2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Inverse Optimal Adaptive Control—The Interplay Between Update Laws, Control Laws, and Lyapunov Functions

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***Abstract*— Approaching the problem of optimal adaptive control as "optimal control made adaptive," namely, as a certainty equivalence combination of linear quadratic optimal control and standard parameter estimation, fails on two counts: numerical (as it requires a solution to a Riccati equation at each time step) and conceptual (as the combination actually does not possess any optimality property). In this note we present a particular form of optimality achievable in Lyapunov-based adaptive control. State and control are subject to positive definite penalties, whereas the parameter estimation error is penalized through an exponential of its square, which means that no attempt is made to enforce the parameter convergence, but the estimation transients are penalized simultaneously with the state and control transients. The form of optimality we reveal here is different from our work in [Z. H. Li and M. Krstic, "Optimal design of adaptive tracking controllers for nonlinear systems," *Automatica*, vol. 33, pp. 1459-1473, 1997] where only the terminal value of the parameter error was penalized. We present our optimality concept on a PDE example—boundary control of a particular parabolic PDE with an unknown reaction coefficient. Two technical ideas are central to the developments in the note: a non-quadratic Lyapunov function and a normalization in the Lyapunov-based update law. The optimal adaptive control problem is fundamentally nonlinear and we explore this aspect through several examples that highlight the interplay between the non-quadratic cost and value functions.**

I. INTRODUCTION

For high-dimensional systems, and for PDEs in particular, solving Riccati equations—even once—is a challenging task. Doing so on a real-time basis, at each time step for each new plant parameter estimate, is not feasible (the numerical difficulty is independent of the dimension of the parameter vector—be it scalar or infinite dimensional—and is associated with the high dynamic order of the plant). Even if one is to indulge in solving a new Riccati equation at each time step, one receives no reward for such excessive effort, as the certainty equivalence combination of the standard parameter estimation schemes with linear quadratic optimal control does not possess any optimality property. In fact, even the transient performance of the certainty equivalence adaptive LQR control can be unpredictably poor, with its stability proof being among the most complicated of any adaptive control scheme [7]. Given the lack of both optimality and numerical feasibility of combining Riccati-based feedbacks with parameter adaptation, it is no surprise then that the vast majority of adaptive control for distributed parameter systems [1], [2], [3], [6], [8], [9], [10], [12], [14], [15], [17], [20], [21], [22], [23] are not based on optimal control ideas (with a notable exception of [4], in the stochastic setting).

This work was supported by NSF.

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The only achievement so far of true optimality in adaptive control has been reported in [13]. This was an inverse optimality result, in the spirit of the classical inverse optimality theory for nonlinear systems [16], [18], stated in the context of a parameter-adaptive tracking problem for globally stabilizable nonlinear finite-dimensional systems with uncertain parameters. The interesting aspect of this result was that the adaptive controller was truly a minimizer of a meaningful cost functional, which included, besides the positive definite penalties on the state and control (with complicated nonlinear scaling in terms of the parameter estimate), a simple *terminal penalty on the parameter estimation error*.

In this note we revisit this problem, but in the context of adaptive control of a linear PDE example considered in [12]. This example is of conceptual significance, as it deals with an unstable plant, but for which backstepping design yields an explicit formula for the feedback law, which allows the main point of the note to be made particularly clearly, without being buried under layers of notation. The idea can be generalized to several other linear adaptive control problems but these extensions are not pursued here.

The note's organization and contributions are as follows. After introducing the plant and adaptive controller in Section II, and proving their stability in Section III, the main result (inverse optimality) is stated in Section IV. The optimality is achieved relative to a *cost that penalizes the infinite-time transient of the exponential of the square of the parameter estimation error*, which in turn acts as a weight on the state and control penalty. No such optimality result (despite being both desirable and nearly obvious in retrospect) has been achieved before in adaptive control. The key technical idea behind the result is that the parameter estimator design is based on a particular form of Lyapunov function (which is also the value function of the optimal control problem) that combines the norms of the plant state and of the parameter error in an unusual, non-additive manner. Such a Lyapunov function leads to a normalization of the parameter update law, which is uncommon in Lyapunov-based adaptive control design [11]. In Section V we discuss the relative merits of normalized and unnormalized update laws by exploring different forms of penalty on control and parameter error, for a particular scalar linear ODE example that is worked out explicitly. In Section VI we quantify the effect of update law normalization on transient performance for the example from Section V.

II. An Adaptive Control Design for a PDE Example

We consider the following plant

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t)+\lambda u(x,t), \quad x\in(0,1) && (1)\\
u(0,t) &= 0, && (2)\\
u_x(1,t) &= U(t), && (3)
\end{aligned}
$$

where λ is an unknown constant parameter that can have any real value and $U(t)$ is the boundary control input to be designed. Consider a change of variable [19]

$$
\begin{aligned}
w(x,t,\lambda) &= u(x,t)-\int_0^x k(x,y,\lambda)u(y,t)\,dy, && (4)\\
k(x,y,\lambda) &= -\lambda y\frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} && (5)
\end{aligned}
$$

and its inverse

$$
\begin{aligned}
u(x,t) &= w(x,t,\lambda)+\int_0^x l(x,y,\lambda)w(y,t,\lambda)dy && (6)\\
l(x,y,\lambda) &= -\lambda y\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, && (7)
\end{aligned}
$$

where $I_1(\cdot)$ and $J_1(\cdot)$ are Bessell functions. The change of variable (4), (5), with the unknown parameter λ replaced by its real-time estimate $\hat\lambda(t)$, transforms the system into

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}w\left(x,t,\hat\lambda(t)\right) &= w_{xx}\left(x,t,\hat\lambda(t)\right)\\
&\quad+\dot{\hat\lambda}(t)\int_0^x\frac{y}{2}w\left(y,t,\hat\lambda(t)\right)dy\\
&\quad+\left(\lambda-\hat\lambda(t)\right)w\left(x,t,\hat\lambda(t)\right), && (8)\\
w\left(0,t,\hat\lambda(t)\right) &= 0, && (9)\\
w_x\left(1,t,\hat\lambda(t)\right) &= U(t)-k\left(1,1,\hat\lambda(t)\right)u(1,t)\\
&\quad-\int_0^1 k_x\left(1,y,\hat\lambda(t)\right)u(y,t)\,dy. && (10)
\end{aligned}
$$

In [12] we showed that the adaptive controller

$$
U^\circ(t)=k\left(1,1,\hat\lambda(t)\right)u(1,t)+\int_0^1 k_x\left(1,y,\hat\lambda(t)\right)u(y,t)dy \qquad (11)
$$

with the parameter update law

$$
\dot{\hat\lambda}(t)=\frac{\int_0^1 w\left(x,t,\hat\lambda(t)\right)^2dx}{1+\int_0^1 w\left(x,t,\hat\lambda(t)\right)^2dx} \qquad (12)
$$

is globally stabilizing and achieves regulation of u to zero.

In this note we propose a very different controller and show its inverse optimality. This controller is given by

$$
\begin{aligned}
U^*(t) &= -\left(1+3\left|\hat\lambda(t)\right|+\left|\hat\lambda(t)\right|^2\right)\\
&\quad\times\left(u(1,t)-\int_0^1 k\left(1,y,\hat\lambda(t)\right)u(y,t)dy\right). && (13)
\end{aligned}
$$

III. Stability

Before we state our results, we introduce the following functional:

$$
\begin{aligned}
\mathscr{L}\left(u,\hat\lambda\right)(t) &= 2\int_0^1 w_x\left(x,t,\hat\lambda(t)\right)^2dx\\
&\quad-\frac{\int_0^1 w\left(x,t,\hat\lambda(t)\right)^2dx}{1+\int_0^1 w\left(x,t,\hat\lambda(t)\right)^2dx}\\
&\quad\times\int_0^1 w\left(x,t,\hat\lambda(t)\right)\int_0^x yw\left(y,t,\hat\lambda(t)\right)dydx\\
&\quad+2l\left(1,1,\hat\lambda(t)\right)w\left(1,t,\hat\lambda(t)\right)^2\\
&\quad+2w\left(1,t,\hat\lambda(t)\right)\int_0^1 l_x\left(1,y,\hat\lambda(t)\right)w\left(y,t,\hat\lambda(t)\right)dy\\
&\quad+\left(1+3\left|\hat\lambda(t)\right|+\left|\hat\lambda(t)\right|^2\right)w\left(1,t,\hat\lambda(t)\right)^2. && (14)
\end{aligned}
$$

Lemma 1: For the functional (14) the following holds

$$
\begin{aligned}
\mathscr{L}\left(u,\hat\lambda\right)(t) &\geq \left(1-\frac{2}{\pi^2\sqrt3}\right)\int_0^1 w_x\left(x,t,\hat\lambda(t)\right)^2dx\\
&\geq \frac{2\int_0^1 u(x,t)^2dx}{\left(1+\sup_{0\le y\le x\le 1}\left|l\left(x,y,\hat\lambda(t)\right)\right|\right)^2}. && (15)
\end{aligned}
$$

Proof: The first line of (15) is established by a lengthy but straightforward calculation starting with

$$
\left|\int_0^1 w(x)\left(\int_0^x yw(y)dy\right)dx\right|\le\frac{1}{2\sqrt3}\|w\|^2\le\frac{2}{\pi^2\sqrt3}\|w_x\|^2, \qquad (16)
$$

(where the first inequality was established in [12, Lemma A.1] and the second inequality follows from the Wirtinger inequality [5, p. 182]), using the fact that $2l\left(1,1,\hat\lambda\right)=-\hat\lambda$, and using the fact, proven in [19], that $\int_0^1\left|l_x\left(1,y,\hat\lambda\right)\right|dy\le\hat\lambda+1$, and hence (with the help of the Agmon and Young inequalities)

$$
\begin{aligned}
&2w(1)\int_0^1 l_x\left(1,y,\hat\lambda\right)w(y)\,dy\\
&\geq-\|w_x\|^2-\left(1+\left|\hat\lambda\right|\right)^2w(1)^2. && (17)
\end{aligned}
$$

The second line of (15) follows from (6), the Wirtinger inequality, and using the fact that $\frac{\pi^2}{4}\left(1-\frac{2}{\pi^2\sqrt3}\right)=2.18>2$. ■

Theorem 2: (Stabilization) The closed-loop system consisting of the plant (1)–(3), parameter update law (12), and the controller (13) is globally Lyapunov stable in the sense of the norm $\int_0^1 u(x,t)^2dx+\left(\hat\lambda(t)-\lambda\right)^2$ and, furthermore, $\int_0^1 u(x,t)^2dx\to0, \hat\lambda(t)\to\lambda_\infty$ as $t\to\infty$, where λ_∞ is some constant.

Proof: We use the Lyapunov functional

$$
V(t)=\left(1+\int_0^1 w\left(x,t,\hat\lambda(t)\right)^2dx\right)\mathrm{e}^{\left(\hat\lambda(t)-\lambda\right)^2}-1. \qquad (18)
$$

An easy calculation yields

$$\dot V(t) = -\mathrm{e}^{\left(\hat\lambda(t)-\lambda\right)^2}\mathscr{L}\left(u,\hat\lambda\right)(t) \\ -\mathrm{e}^{\left(\hat\lambda(t)-\lambda\right)^2}\left(1+3\left|\hat\lambda(t)\right|+\left|\hat\lambda(t)\right|^2\right)w\left(1,t,\hat\lambda(t)\right)^2. \tag{19}$$

By deriving an upper and lower bound on (18), from $\dot V \le 0$ we get that

$$\Upsilon(t) \le \mathrm{e}^{\Upsilon(0)} - 1\,, \tag{20}$$

where $\Upsilon(t) = \int_0^1 w\left(x,t,\hat\lambda(t)\right)^2 dx + \left(\hat\lambda(t)-\lambda\right)^2$. The functions $k(x,y,\lambda)$ and $l(x,y,\lambda)$ are continuous and zero at $\lambda = 0$. Therefore, there exist class-$\mathscr{K}$ functions δ and γ such that

$$\sup_{0\le y\le x\le 1} |k(x,y,\theta)| \le \delta(|\theta|) \tag{21}$$

$$\sup_{0\le y\le x\le 1} |l(x,y,\theta)| \le \gamma(|\theta|)\,. \tag{22}$$

Then a lengthy but routine calculation, starting from (20), (21), (22), yields

$$\Omega(t) \le \Sigma\left(1+\gamma(|\lambda|+\Sigma)\right) \tag{23}$$

where

$$\Omega(t) = \left(\int_0^1 u(x,t)^2 dx + \left(\hat\lambda(t)-\lambda\right)^2\right)^{1/2} \tag{24}$$

$$\Sigma(\Omega_0) = \left(\mathrm{e}^{(1+\delta(|\lambda|+\Omega_0))^2\Omega_0^2} - 1\right)^{1/2} \tag{25}$$

$$\Omega_0 = \Omega(0)\,. \tag{26}$$

Since (23) is a class-$\mathscr{K}$ function of Σ, and Σ is a class-$\mathscr{K}$ function of Ω_0, this proves global stability in the norm $\Omega(t)$. The regulation result is argued in a similar way as in [12] (despite the fact that the Lyapunov function is different), using (19) and (15). ■

IV. Inverse Optimality

Theorem 3: (Inverse Optimality) Consider the system consisting of the plant (1)–(3) with the parameter update law (12). The controller (13) minimizes the cost functional

$$J = \lim_{\sigma\to\infty}\left\{\mathrm{e}^{\left(\hat\lambda(\sigma)-\lambda\right)^2} - 1 + \int_0^\sigma \mathrm{e}^{\left(\hat\lambda(t)-\lambda\right)^2} \times \mathscr{M}\left(u,U,\hat\lambda\right)(t)dt\right\}, \tag{27}$$

where

$$\mathscr{M}\left(u,U,\hat\lambda\right)(t) = \mathscr{L}\left(u,\hat\lambda\right)(t) + \frac{U(t)^2}{1+3\left|\hat\lambda(t)\right|+\left|\hat\lambda(t)\right|^2}. \tag{28}$$

The minimum of the cost functional is

$$J^* = \left(1+\int_0^1 w\left(x,0,\hat\lambda(0)\right)^2 dx\right)\mathrm{e}^{\left(\hat\lambda(0)-\lambda\right)^2} - 1\,. \tag{29}$$

The cost on the state and control $\mathscr{M}\left(u,U,\hat\lambda\right)(t)$ is positive definite and underbounded by the following functional:

$$\mathscr{M}\left(u,U,\hat\lambda\right)(t) \ge \mathrm{e}^{-3\left|\hat\lambda(t)\right|}\left(2\int_0^1 u(x,t)^2 dx + U(t)^2\right). \tag{30}$$

Proof: A straightforward calculation yields

$$J = V(0) + \int_0^\infty \mathrm{e}^{\left(\hat\lambda(t)-\lambda\right)^2} \times \frac{\left[U(t)+\left(1+3\left|\hat\lambda(t)\right|+\left|\hat\lambda(t)\right|^2\right)w\left(1,t,\hat\lambda(t)\right)\right]^2}{1+3\left|\hat\lambda(t)\right|+\left|\hat\lambda(t)\right|^2}dt\,, \tag{31}$$

from which the results in the first two statements follow. To prove the third statement we note from [19, Theorem 3] that $\sup_{0\le y\le x\le 1}\left|l\left(x,y,\hat\lambda\right)\right| \le \left|\hat\lambda\right|\mathrm{e}^{2\left|\hat\lambda\right|}$ and then use Lemma 1 and a simple calculation to obtain a lower bound in terms of an exponential of $\left|\hat\lambda(t)\right|$. ■

The cost functional (27), which is optimized by the controller (13), provides a clue as to what form of cost penalties in state, control, and parameter error are meaningful to pursue in possible developments of direct (rather than inverse) optimal adaptive control. Parameter estimation transients are penalized (through an exponential-of-square penalty) but they are not penalized in a way that would demand convergence of the parameter estimate to the true parameter. This is consistent with the fact that parameter convergence requires persistence of excitation, which is normally not present in problems where the state is being regulated to zero. The cost on the plant state and control are quadratic and positive definite, as indicated by (30), but it also involves scaling by the parameter estimate $\hat\lambda$, which is to be expected, and which is not removable without the actual knowledge of the unknown parameter λ.

We want to emphasize the difference between "adaptive inverse optimal control" [19, Section VII] and "inverse optimal adaptive control" in this note. In [19, Section VII] only a case with known plant parameters was considered and adaptation was used to deal with the conservativeness in the inverse optimal design, namely to tune a control gain to a sufficient (but non-conservative) value. Hence, in [19, Section VII] an inverse optimal design was made *adaptive*. In this note, we deal with an unknown parameter case and design an adaptive controller which is *inverse optimal*. This is clearly a stronger result, for a more challenging problem, and optimality holds for the entire parameter-adaptive *nonlinear* system.

V. Variations of Inverse Optimality

Compared to the cost functional in [13] which imposes only a terminal penalty on the parameter error, this note adds the transient penalty on the parameter error in the cost functional (27). This innovation is closely linked with the choice of Lyapunov function (18). To provide further insight into the interplay between the designer's choices of

Lyapunov function, control and update law, and ultimately the inverse optimal cost function, we revisit a simple scalar adaptive control example worked out in [13] and provide several new inverse optimal designs that incorporate different weights on the state and control. (The quantity $\hat{\theta}_\infty$ denotes $\lim_{t\to\infty}\hat{\theta}(t)$, which is easily shown to exist in all of the results stated in this section.)

Theorem 4: (True optimality of pointwise optimal feedback law) Consider the system

$$\dot{x} = u + \theta x \tag{32}$$

and the associated adaptive feedback law

$$u = -\left(\hat{\theta} + \sqrt{\hat{\theta}^2+1}\right)x \tag{33}$$

where $\hat{\theta}(t)$ is the estimate of the unknown constant parameter θ. With the update law

$$\dot{\hat{\theta}} = x^2 \tag{34}$$

the control law (33) is optimal relative to the following cost functional and value (Lyapunov) function:

$$J_{(34)} = \left(\hat{\theta}_\infty - \theta\right)^2 + \int_0^\infty \frac{x(t)^2 + u(t)^2}{\hat{\theta}(t) + \sqrt{\hat{\theta}(t)^2+1}}dt \tag{35}$$

$$V_{(34)} = x^2 + \left(\hat{\theta} - \theta\right)^2, \tag{36}$$

whereas with the update law

$$\dot{\hat{\theta}} = \frac{x^2}{1+x^2} \tag{37}$$

the control law (33) is optimal relative to the following cost functional and value (Lyapunov) function:

$$J_{(37)} = \mathrm{e}^{\left(\hat{\theta}_\infty-\theta\right)^2} - 1 + \int_0^\infty \mathrm{e}^{\left(\hat{\theta}(t)-\theta\right)^2} \frac{x(t)^2 + u(t)^2}{\hat{\theta}(t) + \sqrt{\hat{\theta}(t)^2+1}}dt \tag{38}$$

$$V_{(37)} = \left(1+x^2\right)\mathrm{e}^{\left(\hat{\theta}-\theta\right)^2} - 1\,. \tag{39}$$

Proof: By direct verification. ■

The first half of Theorem 4 was proved in [13], whereas the second one is new. Note the interesting form of the Lyapunov derivative, $\dot{V}_{(37)} = -2x^2\mathrm{e}^{\left(\hat{\theta}-\theta\right)^2}\sqrt{\hat{\theta}^2+1}$, which, even though dependent on the parameter error, is only negative semidefinite, so, as usual, we only get $x(t)\to 0$ as $t \to 0$.

Next, we explore some variations in the forms of the Lyapunov functions like (39) and the cost functionals like (38). The following two theorems give such examples, where slight changes in the update law and the Lyapunov function result in considerably simplified weights on $u(t)$ in the cost functional.

Theorem 5: (Simpler weight on control effort) Consider the system (32) with a parameter estimator given by

$$\dot{\hat{\theta}} = \frac{\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)x^2}{1+\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)x^2}\,. \tag{40}$$

The feedback law

$$u = -4\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)x \tag{41}$$

is the minimizer of the cost functional

$$\begin{aligned} J = \;& \mathrm{e}^{\left(\hat{\theta}_\infty-\theta\right)^2} - 1 \\ &+ \int_0^\infty \mathrm{e}^{\left(\hat{\theta}(t)-\theta\right)^2} \\ &\times\left[\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)^2\left(1+q\left(x(t),\hat{\theta}(t)\right)\right)x(t)^2 \right.\\ &\left.+\frac{1}{4}u(t)^2\right]dt\,, \end{aligned} \tag{42}$$

where $q\left(x,\hat{\theta}\right)$ is a non-negative function given by

$$\begin{aligned} q\left(x,\hat{\theta}\right) =& \frac{1}{2}\frac{\sqrt{\hat{\theta}^2+1}}{\hat{\theta}+\sqrt{\hat{\theta}^2+1}} + 1 \\ &- \frac{x^2}{\sqrt{\hat{\theta}^2+1}\left(1+\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)x^2\right)} \geq 0\,. \end{aligned} \tag{43}$$

Global stability and regulation of $x(t)$ to zero are achieved relative to the Lyapunov function

$$V = \left(1+\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)x^2\right)\mathrm{e}^{\left(\hat{\theta}-\theta\right)^2} - 1 \tag{44}$$

$$\dot{V} = -\left(\hat{\theta}+\sqrt{\hat{\theta}^2+1}\right)\left(5+q\left(x,\hat{\theta}\right)\right)x^2\mathrm{e}^{\left(\hat{\theta}-\theta\right)^2}\,. \tag{45}$$

Proof: By direct verification. ■

Theorem 6: (Simpler weight on control effort, with simpler control law) Consider the system (32) with a parameter estimator given by

$$\dot{\hat{\theta}} = \frac{\left(1+\left|\hat{\theta}\right|\right)x^2}{1+\left(1+\left|\hat{\theta}\right|\right)x^2}\,. \tag{46}$$

The feedback law

$$u = -2\left(1+\left|\hat{\theta}\right|\right)x \tag{47}$$

is the minimizer of the cost functional

$$\begin{aligned} J = \;& \mathrm{e}^{\left(\hat{\theta}_\infty-\theta\right)^2} - 1 \\ &+ \int_0^\infty \mathrm{e}^{\left(\hat{\theta}(t)-\theta\right)^2} \\ &\times\left[\left(1+q\left(x(t),\hat{\theta}(t)\right)\right)x(t)^2 + \frac{1}{2}u(t)^2\right]dt\,, \end{aligned} \tag{48}$$

where $q\left(x,\hat{\theta}\right)$ is a non-negative function given by

$$q\left(x,\hat{\theta}\right) = 2\left(\left|\hat{\theta}\right| - \hat{\theta}\right) + 1 - \frac{x^2\,\mathrm{sgn}\,\hat{\theta}}{1+\left(1+\left|\hat{\theta}\right|\right)x^2} \geq 0\,. \tag{49}$$

Global stability and regulation of $x(t)$ to zero are achieved relative to the Lyapunov function

$$V = \left(1+\left(1+\left|\hat{\theta}\right|\right)x^2\right)\mathrm{e}^{\left(\hat{\theta}-\theta\right)^2} - 1 \tag{50}$$

$$\dot{V} = -\left(1+q\left(x,\hat{\theta}\right)+2\left(1+\left|\hat{\theta}\right|\right)^2\right)x^2\mathrm{e}^{\left(\hat{\theta}-\theta\right)^2}\,. \tag{51}$$

Proof: By direct verification. ■

In the designs in Theorems 5 and 6, the price paid for the reduced complexity in the weight on $u(t)$ is in the increased complexity in the Lyapunov function, which translates into the increased complexity of the update law and of the weight on $x(t)$ in the cost functional.

The designs in Theorems 5 and 6 both attempt to eliminate the scaling of the costs on x and u in terms of $\hat{\theta}$ but neither succeeds. It is possible in principle to eliminate this scaling, however, the redesigned feedback would have to employ the knowledge of the unknown θ. This obstacle is probably fundamental, and related to the fact that adaptive stabilization is not a full-state stabilization problem (the parameter, which can be treated as a constant state, is unmeasured).

We emphasize that all three control laws, (33), (41), and (47), as well as the simpler, non-optimal control law,

$$u = -\left(1+\hat{\theta}\right)x, \tag{52}$$

are all globally stabilizing also with the update law (34), relative to the Lyapunov function (36), as well as with the update law (37), relative to the Lyapunov functions (39) and $V = \ln\left(1+x^2\right) + \left(\hat{\theta}-\theta\right)^2$.

The above inverse optimality results in Theorems 4, 5, 6 inspire an attempt to approach a direct optimal control problem for the plant (32). In inverse optimal designs the order in which choices are made is as follows: (i) Lyapunov/value function, (ii) feedback law, (iii) cost functional. We now approach a direct optimal control problem where we start with a cost functional

$$J = \lim_{t\to\infty}\Big\{V\left(x(t),\hat{\theta}(t)\right) + \int_0^t e^{\left(\hat{\theta}(t)-\theta\right)^2}\left[x(t)^2+u(t)^2\right]dt\Big\}, \tag{53}$$

and a postulated form of a value function,

$$V = \left(1+p\left(x,\hat{\theta}\right)x^2\right)e^{\left(\hat{\theta}-\theta\right)^2} - 1, \tag{54}$$

where $p\left(x,\hat{\theta}\right)$ is a positive, continuously differentiable function to be found, and then derive the optimal adaptive controller. We first find that the function $p\left(x,\hat{\theta}\right)$ satisfies the nonlinear partial differential equation

$$\left(p+\frac{x}{2}p_x\right)^2 - \frac{2\hat{\theta}+x^2\left(2\hat{\theta}p+p_{\hat{\theta}}\right)}{1+x^2p}\left(p+\frac{x}{2}p_x\right) = 1, \tag{55}$$

with an additional condition that $p\left(0,\hat{\theta}\right) = \hat{\theta}+\sqrt{\hat{\theta}^2+1}$. Then the optimal adaptive controller is given by

$$u^* = -\left(p+\frac{x}{2}p_x\right)x \tag{56}$$

$$\dot{\hat{\theta}} = \frac{\left(p+\frac{x}{2}p_x\right)x^2}{1+px^2}. \tag{57}$$

Obviously, the big open questions are the global existence and the numerical computation of the solution to the nonlinear PDE (55).

VI. Transient Performance Effect of Update Law Nomalization

Theorem 4 highlights the difference between normalized and unnormalized update laws. While update law normalization is a common (and in some cases an essential) tool in swapping (or estimation error/certainty equivalence) based approaches to adaptive control, within the framework of Lyapunov-based adaptive control [11] normalized update laws are uncommon. The form of Lyapunov function in this paper lends justification to the use of normalization with Lyapunov update laws.

Even though it is useful to have one more tool in the design toolkit for adaptive feedback systems, we are not necessarily claiming that there is absolute advantage in using update law normalization. To understand the tradeoff, consider again the scalar ODE plant (32) but with the simpler (non-optimal) control law (52). The Lyapunov functions (36) and (39), respectively, yield the update laws (34) and (37). For the two respective closed-loop systems one can find that the solutions satisfy the following two relations. Under the update law (34) the trajectories satisfy

$$x(t)^2 + \left(\hat{\theta}(t)-\theta+1\right)^2 = x_0^2 + \left(\hat{\theta}_0-\theta+1\right)^2, \tag{58}$$

whereas under the update law (37) the trajectories satisfy

$$\ln\left(1+x(t)^2\right) + \left(\hat{\theta}(t)-\theta+1\right)^2 \tag{59}$$

$$= \ln\left(1+x_0^2\right) + \left(\hat{\theta}_0-\theta+1\right)^2, \tag{60}$$

where $\left(x_0,\hat{\theta}_0\right)$ is the initial condition. Using (58) and (59) the following is obtained.

Theorem 7: (Normalized versus unnormalized update law) Consider the closed-loop systems consisting of the plant (32), the controller (52), and respectively, the update laws (34) and (37). Let $\theta \geq 1$ and $\hat{\theta}_0 = 0$. The following is true:

$$\inf_{x_0\neq 0}\sup_{t\geq 0} x(t,x_0)^2\Big|_{(34)} = (\theta-1)^2 \tag{61}$$

$$\inf_{x_0\neq 0}\sup_{t\geq 0} x(t,x_0)^2\Big|_{(37)} = e^{(\theta-1)^2}-1. \tag{62}$$

Proof: By observing that in both (58) and (59) the peak in $x(t)$ (in the phase space) is achieved at time t for which $\hat{\theta}(t) = \theta - 1$. ■

This theorem says the following. When the instability parameter θ is positive and large, and when the initial parameter estimate $\hat{\theta}_0$ is zero and thus clearly not of stabilizing value, then, the adaptation transient that the system undergoes, as measured by the peak of the state x, is larger with the normalized update law than with the unnormalized update law. This advantage of the unnormalized update law is not unexpected. The absence of normalization allows the update law to act more aggressively and to deliver stabilizing values of the control gain in a shorter period of time, resulting in a smaller peak of the state transient.

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