Fax +44 (0) 116 252 2619. e-mail*: jma28@le.ac.uk and jandrade@scientech.co.uk.

Sarah K. Spurgeon is with Instrumentation, Control and Embedded Systems Research Group, Department of Electronics, University of Kent, Canterbury, CT2 7NT, U.K., Tel: +44 (0) 122 782 3258, Fax: +44 (0) 122 745 6084.

978-1-4244-4524-0/09/$25.00 ©2009 AACC

uncertain plants is described in Section 3. A numerical example in Section 4 demonstrates the new approach. Finally, Section 5 presents some concluding remarks.

Throughout this paper $\|\cdot\|$ stands for the *Euclidean* norm of a vector and the induced spectral norm of a matrix. The index set $I(\varepsilon_1,\varepsilon_2)$ is defined as $I(\varepsilon_1,\varepsilon_2)=\{\varepsilon_1,\varepsilon_1+1,\varepsilon_1+2,\cdots,\varepsilon_2\}$ where $\varepsilon_1,\varepsilon_2\in\mathbb{Z}_+$ and $\varepsilon_1<\varepsilon_2$.

II. System Description and Problem Formulation

Consider an uncertain dynamical system described in state-space form $\forall\, t\geq 0$ by

$$\left.\begin{array}{l}\dot{\mathbf{x}}(t)=\Big(\mathbf{A}+\Delta\mathbf{A}\Big)\mathbf{x}(t)+\mathbf{B}\Big(\mathbf{u}(t)+\xi(t,\mathbf{x},\mathbf{u})\Big)\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)\quad,\quad \mathbf{x}_0=\mathbf{x}(0)\end{array}\right\}\tag{1}$$

where $\mathbf{x}\in\mathscr{X}\subseteq\Re^n$ is the state vector with $\mathscr{X}$ an open set, $\mathbf{u}\in\mathscr{U}\subseteq\Re^m$ is the control input vector with $\mathscr{U}$ the set of all admissible control signals and $\mathbf{y}\in\mathscr{Y}\subseteq\Re^p$ is the output vector with $\mathscr{Y}$ the set of all measurable output signals. The uncertain vector function $\xi(t,\mathbf{x},\mathbf{u}):\Re_+\times\Re^n\times\Re^m\rightarrow\Re^m$ represents the lumped sum of matched nonlinearities and/or uncertainties.

Throughout the paper the following is assumed:

A.1 The order of the system and the number of output and input signals satisfy $n>p>m$.

A.2 The input and output matrices are both full rank, *i.e.* $rank(\mathbf{B})=m$ and $rank(\mathbf{C})=p$.

A.3 In the nominal triple $\big(\mathbf{A},\mathbf{B},\mathbf{C}\big)$, $rank(\mathbf{CB})=m$.

A similarity transformation exists such that the nominal triple $\big(\mathbf{A},\mathbf{B},\mathbf{C}\big)$ in the new coordinates involving the state vector partition $\mathbf{x}=\begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T\end{bmatrix}^T$ where $\mathbf{x}_1\in\Re^{(n-m)}$ and $\mathbf{x}_2\in\Re^m$ has the following structure [5]:

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_2\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}\mathbf{0} & \mathbf{T}\end{bmatrix}\tag{2}$$

where $\mathbf{A}_{11}\in\Re^{(n-m)\times(n-m)}$, $\mathbf{A}_{12}\in\Re^{(n-m)\times m}$, $\mathbf{A}_{21}\in\Re^{m\times(n-m)}$, $\mathbf{A}_{22}\in\Re^{m\times m}$, $\mathbf{B}_2\in\Re^{m\times m}$ and $\mathbf{T}\in\Re^{p\times p}$ are assumed to be known constant matrices. Furthermore, the matrix $\mathbf{B}_2$ is non-singular and $\mathbf{T}$ is orthogonal.

It is assumed that all matched components of the uncertainty associated with the state matrix have been merged into $\xi(t,\mathbf{x},\mathbf{u})$. Commensurate with the structure of the state matrix $\mathbf{A}$ defined in (2), the uncertain state matrix $\mathbf{A}_\Delta=\mathbf{A}+\Delta\mathbf{A}$ is taken to have the form

$$\mathbf{A}_\Delta=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}+\begin{bmatrix}\Delta\mathbf{A}_{11} & \Delta\mathbf{A}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\tag{3}$$

The matrix sub-blocks $\Delta\mathbf{A}_{11}$ and $\Delta\mathbf{A}_{12}$ represent structured uncertainties with bounded uncertain parameters. These parameters are bounded between two extreme real values as follows

$$\theta_i\in\left[\underline{\theta}_i,\overline{\theta}_i\right]\quad\text{for}\quad i\in I(1,r)\tag{4}$$

Define an associated *hyper-rectangle* in the parameter space as $\Theta\subseteq\Re^r$. The uncertain parameter vector $\theta=\begin{bmatrix}\theta_1 & \theta_2 & \cdots & \theta_r\end{bmatrix}^T$ can describe either physical constant parameters or time dependent parameters, *i.e.* $\theta(t):\Re_+\rightarrow\Theta$. In this work, for the sake of generality, uncertain time-varying parameters $\theta_i(t)$ with $i\in I(1,r)$ are considered. Hence, the uncertain matrix sub-blocks in the uncertain state matrix (3) will hereafter be written as $\Delta\mathbf{A}_{11}(t)$ and $\Delta\mathbf{A}_{12}(t)$. Notice that these uncertain matrix sub-blocks depend affinely on the uncertain parameters $\theta_i(t)$ for $i\in I(1,r)$.

For the remainder of the paper, it is assumed that:

A.4 The matched uncertainty term is bounded by

$$\|\xi(t,\mathbf{x},\mathbf{u})\|\leq k_1\|\mathbf{u}(t)\|+\phi(t,\mathbf{y}(t))+k_2\tag{5}$$

where $\phi(t,\mathbf{y}(t))$ is a known function such that $\phi:\Re_+\times\Re^p\rightarrow\Re_+$. Furthermore, $0\leq k_1<1$ and $k_2\in\Re_+$.

The *sliding surface* $\mathscr{S}$ is defined as follows

$$\mathscr{S}=\{\mathbf{x}\in\Re^n:\sigma(t)=\Gamma\mathbf{y}(t)=\Gamma\mathbf{C}\mathbf{x}(t)=0\}\tag{6}$$

where $\sigma(t)\in\Re^m$ is the *switching function* and $\Gamma\in\Re^{m\times p}$ is the *switching gain matrix* to be designed. The sliding surface synthesis problem is the so-called *existence problem*.

Let

$$\Gamma\mathbf{T}=\begin{bmatrix}\Gamma_1 & \Gamma_2\end{bmatrix}\tag{7}$$

where $\Gamma_1\in\Re^{m\times(p-m)}$, $\Gamma_2\in\Re^{m\times m}$ and $\det(\Gamma_2)\neq 0$. Define $\mathbf{C}_1\in\Re^{(p-m)\times(n-m)}$ as

$$\mathbf{C}_1\triangleq\begin{bmatrix}\mathbf{0}_{((p-m)\times(n-p))} & \mathbf{I}_{(p-m)}\end{bmatrix}\tag{8}$$

and the gain matrix $\mathbf{K}\in\Re^{m\times(p-m)}$ as

$$\mathbf{K}\triangleq\Gamma_2^{-1}\Gamma_1\tag{9}$$

During a sliding motion, $\sigma(t)\equiv 0$, and hence $\mathbf{x}_2(t)=-\mathbf{K}\mathbf{C}_1\mathbf{x}_1(t)$. Moreover, considering the null space dynamics, it follows that

$$\dot{\mathbf{x}}_1(t)=\big(\tilde{\mathbf{A}}_{11}(t)-\tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1\big)\mathbf{x}_1(t)\tag{10}$$

where the matrices $\tilde{\mathbf{A}}_{11}(t)=\big(\mathbf{A}_{11}+\Delta\mathbf{A}_{11}(t)\big)$ and $\tilde{\mathbf{A}}_{12}(t)=\big(\mathbf{A}_{12}+\Delta\mathbf{A}_{12}(t)\big)$.

From [5], the switching gain matrix Γ is parameterised straightforwardly from (7) and (9) as follows

$$\Gamma=\Gamma_2\begin{bmatrix}\mathbf{K} & \mathbf{I}_m\end{bmatrix}\mathbf{T}^T\tag{11}$$

The matrix Γ_2 in (11) corresponds to a scaling of the switching matrix Γ. In this paper, it is assumed that $\Gamma_2=\mathbf{B}_2^{-1}$ in order to obtain $\Gamma\mathbf{CB}=\mathbf{I}_m$. This assumption is useful when designing the control law.

In some particular cases, the existence problem for systems given by (1) with matched and mismatched uncertainties cannot be solved, as the following *Bilinear Matrix Inequality* is not feasible

$$\begin{array}{r}\Big(\tilde{\mathbf{A}}_{11}(t)-\tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1\Big)^T\mathbf{P}_1+\\ +\mathbf{P}_1\Big(\tilde{\mathbf{A}}_{11}(t)-\tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1\Big)<0\end{array}\tag{12}$$

An approach to overcome the problem is to design a compensator which introduce additional dynamics and consequently provides further degrees of freedom for synthesising a sliding surface.

Therefore, the first phase of the problem to be addressed consists of the design of a dynamic compensator given by

$$\dot{\mathbf{x}}_c(t)=\Xi\mathbf{x}_c(t)+\Psi\mathbf{y}(t)\tag{13}$$

where $\Xi\in\Re^{q\times q}$ and $\Psi\in\Re^{q\times p}$, and a hyperplane in the augmented state space $\mathscr{X}_a\subseteq\Re^{n+q}$.

$$\mathscr{S}_a=\{\mathbf{x}_a\in\Re^{n\times q}:\sigma_a(t)=\Gamma_c\mathbf{x}_c(t)+\Gamma\mathbf{C}\mathbf{x}(t)=0\}\tag{14}$$

where $\mathbf{x}_a=\begin{bmatrix}\mathbf{x}_c^T & \mathbf{x}^T\end{bmatrix}^T$ is the augmented state vector, $\sigma_a(t)\in\Re^m$ is the *augmented switching function*, whilst $\Gamma_c\in\Re^{m\times q}$ and $\Gamma\in\Re^{m\times p}$ are components of the *augmented switching* *gain matrix* Γ_a to be synthesised.

The control law synthesis for the uncertain dynamical system (1) is dealt with in the second phase of the problem to be solved. Such a control law has to guarantee that the sliding surface $\mathscr{S}_a$ is reached in finite time from any initial point $\mathbf{x}_a(t_0) \notin \mathscr{S}_a$ in the augmented state space $\mathscr{X}_a$ and the sliding motion takes place thereafter. This control law design problem is the so-called *reachability problem*.

III. Sliding Mode Dynamic Output Feedback Control

The SMDOFC design approach developed in this work considers a polytopic formulation for the existence and reachability problems and employs LMI methods.

A. Dynamic Compensator-based Sliding Surface Design

Consider the dynamic compensator given in (13) and let

$$\Psi\mathbf{T} = [\Psi_1 \quad \Psi_2] \tag{15}$$

where $\Psi_1 \in \Re^{q\times(p-m)}$ and $\Psi_2 \in \Re^{q\times m}$.

Consequently, the dynamic compensator (13) considering the state vector partition used in (2) can be written as follows

$$\dot{\mathbf{x}}_c(t) = \Xi\mathbf{x}_c(t) + \Psi_1\mathbf{C}_1\mathbf{x}_1(t) + \Psi_2\mathbf{x}_2(t) \tag{16}$$

where $\mathbf{C}_1 \in \Re^{(p-m)\times(n-m)}$ is defined in (8).

Defining the gain matrix $\mathbf{K}_c \in \Re^{m\times q}$ as

$$\mathbf{K}_c \triangleq \Gamma_2^{-1}\Gamma_c \tag{17}$$

the reduced order sliding motion associated with $\mathscr{S}_a$ is given by

$$\dot{\mathbf{x}}_1(t) = \big(\tilde{\mathbf{A}}_{11}(t) - \tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1\big)\mathbf{x}_1(t) - \tilde{\mathbf{A}}_{12}(t)\mathbf{K}_c\mathbf{x}_c(t) \tag{18}$$

$$\dot{\mathbf{x}}_c(t) = \big(\Psi_1 - \Psi_2\mathbf{K}\big)\mathbf{C}_1\mathbf{x}_1(t) + \big(\Xi - \Psi_2\mathbf{K}_c\big)\mathbf{x}_c(t) \tag{19}$$

The aim is to find Ξ, Ψ_1, Ψ_2, $\mathbf{K}$ and $\mathbf{K}_c$ so that

$$\Phi(t) = \begin{bmatrix} \big(\tilde{\mathbf{A}}_{11}(t) - \tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1\big) & -\tilde{\mathbf{A}}_{12}(t)\mathbf{K}_c \\ \big(\Psi_1 - \Psi_2\mathbf{K}\big)\mathbf{C}_1 & \big(\Xi - \Psi_2\mathbf{K}_c\big) \end{bmatrix} \tag{20}$$

is stable.

The problem of designing the compensator gains Ψ_1, Ψ_2, and Ξ as well as the gain matrices $\mathbf{K}$ and $\mathbf{K}_c$ can be written in a static output feedback fashion as follows:

$$\Phi(t) = \mathscr{A}(t) - \mathscr{B}(t)\mathscr{K}\mathscr{C} \tag{21}$$

where

$$\mathscr{A}(t) = \begin{bmatrix} \tilde{\mathbf{A}}_{11}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathscr{B}(t) = \begin{bmatrix} \tilde{\mathbf{A}}_{12}(t) & \mathbf{0} \\ \Psi_2 & -\mathbf{I}_q \end{bmatrix} \tag{22}$$

$$\mathscr{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q \end{bmatrix}, \quad \mathscr{K} = \begin{bmatrix} \mathbf{K} & \mathbf{K}_c \\ \Psi_1 & \Xi \end{bmatrix} \tag{23}$$

Let $\mathscr{P}$ be a *polyhedric closed convex sub-set*: a so-called *polytope*. Let $\Pi(t) \in \mathscr{P} \subset \Re^{(n+p-2m+2q)\times(n+2q)}$ be an uncertain matrix of the form

$$\Pi(t) = \left[\begin{array}{c|c} \mathscr{A}(t) & \mathscr{B}(t) \\ \hline \mathscr{C} & \mathbf{0} \end{array}\right] \tag{24}$$

then $\mathscr{P}$ is the convex hull $\mathscr{P} = \mathrm{Co}\{\Pi_1, \Pi_2, \cdots, \Pi_N\}$ defined as follows

$$\mathscr{P} = \left\{ \sum_{j=1}^N \mu_j \left[\begin{array}{c|c} \mathscr{A}_j & \mathscr{B}_j \\ \hline \mathscr{C} & \mathbf{0} \end{array}\right] : \sum_{j=1}^N \mu_j = 1, \; \mu_j \geq 0 \text{ for } j \in I(1,N) \right\} \tag{25}$$

where N is the number of vertices of $\mathscr{P}$, and μ_j with $j \in I(1,N)$ are the polytopic coordinates of Π. The vertices of $\mathscr{P}$ in (25) are given by $\mathscr{C}$ defined in (23) and

$$\mathscr{A}_j = \mathscr{A}_0 + \sum_{i=1}^r \theta_i \Delta\mathscr{A}_j \Big|_{\theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}} \text{ for } j \in I(1, N = 2^r) \tag{26}$$

$$\mathscr{B}_j = \mathscr{B}_0 + \sum_{i=1}^r \theta_i \Delta\mathscr{B}_j \Big|_{\theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}} \text{ for } j \in I(1, N = 2^r) \tag{27}$$

where

$$\mathscr{A}_0 \triangleq \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \Delta\mathscr{A}_j \triangleq \begin{bmatrix} \Delta\tilde{\mathbf{A}}_{11j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{28}$$

$$\mathscr{B}_0 \triangleq \begin{bmatrix} \tilde{\mathbf{A}}_{12} & \mathbf{0} \\ \Psi_2 & -\mathbf{I}_q \end{bmatrix}, \quad \Delta\mathscr{B}_j \triangleq \begin{bmatrix} \Delta\tilde{\mathbf{A}}_{12j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{29}$$

The following is assumed:

A.5 The triples $(\mathscr{A}_j, \mathscr{B}_j, \mathscr{C})$ for $j \in I(1,N)$ is stabilisable and detectable for all admissible uncertainties in the hyper-rectangle Θ.

The reduced-order system (18)-(19) is output feedback stabilisable if there exists a positive definite *Lyapunov* matrix $\mathbf{P}_1 = \mathbf{P}_1^T \in \Re^{(n+q-m)\times(n+q-m)}$ and a gain matrix $\mathscr{K}$ such that

$$\Big(\mathscr{A}_j - \mathscr{B}_j\mathscr{K}\mathscr{C}\Big)^T\mathbf{P}_1 + \mathbf{P}_1\Big(\mathscr{A}_j - \mathscr{B}_j\mathscr{K}\mathscr{C}\Big) < 0 \tag{30}$$

for $j \in I(1, N = 2^r)$.

The vertices of the polytope $\mathscr{P}$ are said to be simultaneously stabilised by the gain matrix $\mathscr{K}$ if (30) holds.

As the synthesis of the gain $\mathscr{K}$ corresponds to a static output feedback problem for the system triple $\big(\mathscr{A}(t), \mathscr{B}(t), \mathscr{C}\big)$ any available LMI approach could be employed. In this work, the non-iterative LMI-based algorithm proposed by Benton and Smith in [1] is re-formulated in the context of the existence problem for the SMDOFC as follows:

Step 1: Define N vertices of the polytope $\mathscr{P}$.

Step 2: Define a degree of stability such that

$$\mathscr{A}_{\alpha j} = \mathscr{A}_j + \alpha\mathbf{I}$$

Step 3 : Solve the following optimization problem

$$\begin{aligned} &\mathit{min} \quad \mathit{trace}(\mathbf{Q}_{sf}) \\ &\quad s.t. \\ &\quad\quad \mathbf{Q}_{sf} - \mathbf{I} > \mathbf{0} \\ &\quad\quad \mathbf{Q}_{sf}\mathscr{A}_{\alpha j}^T + \mathscr{A}_{\alpha j}\mathbf{Q}_{sf} + \mathbf{Y}_{sf}^T\mathscr{B}_j^T + \mathscr{B}_j\mathbf{Y}_{sf} < 0 \end{aligned}$$

Step 4 : Set $\mathscr{K}_{sf} = \mathbf{Y}_{sf}\mathbf{Q}_{sf}^{-1}$.

Step 5 : Solve the LMI feasibility problem

$$\begin{aligned} &\mathit{find} \ \varepsilon \ \mathit{and} \ \mathbf{P}_1 \\ &\quad s.t. \\ &\quad\quad \mathbf{P}_1 > \mathbf{I}, \ \varepsilon > 0 \\ &\quad\quad \Big(\mathscr{A}_{\alpha j} + \mathscr{B}_j\mathscr{K}_{sf}\Big)^T\mathbf{P}_1 + \mathbf{P}_1\Big(\mathscr{A}_{\alpha j} + \mathscr{B}_j\mathscr{K}_{sf}\Big) < 0 \\ &\quad\quad \mathscr{A}_{\alpha j}^T\mathbf{P}_1 + \mathbf{P}_1\mathscr{A}_{\alpha j} - \varepsilon\mathscr{C}^T\mathscr{C} < 0 \end{aligned}$$

Step 6 : Solve the following LMI problem

$$\begin{aligned} &find\ \mathscr{K} \\ &\quad s.t. \\ &\left(\mathscr{A}_{\alpha j}-\mathscr{B}_j\mathscr{K}\mathscr{C}\right)^T\mathbf{P}_1+\mathbf{P}_1\left(\mathscr{A}_{\alpha j}-\mathscr{B}_j\mathscr{K}\mathscr{C}\right)<0 \end{aligned}$$

where $j\in I(1,N=2^r)$.

The feasibility problem formulated in the step 6 can be replaced by an optimization problem involving the minimization of a norm defined by the designer.

B. Control Law Synthesis

Consider the following system augmented matrices

$$\mathscr{A}_{a\Delta}(t)=\begin{bmatrix} \Xi & \Psi\mathbf{C} \\ \mathbf{0} & \tilde{\mathbf{A}}_{\Delta}(t) \end{bmatrix} \tag{31}$$

$$\mathscr{B}_a=\begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix} \quad , \quad \mathscr{C}_a=\begin{bmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \tag{32}$$

If a switching gain matrix $\Gamma_a=[\Gamma_c \quad \Gamma]$ exists such that the sliding dynamics (18)-(19) is stable, then a nonsingular change of coordinates $\mathbf{x}\mapsto\hat{\mathbf{T}}\mathbf{x}$ exists such that the triple $\left(\mathscr{A}_{a\Delta}(t),\mathscr{B}_a,\mathscr{C}_a\right)$ above can be transformed into

$$\hat{\mathscr{A}}_{a\Delta}(t)=\hat{\mathscr{A}}_a+\Delta\hat{\mathscr{A}}_a(t)=\begin{bmatrix} \hat{\mathscr{A}}_{a\Delta 11}(t) & \hat{\mathscr{A}}_{a\Delta 12}(t) \\ \hat{\mathscr{A}}_{a\Delta 21}(t) & \hat{\mathscr{A}}_{a\Delta 22}(t) \end{bmatrix} \tag{33}$$

$$\hat{\mathscr{B}}_a=\begin{bmatrix} 0 & \mathbf{I}_m \end{bmatrix}^T \tag{34}$$

$$\Gamma_a\hat{\mathscr{C}}_a=\begin{bmatrix} 0 & \mathbf{I}_m \end{bmatrix} \quad \text{where} \quad \hat{\mathscr{C}}_a=\begin{bmatrix} 0 & \overline{\mathbf{T}} \end{bmatrix} \tag{35}$$

with $\overline{\mathbf{T}}\in\Re^{(p+q)\times(p+q)}$ such that $\det\{\overline{\mathbf{T}}\}\neq 0$. The structure of $\Gamma_a\hat{\mathscr{C}}_a$ follows since by construction $\Gamma_a\hat{\mathscr{C}}_a\hat{\mathscr{B}}_a=\Gamma\mathbf{C}\mathbf{B}=\mathbf{I}_m$.

Define

$$\hat{\Pi}(t)=\left[\begin{array}{c|c} \hat{\mathscr{A}}_{a\Delta}(t) & \hat{\mathscr{B}}_a \\ \hline \hat{\mathscr{C}}_a & \mathbf{0} \end{array}\right] \tag{36}$$

consequently, the corresponding polytope $\hat{\mathscr{P}}$ is defined as follows

$$\left.\begin{aligned} \hat{\mathscr{P}}=\Bigg\{\textstyle\sum_{j=1}^N\mu_j\left[\begin{array}{c|c} \hat{\mathscr{A}}_{a\Delta j} & \hat{\mathscr{B}}_a \\ \hline \hat{\mathscr{C}}_a & \mathbf{0} \end{array}\right]:\sum_{j=1}^N\mu_j=1, \\ \mu_j\geq 0 \ \text{ for } \ j\in I(1,N)\Bigg\} \end{aligned}\right\} \tag{37}$$

where the vertices are given by

$$\left.\begin{aligned} \hat{\mathscr{A}}_{a\Delta j}&=\hat{\mathscr{A}}_a+\textstyle\sum_{i=1}^r\theta_i\Delta\hat{\mathscr{A}}_{ai}\Big|_{\theta_i=\{\underline{\theta}_i,\overline{\theta}_i\}} \\ &=\begin{bmatrix} \hat{\mathscr{A}}_{a\Delta 11j} & \hat{\mathscr{A}}_{a\Delta 12j} \\ \hat{\mathscr{A}}_{a\Delta 21j} & \hat{\mathscr{A}}_{a\Delta 22j} \end{bmatrix} \end{aligned}\right\} \tag{38}$$

for $j\in I(1,N=2^r)$ and $\hat{\mathscr{B}}_a$ and $\hat{\mathscr{C}}_a$ are defined in (34) and (35) respectively.

The sliding mode dynamics are represented by a convex combination of

$$\hat{\mathscr{A}}_{a\Delta 11j}=\mathscr{A}_j-\mathscr{B}_j\mathscr{K}\mathscr{C} \ \text{ for } \ j\in I(1,N) \tag{39}$$

which are *Hurwitz* by design, and in turn

$$\hat{\mathscr{A}}_{a\Delta 11}(t)=\mathscr{A}(t)-\mathscr{B}(t)\mathscr{K}\mathscr{C} \tag{40}$$

is stable by the convexity property of the polytope $\hat{\mathscr{P}}$ defined in (37).

Consider the control law

$$\mathbf{u}(t)=\mathbf{u}_L(t)+\mathbf{u}_{NL}(t) \tag{41}$$

with the linear component $\mathbf{u}_L(t)$ of the form

$$\mathbf{u}_L(t)=-\mathbf{G}\mathbf{y}_a(t) \tag{42}$$

where $\mathbf{G}\in\Re^{m\times(p+q)}$ and $\mathbf{y}_a(t)=\begin{bmatrix} \mathbf{x}_c^T(t) & \mathbf{y}^T(t) \end{bmatrix}^T\in\Re^{(p+q)}$ is the augmented output vector; the nonlinear component given by

$$\mathbf{u}_{NL}(t)=\begin{cases} -\rho(\cdot)\mathbf{P}_2^{-1}\dfrac{\Gamma_a\mathbf{y}_a(t)}{\|\Gamma_a\mathbf{y}_a(t)\|} & \text{, if } \Gamma_a\mathbf{y}_a(t)\neq 0 \\ 0 & \text{, otherwise} \end{cases} \tag{43}$$

where

$$\rho\big(t,\mathbf{y}(t),\mathbf{u}(t)\big)=\frac{k_1\|\mathbf{u}_L(t)\|+\phi(t,\mathbf{y}(t))+k_2+\eta}{(1-k_1)} \tag{44}$$

PROPOSITION: Let $\mathbf{P}$ be a Lyapunov matrix partitioned as follows

$$\mathbf{P}=\begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{bmatrix}>\mathbf{I} \tag{45}$$

where $\mathbf{P}_1\in\Re^{(n+q-m)\times(n+q-m)}$ is the Lyapunov matrix in (30) calculated by means of the Benton and Smith algorithm, and $\mathbf{P}_2\in\Re^{m\times m}$. Let the gain matrix $\mathbf{G}$ be parametrised as

$$\mathbf{G}=\begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix}\overline{\mathbf{T}}^{-1} \tag{46}$$

where $\mathbf{G}_1\in\Re^{m\times(p+q-m)}$ and $\mathbf{G}_2\in\Re^{m\times m}$ such that the following matrix inequality holds for $j\in I(1,N)$

$$\mathscr{A}_j^T\mathbf{P}+\mathbf{P}\mathscr{A}_j<0 \tag{47}$$

where

$$\mathscr{A}_j=\begin{bmatrix} \hat{\mathscr{A}}_{a\Delta 11j} & \hat{\mathscr{A}}_{a\Delta 12j} \\ \hat{\mathscr{A}}_{a\Delta 21j}-\mathbf{G}_1\mathscr{C} & \hat{\mathscr{A}}_{a\Delta 22j}-\mathbf{G}_2 \end{bmatrix} \tag{48}$$

Then the control law in (41) guarantees a sliding motion on the surface $\mathscr{S}_a$ inside *the sliding patch*

$$\Omega=\left\{\left(\hat{\mathbf{x}}_1\in\Re^{n+q-m},\hat{\mathbf{x}}_2\in\Re^m\right) \ : \ \|\hat{\mathbf{x}}_1\|<\eta\gamma^{-1}\right\} \tag{49}$$

where $\eta>0$ is a design scalar and

$$\gamma=max_{j\in(1,N=2^r)}\left\{\|\mathbf{P}_2\big(\hat{\mathscr{A}}_{a\Delta 21j}-\mathbf{G}_1\mathscr{C}\big)\|\right\} \tag{50}$$

$\triangle$

PROOF: Consider *the Lyapunov function* $V(t)=\hat{\mathbf{x}}^T(t)\mathbf{P}\hat{\mathbf{x}}(t)$. Since $\mathbf{P}$ has the partition shown in (45) then $\mathbf{P}\hat{\mathscr{B}}_a=\left(\Gamma\hat{\mathscr{C}}_a\right)^T\mathbf{P}_2$. After manipulations involving (43), (44) and (47), it can be shown $\dot{V}<0 \quad \forall\,\hat{\mathbf{x}}(t)\neq 0$. Therefore, the closed-loop system is *quadratically stable*.

Partition the state vector $\hat{\mathbf{x}}(t)$ as $\begin{bmatrix} \hat{\mathbf{x}}_1^T(t) & \hat{\mathbf{x}}_2^T(t) \end{bmatrix}^T$. Using matrix inequality (47) with the previous partition, the following quadratic form can be obtained

$$\hat{\mathbf{x}}_2^T(t)\Big(\big(\hat{\mathscr{A}}_{a\Delta 22j}-\mathbf{G}_2\big)^T\mathbf{P}_2+\mathbf{P}_2\big(\hat{\mathscr{A}}_{a\Delta 22j}-\mathbf{G}_2\big)\Big)\hat{\mathbf{x}}_2(t)<0$$

for $j\in I(1,N=2^r)$.

Consider the Lyapunov function $\hat{V}(t) = \hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\hat{\mathbf{x}}_2(t)$. Its derivative along the closed-loop trajectories is given by

$$\dot{\hat{V}}(t) = \sum_{j=1}^{N} \mu_j \Big(2\hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\big(\hat{\mathscr{A}}_{a\Delta 21j} - \mathbf{G}_1\mathscr{C}\big)\hat{\mathbf{x}}_1(t) + \hat{\mathbf{x}}_2^T(t)\Big(\big(\hat{\mathscr{A}}_{a\Delta 22j} - \mathbf{G}_2\big)^T\mathbf{P}_2 + \mathbf{P}_2\big(\hat{\mathscr{A}}_{a\Delta 22j} - \mathbf{G}_2\big)\Big)\hat{\mathbf{x}}_2(t) + 2\hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\big(\mathbf{u}_{NL}(t) + \xi(t,\mathbf{x},\mathbf{u})\big)\Big)$$

Since (44) implies $\|\xi(t,\mathbf{y}(t),\mathbf{u}(t))\| \le \rho(t,\mathbf{y}(t),\mathbf{u}(t)) - \eta$ then

$$\dot{\hat{V}}(t) < \sum_{j=1}^{N} \mu_j \Big(2\hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\big(\hat{\mathscr{A}}_{a\Delta 21j} - \mathbf{G}_1\mathscr{C}\big)\hat{\mathbf{x}}_1(t) - 2\eta\|\hat{\mathbf{x}}_2(t)\|\Big)$$

which means that the sliding motion occurs inside the sliding patch Ω defined in (49). Since the closed-loop system is quadratically stable, the sliding patch Ω is reached in finite time and a sliding motion takes place thereafter. *Q.E.D.*

The matrix $\mathscr{A}_j$ in (48) can be expressed as

$$\mathscr{A}_j \triangleq \hat{\mathscr{A}}_{a\Delta j} - \hat{\mathscr{B}}_a\mathbf{G}\hat{\mathscr{C}}_a \tag{51}$$

then it follows that

$$\mathscr{A}_j^T\mathbf{P} + \mathbf{P}\mathscr{A}_j = \begin{bmatrix} \Lambda_{11j} & \Lambda_{12j} \\ \Lambda_{12j}^T & \Lambda_{22j} \end{bmatrix} < 0 \tag{52}$$

where

$$\left.\begin{array}{l} \Lambda_{11j} = \hat{\mathscr{A}}_{a\Delta 11j}^T\mathbf{P}_1 + \mathbf{P}_1\hat{\mathscr{A}}_{a\Delta 11j} \\ \Lambda_{12j} = \mathbf{P}_1\hat{\mathscr{A}}_{a\Delta 12j} + \hat{\mathscr{A}}_{a\Delta 21j}^T\mathbf{P}_2 - \mathscr{C}^T\mathbf{L}_1^T \\ \Lambda_{22j} = \mathbf{P}_2\hat{\mathscr{A}}_{a\Delta 22j} - \mathbf{L}_2 + \hat{\mathscr{A}}_{a\Delta 22j}^T\mathbf{P}_2 - \mathbf{L}_2^T \end{array}\right\} \tag{53}$$

for $j \in I(1,N)$ with $\mathbf{L}_1 \triangleq \mathbf{P}_2\mathbf{G}_1$ and $\mathbf{L}_2 \triangleq \mathbf{P}_2\mathbf{G}_2$.

The Lyapunov inequality (52) with (53) depends affinely on the matrix variables $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{L}_1$ and $\mathbf{L}_2$. Therefore, an LMI problem can be formulated in order to design a gain matrix $\mathbf{G}$ such that

$$\|\mathbf{G}\| < \zeta \tag{54}$$

and

$$\|\mathbf{P}_2\hat{\mathscr{A}}_{a\Delta 21j} - \mathbf{L}_1\mathscr{C}\| < \gamma \tag{55}$$

Inequality (54) has to be formulated in terms of the matrix variables $\mathbf{L}_1$ and $\mathbf{L}_2$: by considering $\mathbf{P}_2$ from (45) and the parameterisation of $\mathbf{G}$ given in (46), after straightforward computation it follows

$$\|\mathbf{G}\| \le \left\| \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \end{bmatrix} \overline{\mathbf{T}}^{-1} \right\| \tag{56}$$

By ensuring by design

$$\left\| \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \end{bmatrix} \overline{\mathbf{T}}^{-1} \right\| < \zeta \tag{57}$$

inequality (54) is satisfied.

The poles of $\mathscr{A}_j$ for $j \in I(1,N)$ can be placed in an convex region of the complex plane $\mathbb{C}$ established by the designer as in [3]. In this paper, a convex region characterised by the intersection of the disk $D(c_n, r_d)$ centered at $(-c_n, 0)$ with radius r_d and a half-plane $H(h)$ delimited by a vertical line at $(-h, 0)$ are considered.

In order to formulate an optimization problem for synthesising the gain matrix $\mathbf{G}$, consider a partition of

$$\hat{\mathscr{A}}_{a\Delta 21j} = \begin{bmatrix} \hat{\mathscr{A}}_{a\Delta 211j} & \hat{\mathscr{A}}_{a\Delta 212j} \end{bmatrix} \text{ for } j \in I(1,N) \tag{58}$$

where $\hat{\mathscr{A}}_{a\Delta 211j} \in \Re^{m\times(n+q-p)}$ and $\hat{\mathscr{A}}_{a\Delta 212j} \in \Re^{m\times(p-m)}$.

Then, choose any

$$\gamma > \max_{j\in I(1,N)} \left\{ ||\hat{\mathscr{A}}_{a\Delta 211j}|| \right\} \tag{59}$$

and solve the following LMI problem:

$$\left.\begin{array}{l} \min\ \zeta \\ s.t. \\ \begin{bmatrix} -\zeta\mathbf{I} & \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \end{bmatrix}\overline{\mathbf{T}}^{-1} \\ \left(\begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \end{bmatrix}\overline{\mathbf{T}}^{-1}\right)^T & -\zeta\mathbf{I} \end{bmatrix} < 0 \\ \begin{bmatrix} -\gamma\mathbf{I} & \mathbf{P}_2\hat{\mathscr{A}}_{a\Delta 21j} - \mathbf{L}_1\mathscr{C} \\ \hat{\mathscr{A}}_{a\Delta 21j}^T\mathbf{P}_2 - \mathscr{C}^T\mathbf{L}_1^T & -\gamma\mathbf{I} \end{bmatrix} < 0 \\ \begin{bmatrix} -r_d\mathbf{P} & \mathbf{P}\mathscr{A}_j + c_n\mathbf{P} \\ c_n\mathbf{P} + \mathscr{A}_j^T\mathbf{P} & -r_d\mathbf{P} \end{bmatrix} < 0 \\ \mathbf{P}\mathscr{A}_j + \mathscr{A}_j^T\mathbf{P} + 2h\mathbf{P} < 0 \\ \mathbf{P} > \mathbf{I} \end{array}\right\} \tag{60}$$

for $j \in I(1,N)$ where $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are the decision variables.

If there exists a feasible solution to the optimization problem (60) then

$$\mathbf{G}_1 = \mathbf{P}_2^{-1}\mathbf{L}_1 \quad \text{and} \quad \mathbf{G}_2 = \mathbf{P}_2^{-1}\mathbf{L}_2 \tag{61}$$

and the proposed control law (41) with (42) and (43) guarantees that the sliding mode takes place inside the sliding patch. Furthermore, the state trajectories will reach the sliding patch in finite time and will remain on it.

IV. Numerical Example

Consider the uncertain dynamical plant

$$\left.\begin{array}{l} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1+\theta(t) & 1 & -1 \\ 1 & -1 & 0 \\ 4 & 0 & 2 \end{bmatrix}\mathbf{x}(t) + \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\big(u(t) + \xi(t,\mathbf{x}(t),\mathbf{u}(t))\big) \\ \mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}(t) \end{array}\right\} \tag{62}$$

where $\theta(t) = 0.2\sin(t)$ is the mismatched uncertain parameter and $\xi(t,\mathbf{x}(t),\mathbf{u}(t)) = 0.5\big(\sin(2\pi t)x_2(t) + \sin(4\pi t)x_3(t)\big)$ corresponds to the matched uncertainty.

The root loci of $(\tilde{\mathbf{A}}_{11j}, \tilde{\mathbf{A}}_{12j}, \mathbf{C}_{1j})$ for $j \in I(1,2)$, considering the extreme values of the mismatched parameter $\theta(t)$, are shown in Fig. 1 and Fig. 2 respectively. Fig. 2 demonstrates that the reduced-order system is not static output feedback stabilisable. Therefore, a dynamical compensator is required to solve the SMC problem.

Defining $\Psi_2 = 1$ and $\Gamma_2 = 1$, the LMI approach proposed in this paper generates the following matrix

$$\mathscr{K} = \begin{bmatrix} -5.8603 & 4.6965 \\ -4.7174 & 3.0759 \end{bmatrix} \tag{63}$$

which determines the compensator and the switching gain matrix Γ.

The convex region is defined through $c_n = 0$, $r_d = 5$ and $h = 0.10$. The gain matrix $\mathbf{G}$ designed using the LMI method developed in this paper is given by

$$\mathbf{G} = \begin{bmatrix} 25.3915 & -23.5923 & 6.8839 \end{bmatrix} \quad (64)$$

where $\mathbf{P}_2 = 1.0000$. The nonlinear part of the control law can be straightforwardly computed from the matched uncertainty $\xi(t,\mathbf{x},\mathbf{u})$.

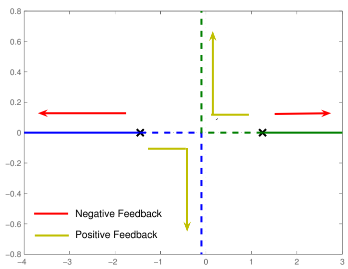

Fig. 1. Root locus for the system triple $(\tilde{\mathbf{A}}_{111},\tilde{\mathbf{A}}_{121},\mathbf{C}_{11})$ for $\theta = \underline{\theta} = -0.2$

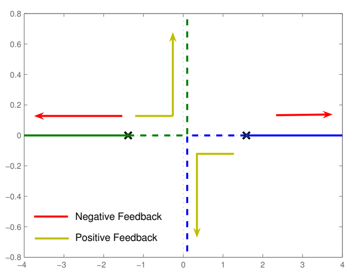

Fig. 2. Root locus for the system triple $(\tilde{\mathbf{A}}_{112},\tilde{\mathbf{A}}_{122},\mathbf{C}_{12})$ for $\theta = \overline{\theta} = +0.2$

Computer simulations were carried out using the initial condition $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$. Time evolution of the output signals $\mathbf{y}(t)$ and the unmeasurable state variable $x_1(t)$ is shown in Fig. 3. The designed SMDOFC stabilises the plant (62) in spite of the mismatched uncertain parameter $\theta(t)$. Fig. 4 depicts the corresponding control signal whilst Fig. 5 shows the switching function.

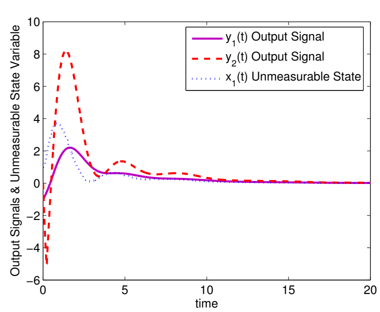

Fig. 3. Response of the uncertain plant using the SMDOF controller

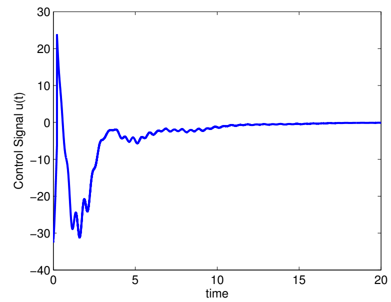

Fig. 4. Control signal $u(t)$

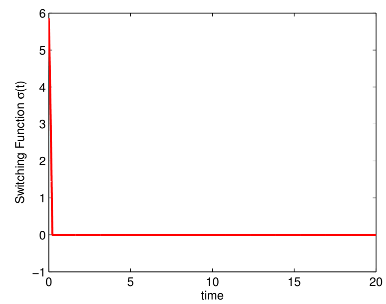

Fig. 5. Time evolution of the switching function $\sigma(t)$

V. Conclusion

This paper has described an LMI design framework for SMDOFCs. Plants with both matched and mismatched uncertainties and partial state information can be dealt with. The SMDOFC is compensator-based and represents an alternative when SMSOF cannot be applied.

VI. ACKNOWLEDGMENTS

J.M. ANDRADE–DA SILVA gratefully acknowledges the support received from the Roberto Rocca Education Program (*www.robertorocca.org*).

References

[1] R. E. Benton Jr. and D. Smith. A non-iterative LMI-based algorithm for robust static-outout-feedback stabilization. *International Journal of Control*, 72(14):1322–1330, 1999.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *SIAM Studies in Applied Mathematics*. Philadelphia - U.S.A., 1994.

[3] M. Chilali and P. Gahinet. $\mathscr{H}_\infty$ design with pole placement constraints: An LMI approach. *IEEE Transactions on Automatic Control*, pages 358–367, 1996.

[4] H. Choi. Variable structure output feedback control design for a class of uncertain dynamic systems. *Automatica*, 38:335–341, 2002.

[5] C. Edwards and S. K. Spurgeon. Sliding mode stabilization of uncertain systems using only output information. *International Journal of Control*, 62(5):1129–1144, 1995.

[6] C. Edwards and S. K. Spurgeon. Linear matrix inequality methods for designing sliding mode output feedback controllers. *IEE Proceedings on Control Theory Applications*, 150(5):539–545, September 2003.

[7] P. Park, D. J. Choi, and S. G Kong. Output feedback variable structure control for linear systems with uncertainties and disturbances. *Automatica*, 43:72–79, 2007.

[8] J. Xiang, W. Wei, and H. Su. An ILMI approach to robust static output feedback sliding mode control. *International Journal of Control*, 79(8):959–967, August 2006.