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Adaptive Robust Control of Uncertain Neutral Time-Delay Systems

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Abstract— **In this paper, the problem of adaptive robust stabilization of neutral time-delay systems with uncertainties is investigated. It is assumed that the system parameters are subject to uncertainties with unknown bounds, but with known functional properties. A novel memoryless adaptive robust state feedback controller is proposed, and an adaptive scheme is introduced to estimate the bounds on uncertainties. A set of time-dependent trajectories which follow specific switching dynamic equations are utilized in the adaptive scheme and in the adaptive robust control input to achieve the stability of the closed-loop system. It is shown that by applying the proposed adaptive robust control input the states of the resultant closed-loop uncertain time-delay system will be uniformly ultimately bounded. The simulation results elucidate the effectiveness of the proposed approach.**

I. INTRODUCTION

In recent years, there has been a burst of research activities in the area of time-delay control systems. Problems of this type appear, for example, in process control systems, communication networks and power systems, to name only a few [2], [3], [4]. It is known that neglecting time-delay in the dynamics of the system in control design can lead to the degradation of the control performance, and may even cause instability.

Time-delay systems can be classified in two different categories: the ones expressed by retarded functional differential equations (RFDE) and the ones described by neutral functional differential equations (NFDE). RFDEs involve only delay in the state, whereas NFDEs involve also derivatives of the state with delays (see, e.g., [1], [2]).

In a practical environment, time-delay in the dynamics of the system can be uncertain or time-varying. Furthermore, the parameters of the system are often subject to perturbation, uncertainties, and unmodeled dynamics. Robust stabilization of time-delay systems has been extensively investigated in the literature; e.g., see [4], [5], [6], [7], [8], [9], [10] and the references therein.

Uncertain RFDE systems have been studied in [11], [12], [13], [14]. A typical assumption in such problems is that the upper bounds of the uncertainties are unknown. Thus, an adaptive robust controller is employed to estimate the upper bound of the uncertainties and consequently stabilize the uncertain time-delay system. Various techniques are proposed in the prior literature to design adaptive robust controllers to stabilize the uncertain RFDE systems asymptotically as well as exponentially. To the best of the knowledge of the authors, however, there are very few results on the stabilization of NFDE systems (due to their complexity) when the information on the upper bounds of the uncertainties is unknown. An adaptive robust control scheme was proposed in [15] to stabilize uncertain neutral-type time-delay systems. However, the drawbacks of that work are that it assumes the delay is known, and that it utilizes the delayed state of the system in the adaptation process.

In this paper, the uncertain neutral time-delay systems are investigated, and a novel adaptive memoryless state feedback control scheme is proposed to uniformly ultimately stabilize the system. It is assumed that the upper bounds of the uncertainties are unknown. First, using the improved adaptation laws with σ-modification and a set of time-varying parameters which are governed by some dynamic equations, the upper bounds of the uncertainties are estimated. Then, using the updated parameters and the above mentioned time-varying parameters, a new memoryless state feedback adaptive control input is introduced to robustly stabilize the uncertain neutral time-delay system. It is proved that the state of the resultant closed-loop system is uniformly ultimately bounded.

The remainder of the paper is organized as follows. The problem statement and some essential assumptions are given in Section II. An adaptive robust control design technique is introduced in Section III as the main result of the paper. Simulations are presented in Section IV to demonstrate the efficacy of the proposed method, and finally some concluding remarks are drawn in Section V.

II. PROBLEM FORMULATION

Consider the uncertain neutral type time-delay system described by the following differential equation

$$\begin{aligned}\dot{x}(t)+D\dot{x}(t-h) &= (A_0+\Delta A_0(\xi(t)))x(t)\\ &+ (A_1+\Delta A_1(\xi(t)))x(t-h)+Bu(t)+\omega(t)\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector, $\omega(t) \in R^n$ is the disturbance vector, $D,\ A_0,\ A_1 \in R^{n\times n}$ are system matrices, and $\Delta A_0(\cdot),\ \Delta A_1(\cdot)$ represent the system uncertainties. Moreover, h is the unknown time-delay, which is assumed to be a positive constant value.

The initial condition for the system (1) is expressed as

$$x(t)=\phi(t), \qquad t\in[t_0-h,\ t_0] \tag{2}$$

where $\phi(t)$ is a continuous function on $[t_0-h,\ t_0]$. To proceed further, the following assumptions are made.

Assumption 1: The pair $\{A_0,\ B\}$ given in (1) is completely controllable.

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Assumption 2: All admissible uncertainties can be described as follows

$$\Delta A_0(\xi(t)) = E_1(\xi(t))F \tag{3a}$$
$$\Delta A_1(\xi(t)) = E_2(\xi(t))F \tag{3b}$$
$$\omega(t) = BE_3(t) \tag{3c}$$

where $E_1(\cdot), E_2(\cdot)$ and $E_3(\cdot)$ are unknown matrices with appropriate dimensions and the following bounds

$$\max_{\xi\in\Omega} ||E_1(\xi)|| \leq \alpha_1, \quad \max_{\xi\in\Omega} ||E_2(\xi)|| \leq \alpha_2, \tag{4a}$$

$$\max ||E_3(t)|| \leq \alpha_3 \tag{4b}$$

(α_1, α_2 and α_3 are positive constants), and F is a known matrix with appropriate dimension. The uncertainty $\xi(t) \in \Omega \subset R^k$ is Lebesgue measurable and takes values in known compact bounded set Ω.

It is to be noted, from Assumption 1, that for any given positive definite matrix $Q \in R^{n\times n}$, there exists a unique positive definite symmetric matrix $P \in R^{n\times n}$ such that the following algebraic Riccati equation is satisfied

$$A_0^T P + PA_0 - \omega PBB^T P = -Q \tag{5}$$

where ω is a given positive constant. Define

$$\rho := \frac{\omega}{2} \tag{6}$$

It is desired now to design a state feedback controller to stabilize the closed-loop uncertain time-delay system.

III. Adaptive Robust Controller Design

Consider the system (1) and suppose that the conditions of Assumptions 1 and 2 are met. The following state feedback controller is proposed

$$u(t) = -\rho B^T Px(t) + \tilde{p}_1(t) + \tilde{p}_2(t) \tag{7}$$

where ρ is defined in (6) and

$$\tilde{p}_1(t) = -\frac{1}{2}\hat{\theta}_1(t)B^T Px(t) - \frac{1}{2}\mu_1(t)B^T Px(t) \tag{8a}$$

$$\tilde{p}_2(t) = \begin{cases} \frac{-\hat{\theta}_2(t)B^T Px(t)}{||B^T Px(t)||} - \mu_2(t)B^T Px(t), \\ \qquad\qquad ||B^T Px(t)|| \geq \delta_x \\ \frac{-\hat{\theta}_2(t)B^T Px(t)}{||B^T Px(t)||+\sigma^2(t)} - \mu_2(t)B^T Px(t), \\ \qquad\qquad ||B^T Px(t)|| < \delta_x \end{cases} \tag{8b}$$

($\hat{\theta}_1(t), \hat{\theta}_2(t), \mu_1(t), \mu_2(t)$ and $\sigma(t)$ will be introduced later). δ_x in the above equation is a positive constant design parameter, $P \in R^{n\times n}$ is the solution of the Riccati equation (5), such that the following LMIs with respect to positive constants ϵ_i, $i = 1, ..., 9$, the positive definite matrices Q_1, Q_2, $Q_3 \in R^{n\times n}$ and given positive definite matrix $Q \in R^{n\times n}$ and positive constant ρ is satisfied

$$\tilde{\mathcal{E}} = \begin{bmatrix} \tilde{\mathcal{E}}_{11} & \tilde{\mathcal{E}}_{12} & \tilde{\mathcal{E}}_{13} \\ * & \tilde{\mathcal{E}}_{22} & \tilde{\mathcal{E}}_{23} \\ * & * & \tilde{\mathcal{E}}_{33} \end{bmatrix} < 0 \tag{9a}$$

$$PBQ_3B^T P \geq F^T F \tag{9b}$$

where $\tilde{\mathcal{E}}_{11} = -Q + D^T Q_2 D + P(\epsilon_1 + \epsilon_2)P$, $\tilde{\mathcal{E}}_{12} = PA_1$, $\tilde{\mathcal{E}}_{13} = -PB\rho B^T P + A_0^T P$, $\tilde{\mathcal{E}}_{22} = -Q_1$, $\tilde{\mathcal{E}}_{23} = A_1^T P$, $\tilde{\mathcal{E}}_{33} = -Q_2 + \frac{1}{2}\sum_{i=5}^{10} PB\epsilon_i B^T P + \sum_{i=3}^{4} P\epsilon_i P + \frac{1}{2}PB\epsilon_9 B^T P$.

Define now

$$\theta_1^* := \lambda_{max}(Q_3)[(\epsilon_1^{-1} + \epsilon_4^{-1})\alpha_1^2 + (\epsilon_2^{-1} + \epsilon_3^{-1})\alpha_2^2] \tag{10a}$$
$$\theta_2^* := 2\alpha_3 \tag{10b}$$

where α_1, α_2 and α_3 are defined in (4). Furthermore, $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ in (8) represent the estimates of the unknown parameters θ_1^* and θ_2^*, respectively, which are updated through the following adaptation laws

$$\frac{d\hat{\theta}_1(t)}{dt} = \frac{1}{2}\gamma_1||x^T(t)PB|| - \gamma_1\hat{\theta}_1(t) \tag{11a}$$

$$\frac{d\hat{\theta}_2(t)}{dt} = \begin{cases} \frac{1}{2}\gamma_2||x^T(t)PB|| - \gamma_2\hat{\theta}_2(t), & ||B^T Px(t)|| \geq \delta_x \\ \frac{1}{2}\gamma_2\frac{||x^T(t)PB||}{||x^T(t)PB||+\sigma^2(t)} + \frac{1}{2}\gamma_2\frac{\sigma^2(t)}{||x^T(t)PB||+\sigma^2(t)} - \gamma_2\hat{\theta}_2(t), \\ \qquad ||B^T Px(t)|| < \delta_x \end{cases} \tag{11b}$$

where $\sigma(t)$ follows the dynamic equation given below

$$\dot{\sigma}(t) = \begin{cases} -\sigma(t), & ||B^T Px(t)|| \geq \delta_x \\ -\frac{\sigma(t)\hat{\theta}_2(t)}{||x^T(t)PB||+\sigma^2(t)} - \sigma(t), & ||B^T Px|| < \delta_x,\ \sigma(t) \geq \delta_\sigma \\ -\frac{\sigma(t)\hat{\theta}_2(t)}{||x^T(t)PB||+\sigma^2(t)} - \sigma(t) + \epsilon_\sigma, & ||B^T Px|| < \delta_x, \sigma(t) < \delta_\sigma \end{cases} \tag{12}$$

In addition, $\mu_1(t)$ and $\mu_2(t)$ in (8) are updated through the dynamic equations in (13). In (13) ϵ_{μ_1}, ϵ_{μ_2}, δ_{μ_1} and δ_{μ_2} are design positive constants. Moreover, $\eta_1(t)$ and $\eta_2(t)$ in (13) are updated through the dynamic equations in (14). Note in (14) that ϵ_{η_1}, ϵ_{η_2}, δ_{η_1}, δ_{η_2} are positive constant to be designed. Note also that the initial conditions $\eta_1(t_0)$, $\eta_2(t_0)$, $\mu_1(t_0)$, $\mu_2(t_0)$ and $\sigma(t_0)$ can be any positive real numbers.

Remark 1: It is to be noted that the dynamic equations introduced in (12)-(14) guarantee that the time-varying variables $\mu_1(t)$, $\mu_2(t)$, $\eta_1(t)$, $\eta_2(t)$ and $\sigma(t)$ remain positive and do not cross zero.

The following theorem addresses the stability of the system (1) with the input defined in (7).

Theorem 1: Consider the uncertain time-delay system (1). Suppose that the conditions of Assumptions 1 and 2 are satisfied. The system (1) with the control input (7) is uniformly ultimately stable if there exist a positive definite matrix $P \in R^{n\times n}$ which satisfies the Riccati equation (5), positive constants ϵ_i, and positive definite matrices $Q_1 \in R^{n\times n}$ and $Q_2 \in R^{n\times n}$ such that the LMIs in (9) becomes

$$\frac{d\mu_1(t)}{dt} = \begin{cases} ||x^T(t)PB||^2 - \frac{1}{2\epsilon_6}\mu_1(t)||x^T(t)PB||^2 - \frac{1}{2\epsilon_5}\frac{\mu_1(t)\hat{\theta}_1^2(t)||x^T(t)PB||^2}{\mu_1^2(t)+\eta_1(t)^2} - \mu_1(t), & \mu_1(t) \geq \delta_{\mu_1} \\ ||x^T(t)PB||^2 - \frac{1}{2\epsilon_6}\mu_1(t)||x^T(t)PB||^2 - \frac{1}{2\epsilon_5}\frac{\mu_1(t)\hat{\theta}_1^2(t)||x^T(t)PB||^2}{\mu_1^2(t)+\eta_1(t)^2} - \mu_1(t) + \epsilon_{\mu_1}, & \mu_1(t) < \delta_{\mu_1} \end{cases} \tag{13a}$$

$$\frac{d\mu_2(t)}{dt} = \begin{cases} ||x^T(t)PB||^2 - \frac{1}{2\epsilon_7}\mu_2(t)||x^T(t)PB||^2 - \frac{1}{2\epsilon_8}\frac{\mu_2(t)\hat{\theta}_2^2(t)}{\mu_2^2(t)+\eta_2(t)^2} - \mu_2(t), & \mu_2(t) \geq \delta_{\mu_2} \\ ||x^T(t)PB||^2 - \frac{1}{2\epsilon_7}\mu_2(t)||x^T(t)PB||^2 - \frac{1}{2\epsilon_8}\frac{\mu_2(t)\hat{\theta}_2^2(t)}{\mu_2^2(t)+\eta_2(t)^2} - \mu_2(t) + \epsilon_{\mu_2}, & \mu_2(t) < \delta_{\mu_2} \end{cases} \tag{13b}$$

$$\frac{d\eta_1(t)}{dt} = \begin{cases} -\frac{1}{\epsilon_5}\frac{\eta_1(t)\hat{\theta}_1^2(t)||x^T(t)PB||^2}{\eta_1(t)^2+\mu_1(t)^2} - \eta_1(t) & , \ \eta_1(t) \geq \delta_{\eta_1} \\ -\frac{1}{\epsilon_5}\frac{\eta_1(t)\hat{\theta}_1^2(t)||x^T(t)PB||^2}{\eta_1(t)^2+\mu_1(t)^2} - \eta_1(t) + \epsilon_{\eta_1} & , \ \eta_1(t) < \delta_{\eta_1} \end{cases} \tag{14a}$$

$$\frac{d\eta_2(t)}{dt} = \begin{cases} -\frac{1}{\epsilon_8}\frac{\eta_2(t)\hat{\theta}_2^2(t)}{\eta_2(t)^2+\mu_2(t)^2} - \eta_2(t) & , \ \eta_2(t) \geq \delta_{\eta_2} \\ -\frac{1}{\epsilon_8}\frac{\eta_2(t)\hat{\theta}_2^2(t)}{\eta_2(t)^2+\mu_2(t)^2} - \eta_2(t) + \epsilon_{\eta_2} & , \ \eta_2(t) < \delta_{\eta_2} \end{cases} \tag{14b}$$

negative definite for a given positive constant ρ and a positive definite matrix $Q \in R^{n\times n}$.

Proof: In the following the proof is provided separately for the two cases: $||x^T(t)PB|| \geq \delta_x$ and $||x^T(t)PB|| < \delta_x$.

Case 1: $||x^T(t)PB|| \geq \delta_x$. Substituting control input (7) with $||x^T(t)PB|| \geq \delta_x$ into (1) leads to the following closed-loop system

$$\begin{aligned}\dot{x}(t) + D\dot{x}(t-h) = &(A_0 + \Delta A_0(t))x(t) - \rho BB^T Px(t) \\ &+ (A_1 + \Delta A_1(t))x(t-h) - \hat{\theta}_1(t)BB^TPx(t) \\ &- \mu_1(t)BB^TPx(t) - \frac{\hat{\theta}_2(t)BB^TPx(t)}{||B^TPx(t)||} \\ &- \mu_2(t)BB^TPx(t) + \omega(t)\end{aligned} \tag{15}$$

Choosing the following positive definite Lyapunov-Krasovsky functional candidate

$$\begin{aligned}V =&[x(t)+Dx(t-h)]^TP[x(t)+Dx(t-h)] \\ &+ \int_{t-h}^t x^T(s)Q_1x(s)ds + \int_{t-h}^t x^T(s)D^TQ_2Dx(s)ds \\ &+ (\epsilon_2^{-1}+\epsilon_3^{-1})\alpha_2^2\int_{t-h}^t x^T(s)PBQ_3B^TPx(s)ds \\ &+ \gamma_1^{-1}\tilde{\theta}_1^2(t) + \gamma_2^{-1}\tilde{\theta}_2^2(t) \\ &+ \frac{1}{2}\mu_1^2(t) + \frac{1}{2}\mu_2^2(t) + \frac{1}{4}\eta_1^2(t) + \frac{1}{4}\eta_2^2(t) + \frac{1}{2}\sigma^2(t)\end{aligned} \tag{16}$$

where P, Q_1 and Q_2 are symmetric positive definite matrices and ϵ_3 is a strictly positive constant, as mentioned earlier. Define the parameter estimation errors as follows

$$\tilde{\theta}_1(t) := \hat{\theta}_1(t) - \theta_1^*, \qquad \tilde{\theta}_2(t) := \hat{\theta}_2(t) - \theta_2^* \tag{17}$$

(note that θ_1^* and θ_2^* are defined in (10), and that $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ are their corresponding estimates).

For $t \geq t_0$, the derivative of the Lyapunov-Krasovsky functional introduced in (16) is obtained as follows

$$\begin{aligned}\dot{V} =&2\left[x(t)+Dx(t-h)\right]^T P\left[\dot{x}(t)+D\dot{x}(t-h)\right] \\ &+ x^T(t)Q_1x(t) - x^T(t-h)Q_1x(t-h) \\ &+ x^T(t)D^TQ_2Dx(t) - x^T(t-h)D^TQ_2Dx(t-h) \\ &+ x^T(t)PBQ_3B^TPx(t)(\epsilon_2^{-1}+\epsilon_3^{-1})\alpha_2^2 \\ &- x^T(t-h)PBQ_3B^TPx(t-h)(\epsilon_2^{-1}+\epsilon_3^{-1})\alpha_2^2 \\ &+ 2\gamma_1^{-1}\tilde{\theta}_1(t)\dot{\hat{\theta}}_1(t) + 2\gamma_2^{-1}\tilde{\theta}_2(t)\dot{\hat{\theta}}_2(t) + \mu_1(t)\dot{\mu}_1(t) \\ &+ \mu_2(t)\dot{\mu}_2(t) + \frac{1}{2}\eta_1(t)\dot{\eta}_1(t) + \frac{1}{2}\eta_2(t)\dot{\eta}_2(t) + \sigma(t)\dot{\sigma}(t)\end{aligned} \tag{18}$$

Substituting (15) (with $||x^T(t)PB|| \geq \delta_x$) in (18) and using (3) yield

$$\begin{aligned}\dot{V} =&\mathcal{X}^T(t)\tilde{\mathcal{E}}\mathcal{X}(t) + 2x^T(t)PE_1Fx(t) + 2x^T(t)PBE_2Fx(t-h) \\ &+ 2x^T(t-h)D^TPE_1Fx(t) \\ &+ 2x^T(t-h)D^TPE_2Fx(t-h) \\ &+ x^T(t)D^TQ_2Dx(t) - \hat{\theta}_1(t)||x^T(t)PB||^2 \\ &- \mu_1(t)||x^T(t)PB||^2 - \hat{\theta}_2(t)||x^T(t)PB|| \\ &- \mu_2(t)||x^T(t)PB||^2 - \hat{\theta}_1(t)x^T(t-h)D^TPBB^TPx(t) \\ &- \mu_1(t)x^T(t-h)D^TPBB^TPx(t) \\ &- \frac{\hat{\theta}_2(t)x^T(t-h)D^TPBB^TPx(t)}{||x^T(t)PB||} \\ &- \mu_2(t)x^T(t-h)D^TPBB^TPx(t) \\ &+ x^T(t)PBQ_3B^TPx(t)(\epsilon_2^{-1}+\epsilon_3^{-1})\alpha_2^2 \\ &- x^T(t-h)PBQ_3B^TPx(t-h)(\epsilon_2^{-1}+\epsilon_3^{-1})\alpha_2^2 \\ &+ 2x^T(t)PBE_3(t) + 2x^T(t-h)D^TPBE_3(t) \\ &+ 2\gamma_1^{-1}\tilde{\theta}_1(t)\dot{\hat{\theta}}_1(t) + 2\gamma_2^{-1}\tilde{\theta}_2(t)\dot{\hat{\theta}}_2(t) + \mu_1(t)\dot{\mu}_1(t) \\ &+ \mu_2(t)\dot{\mu}_2(t) + \frac{1}{2}\eta_1(t)\dot{\eta}_1(t) + \frac{1}{2}\eta_2(t)\dot{\eta}_2(t) + \sigma(t)\dot{\sigma}(t)\end{aligned} \tag{19}$$

where

$$\mathcal{X}(t) = [x^T(t), x^T(t-h), x^T(t-h)D^T]^T \tag{20a}$$

$$\tilde{\mathcal{E}} = \begin{bmatrix} -Q + D^TQ_2D & PA_1 & -PB\rho B^TP + A_0^TP \\ * & -Q_1 & A_1^TP \\ * & * & -Q_2 \end{bmatrix} \tag{20b}$$

Note that there exist positive constants ϵ_i, $i = 1, ..., 8$ such that using (4), one can write the following inequalities

$$2x^T(t)PE_1Fx(t) \leq \lambda_{max}(Q_3)\alpha_1^2\epsilon_1^{-1}||x^T(t)PB||^2 + x^T(t)P\epsilon_1 Px(t) \tag{21a}$$

$$2x^T(t)PE_2Fx(t-h) \leq \alpha_2^2\epsilon_2^{-1}x^T(t-h)PBQ_3B^TPx(t-h) + x^T(t)P\epsilon_2 Px(t) \tag{21b}$$

$$2x^T(t-h)D^TPE_2Fx(t-h) \leq x^T(t-h)D^TP\epsilon_3 PDx(t-h) + \alpha_2^2\epsilon_3^{-1}x^T(t-h)^TPBQ_3B^TPx(t-h) \tag{21c}$$

$$2x^T(t-h)D^TPE_1Fx(t) \leq x^T(t-h)D^TP\epsilon_4 PDx(t-h) + \alpha_1^2\epsilon_4^{-1}\lambda_{max}(Q_3)||x^T(t)PB||^2 \tag{21d}$$

$$-\hat{\theta}_1(t)x^T(t-h)DPBB^TPx(t) \leq \frac{1}{2}x^T(t-h)D^TPB\epsilon_5 B^TPDx(t-h) + \frac{1}{2}\epsilon_5^{-1}\hat{\theta}_1^2(t)||x^T(t)PB||^2 \tag{21e}$$

$$-\mu_1(t)x^T(t-h)D^TPBB^TPx(t) \leq \frac{1}{2}x^T(t-h)D^TPB\epsilon_6 B^TPBDx(t-h) + \frac{1}{2}\epsilon_6^{-1}\mu_1^2(t)||x^T(t)PB||^2 \tag{21f}$$

$$-\mu_2(t)x^T(t-h)D^TPBB^TPx(t) \leq \frac{1}{2}x^T(t-h)D^TPB\epsilon_7 B^TPDx(t-h) + \frac{1}{2}\epsilon_7^{-1}\mu_2^2(t)||x^T(t)PB||^2 \tag{21g}$$

$$-\frac{\hat{\theta}_2(t)x^T(t-h)D^TPBB^TPx(t)}{||B^TPx(t)||} \leq \frac{1}{2}x^T(t-h)D^TPB\epsilon_8 B^TPDx(t-h) + \frac{1}{2}\epsilon_8^{-1}\hat{\theta}_2^2 \tag{21h}$$

$$2x^T(t)PBE_3 \leq 2||x^T(t)PB||\alpha_3 \tag{21i}$$

$$2x^T(t-h)D^TPBE_3 \leq \epsilon_9^{-1}\alpha_3^2 + x^T(t-h)D^TPB\epsilon_9 B^TPDx(t-h) \tag{21j}$$

Substituting (21) in (19) yields

$$\begin{aligned}
\dot{V} \leq& \mathcal{X}^T(t)\hat{\mathcal{E}}\mathcal{X}(t) - \tilde{\theta}_1(t)||x^T(t)PB||^2 + 2\gamma_1^{-1}\tilde{\theta}_1(t)\dot{\hat{\theta}}_1(t) \\
&- \tilde{\theta}_2(t)||x^T(t)PB|| + 2\gamma_2^{-1}\tilde{\theta}_2(t)\dot{\hat{\theta}}_2(t) \\
&- \mu_1(t)||x^T(t)PB||^2 + \frac{1}{2}\epsilon_5^{-1}\hat{\theta}_1^2(t)||x^T(t)PB||^2 \\
&+ \frac{1}{2}\epsilon_6^{-1}\mu_1^2(t)||x^T(t)PB||^2 + \mu_1(t)\dot{\mu}_1(t) \\
&- \mu_2(t)||x^T(t)PB||^2 + \frac{1}{2}\epsilon_7^{-1}\mu_2^2(t)||x^T(t)PB||^2 \\
&+ \frac{1}{2}\epsilon_8^{-1}\hat{\theta}_2^2(t) + \mu_2(t)\dot{\mu}_2(t) \\
&+ \frac{1}{2}\eta_1(t)\dot{\eta}_1(t) + \frac{1}{2}\eta_2(t)\dot{\eta}_2(t) + \sigma(t)\dot{\sigma}(t) + \epsilon_9^{-1}\alpha_3^2
\end{aligned} \tag{22}$$

where $\tilde{\theta}_1(t)$ and $\tilde{\theta}_2(t)$ are defined in (17). Furthermore,

$$\hat{\mathcal{E}} = \begin{bmatrix} \hat{\mathcal{E}}_{11} & PA_1 & -PB\rho B^TP + A_0^TP \\ * & -Q_1 & A_1^TP \\ * & * & \hat{\mathcal{E}}_{33} \end{bmatrix} \tag{23}$$

where $\hat{\mathcal{E}}_{11} = -Q + Q_1 + D^TQ_2D + P(\epsilon_1 + \epsilon_2)P$, $\hat{\mathcal{E}}_{33} = -Q_2 + \frac{1}{2}\sum_{i=5}^{8} PB\epsilon_i B^TP + \sum_{i=3}^{4} P\epsilon_i P + PB\epsilon_9 B^TP$.

Notice that in this case (i.e. $||x^T(t)PB|| \geq \delta_x$), the dynamic equations governing $\mu_1(t)$, $\mu_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ depend on their current values, and that, $\sigma(t)$ is governed by only one dynamic equation. There will be 16 different scenarios based on the relative values of $\mu_1(t)$, $\mu_2(t)$, $\eta_1(t)$ and $\eta_2(t)$. Two of these scenarios characterized by (i) $\mu_1(t) \geq \delta_{\mu_1}$, $\mu_2(t) \geq \delta_{\mu_2}$, $\eta_1(t) \geq \delta_{\eta_1}$, $\eta_2(t) \geq \delta_{\eta_2}$; and (ii) $\mu_1(t) < \delta_{\mu_1}$, $\mu_2(t) < \delta_{\mu_2}$, $\eta_1(t) < \delta_{\eta_1}$, $\eta_2(t) < \delta_{\eta_2}$ will be investigated here, and the other scenarios will be addressed briefly, as they can be treated in a similar fashion. Note that the controller switches to the appropriate scenario at the propoer time instant based on the values of the aforementioned parameters. Therefore, the objective here is to find the upper bound of the derivative of the Lyapunov-Krasovsky functional (16).

Regarding scenario (i) defined above, substitute (11), (12) and (13) in (22), and use the inequalities $\mu_1(t) \geq \delta_{\mu_1}$, $\mu_2(t) \geq \delta_{\mu_2}$ and $||x^T(t)PB|| \geq \delta_x$. Then, the derivative of the Lyapunov-Krasovsky functional (16) (associated to scenario (i)) can be expressed as

$$\begin{aligned}
\dot{V} \leq& \mathcal{X}^T(t)\hat{\mathcal{E}}\mathcal{X}(t) \\
&- 2\tilde{\theta}_1(t)\hat{\theta}_1(t) - 2\tilde{\theta}_2(t)\hat{\theta}_2(t) - \mu_1^2(t) - \mu_2^2(t) - \sigma^2(t) \\
&+ \frac{1}{2\epsilon_5}\frac{\eta_1(t)\hat{\theta}_1^2(t)||x^T(t)PB||^2}{\eta_1^2(t) + \mu_1^2(t)} \\
&+ \frac{1}{2\epsilon_8}\frac{\eta_2(t)\hat{\theta}_2^2(t)}{(\eta_2^2(t) + \mu_1^2(t))} \\
&+ \frac{1}{2}\eta_1(t)\dot{\eta}_1(t) + \frac{1}{2}\eta_2(t)\dot{\eta}_2(t) + \epsilon_9^{-1}\alpha_3^2
\end{aligned} \tag{24}$$

Furthermore, by substituting (14) in (24), and considering $\eta_1(t) \geq \delta_{\eta_1}$, $\eta_2(t) \geq \delta_{\eta_2}$ and $||x^T(t)PB|| \geq \delta_x$, one can write

$$\begin{aligned}
\dot{V} \leq& \mathcal{X}^T(t)\hat{\mathcal{E}}\mathcal{X}(t) - 2\tilde{\theta}_1(t)\hat{\theta}_1(t) - 2\tilde{\theta}_2(t)\hat{\theta}_2(t) \\
&- \mu_1^2(t) - \mu_2^2(t) - \frac{1}{2}\eta_1^2(t) - \frac{1}{2}\eta_2^2(t) - \sigma^2(t) + \epsilon_9^{-1}\alpha_3^2
\end{aligned} \tag{25}$$

Note that since the matrix inequality in (9a) is negative-definite, it is straightforward to show that $\hat{\mathcal{E}}$ is negative-definite as well. Note also that

$$-2\tilde{\theta}_1(t)\hat{\theta}_1(t) \leq -\tilde{\theta}_1^2(t) + \theta_1^{*2}, -2\tilde{\theta}_2(t)\hat{\theta}_2(t) \leq -\tilde{\theta}_2^2(t) + \theta_2^{*2} \tag{26}$$

Cosidering the above inequalities and on noting that all eigenvalues of $\hat{\mathcal{E}}$ are in the open left-half plane, the inequality (25) can be rewritten as follows

$$\dot{V} \leq \lambda_{max}(\hat{\mathcal{E}})||x(t)||^2 - \tilde{\theta}_1^2(t) - \tilde{\theta}_2^2(t) - \mu_1^2(t) - \mu_2^2(t) - \frac{1}{2}\eta_1^2(t) - \frac{1}{2}\eta_2^2(t) - \sigma^2(t) + \theta_1^{*2} + \theta_2^{*2} + \epsilon_9^{-1}\alpha_3^2 \tag{27}$$

(notice that $\lambda_{max}(\hat{\mathcal{E}})$ is in the open left-half complex plane). Define

$$\omega(||\tilde{x}(t)||) := -\lambda_{max}(\hat{\mathcal{E}})||x(t)||^2 + \tilde{\theta}_1^2(t) + \tilde{\theta}_2^2(t) + \mu_1^2(t) + \mu_2^2(t) + \frac{1}{2}\eta_1^2(t) + \frac{1}{2}\eta_2^2(t) + \sigma^2(t) \tag{28a}$$

$$\tilde{x}(t) := [x^T(t), \tilde{\theta}_1(t), \tilde{\theta}_2(t), \mu_1(t), \mu_2(t), \eta_1(t), \eta_2(t), \sigma(t)]^T \tag{28b}$$

$$\kappa_1 := \theta_1^{*2} + \theta_2^{*2} + \epsilon_9^{-1}\alpha_3^2 \tag{28c}$$

Substituting (28) in (27) results in

$$\dot{V} \leq -\omega(||\tilde{x}(t)||) + \kappa_1 \tag{29}$$

Now, consider scenario (ii), i.e., the case when $\mu_1(t) \leq \delta_{\mu_1}$, $\mu_2(t) \leq \delta_{\mu_2}$, $\eta_1(t) \leq \delta_{\eta_1}$, $\eta_2(t) \leq \delta_{\eta_2}$ (with $||x^T(t)PB|| \geq \delta_x$). By substituting (11)-(14) into the derivative of the Lyapunov-Krasovsky functional defined in (16) and following a procedure similar to the one presented in scenario (i) with the new set of inequalities given above, the following will be obtained

$$\dot{V} \leq -\omega(||\tilde{x}(t)||) + \theta_1^{*2} + \theta_2^{*2} + \mu_1(t)\epsilon_{\mu_1} + \mu_2(t)\epsilon_{\mu_2} + \frac{1}{2}\eta_1(t)\epsilon_{\eta_1} + \frac{1}{2}\eta_2(t)\epsilon_{\eta_2} + \epsilon_9^{-1}\alpha_3^2 \tag{30}$$

where $\omega(\cdot)$ and $\tilde{x}(t)$ are defined in (28a) and (28b). Note that with the set of inequalities in scenario (ii), it is easy to verify that

$$\dot{V} \leq -\omega(||\tilde{x}(t)||) + \kappa_2 \tag{31}$$

where,

$$\kappa_2 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2} \tag{32}$$

Recall that the controller can switch to any of the 16 different scenarios at any time, depending on the values of the aforementioned parameters. Hence, following an argument similar to the one presented in the first two scenarios, one can obtain the upper bound of the derivative of the Lyapunov-Krasovsky functional (16) for each active scenario as follows

$$\dot{V} \leq -\omega(||\tilde{x}(t)||) + \kappa_i, \quad \kappa_i > 0, \quad i = 1, \ ..., \ 16 \tag{33}$$

where κ_i's are defined below

$$\kappa_3 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1}, \kappa_4 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \delta_{\mu_2}\epsilon_{\mu_2}$$
$$\kappa_5 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1}$$
$$\kappa_6 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$
$$\kappa_7 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1}$$
$$\kappa_8 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$
$$\kappa_9 := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$
$$\kappa_{10} := \kappa_1 + \delta_{\mu_2}\epsilon_{\mu_2}, \kappa_{11} := \kappa_1 + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1}$$
$$\kappa_{12} := \kappa_1 + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$
$$\kappa_{13} := \kappa_1 + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$
$$\kappa_{14} := \kappa_1 + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1}, \kappa_{15} := \kappa_1 + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$
$$\kappa_{16} := \kappa_1 + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2}$$

It is easy to verify that $\kappa_2 > \kappa_i$, $i = 1, 3, 4, \ ..., \ 16$. Therefore, it can be concluded that the inequality (31) hold in all scenarios (with $||x^T(t)PB|| \geq \delta_x$).

Case 2: The rest of the proof for the case when $||x^T(t)PB|| < \delta_x$ is similar to the previous case, and is presenterd here in brief.

It can be shown that the upper bound on the derivative of Lyapunov-Krasovsky functional (16) with $||x^T(t)PB|| < \delta_x$, is obtained as follows

$$\dot{V} \leq -\omega^*(||\tilde{x}(t)||) + \kappa_2^* \tag{34}$$

where $\tilde{x}(t)$ is defined in (28a) and

$$\omega^*(||\tilde{x}(t)||) := -\lambda_{max}(\bar{\mathcal{E}})||x(t)||^2 + \tilde{\theta}_1^2(t) + \tilde{\theta}_2^2(t) + \mu_1^2(t) + \mu_2^2(t) + \frac{1}{2}\eta_1^2(t) + \frac{1}{2}\eta_2^2(t) + \sigma^2(t) \tag{35a}$$

$$\kappa_2^* := \kappa_1 + \delta_{\mu_1}\epsilon_{\mu_1} + \delta_{\mu_2}\epsilon_{\mu_2} + \frac{1}{2}\delta_{\eta_1}\epsilon_{\eta_1} + \frac{1}{2}\delta_{\eta_2}\epsilon_{\eta_2} + \delta_\sigma\epsilon_\sigma \tag{35b}$$

$$\bar{\mathcal{E}} = \begin{bmatrix} \bar{\mathcal{E}}_{11} & PA_1 & -PB\rho B^T P + A_0^T P \\ * & -Q_1 & A_1^T P \\ * & * & \bar{\mathcal{E}}_{33} \end{bmatrix} \tag{36}$$

where $\bar{\mathcal{E}}_{11} = -Q + Q_1 + D^T Q_2 D + P(\epsilon_1 + \epsilon_2)P$ and $\bar{\mathcal{E}}_{33} = -Q_2 + \frac{1}{2}\sum_{i=5}^{7} PB\epsilon_i B^T P + \sum_{i=3}^{4} P\epsilon_i P + PB\epsilon_9 B^T P + \frac{1}{2}PB\epsilon_{10}B^T P$.

Note that the upper bound on the derivative of the Lyapunov function (16) is obtained in (31) and (34), for $||x^T(t)PB|| \geq \delta_x$ and $||x^T(t)PB|| < \delta_x$, respectively. Define now

$$\tilde{\lambda} := \max(\lambda_{max}(\bar{\mathcal{E}}), \lambda_{max}(\hat{\mathcal{E}})) \tag{37a}$$

$$\tilde{\kappa} := \max(\kappa_2^*, \kappa_2) \tag{37b}$$

$$\begin{aligned}\tilde{\omega}(||\tilde{x}(t)||) :=& -\tilde{\lambda}||x(t)||^2 + \tilde{\theta}_1^2(t) + \tilde{\theta}_2^2(t) + \mu_1^2(t) \\ &+ \frac{1}{2}\eta_1^2(t) + \frac{1}{2}\eta_2^2(t) + \sigma^2(t) + \mu_2^2(t)\end{aligned} \tag{37c}$$

It can be inferred from (37) for all $t \geq t_0$, that the upper bound on the derivative of the Lyapunov function (16) is expressed by

$$\dot{V} \leq -\tilde{\omega}(||\tilde{x}(t)||) + \tilde{\kappa} \tag{38}$$

Hence, using the Lyapunov stability theory for retarded functional differential equations [1], [2] it can be shown that the solution of the system $(x(t), \tilde{\theta}_i(t), \mu_i(t), \eta_i(t), \sigma(t))$ $(t; t_0, x(t_0), \tilde{\theta}_i(t_0), \mu_i(t_0), \eta_i(t_0), \sigma(t_0))$, $i = 1, 2$ is uniformly ultimately bounded.

IV. Numerical Example

Consider the uncertain time-delay system described by (1) where

$$D = \begin{bmatrix} 0.5 & 0 \\ -0.2 & -0.1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1.1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Delta A_0 = \begin{bmatrix} 0 & 5 - 5\cos(2t) \\ 0 & 5\sin(3t) \end{bmatrix}, \quad \Delta A_1 = \begin{bmatrix} 0 & 5\cos(3t) \\ 0 & 5\sin(2t) \end{bmatrix}$$

Let h be equal to 2. In order to stabilize the system, the control input (7) is applied here by the following parameters

$$\gamma_1 = 2, \quad \gamma_2 = 2, \quad \rho = 1,$$

and the following initial conditions

$$\mu_1(t_0) = 10, \mu_2(t_0) = 10, \eta_1(t_0) = 10, \eta_2(t_0) = 10, \sigma(t_0) = 10$$

$$x(t) = [10 \quad 25]^T, \forall t \in [t_0 - h, \quad t_0], \hat{\theta}_1(t_0) = 20, \hat{\theta}_2(t_0) = 20$$

where the design parameters in (11)-(14) are chosen as follows

$$\delta_x = .01, \delta_{\mu_1} = .01, \delta_{\mu_2} = .01, \delta_{\eta_1} = .01, \delta_{\eta_2} = .01, \delta_\sigma = .01$$

$$\epsilon_{\mu_1} = .01, \quad \epsilon_{\mu_2} = .01, \quad \epsilon_{\eta_1} = .01, \quad \epsilon_{\eta_2} = .01, \quad \epsilon_\sigma = .01$$

Using the above parameters, the results sketched in Figs. 1-2 are obtained. The state trajectories are given in Fig. 1, which show that using the proposed adaptive control law, uniform ultimate boundedness is achieved, as expected.

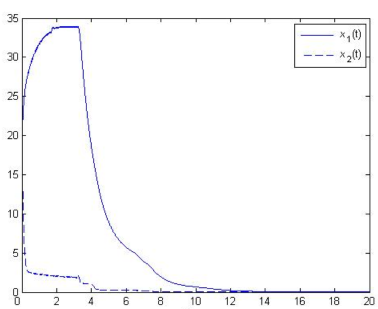

Fig. 1. State trajectoires for $x_1(t)$ and $x_2(t)$

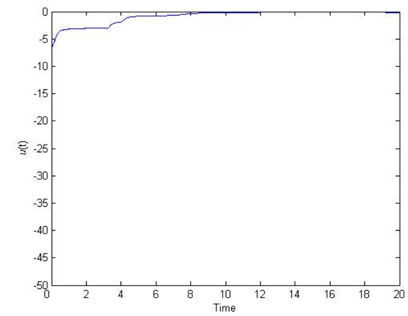

Fig. 2. The adaptive control input applied to the system

V. Conclusions

In this paper, robust stabilization of neutral time-delay systems with uncertainties has been investigated. It is supposed that the uncertainties in the system matrices are bounded with unknown bounds. A memoryless adaptive robust state feedback controller has been developed which guarantees the stability of the uncertain time-delay system. Simulation results demonstrate the efficacy of this approach.

References

[1] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, New York: Springer-Verlag, 1993.

[2] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*. New York: Academic, 1986.

[3] S. Bibian and H. Jin, "Time delay compensation of digital control for DC switch mode power supplies using prediction techniques," *IEEE Transactions on Power Electronics*, vol. 15, no. 5, pp. 835-842, 2000.

[4] W. Zhang, M. S. Branicky, and S. M. Philip, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 21, no. 1, pp. 84-99, 2001.

[5] Q.-L. Han and L. Yu, "On robust stability of linear neutral systems with nonlinear parameter perturbations," *in Proceedings of American Control Conference*, vol. 3, pp. 2027-2032, 2004.

[6] M. S. Mahmoud, "Robust H_∞ Control of linear neutral systems," *Automatica*, vol. 36, no. 5, pp. 757-764, 2000.

[7] S. Xu, J. Lam, and C. Yang, "H_∞ and positive real control for linear neutral delay systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1321-1326, 2001.

[8] H. Wang, J. Lam, S. Xu, S. Huang, "Robust H_∞ reliable control for a class of uncertain neutral delay systems," *International Journal of Control*, vol. 33, no. 7, pp. 611-622, 2002.

[9] V. Kharitonov, S. Mondie and J. Collado, "Exponential estimates for neutral time-delay systems: an LMI approach," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 666-670, 2005.

[10] M. Wu, Y. He, and J.-H. She, "New delay-dependent stability criteria and stabilizing method for neutral systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2266-2271, 2004.

[11] H. Wu, "Adaptive robust tracking and model following of uncertain dynamical systems with multiple time delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 611-616, 2004.

[12] H. Wu, "Adaptive stabilizing state feedback controllers of uncertain dynamical systems with multiple time delays," *IEEE Transactions on Automatic Control*, vol. 45, no. 9, pp. 1697-1701, 2000.

[13] F. Zheng, Q. G. Wang, and T. H. Lee, "Adaptive robust control of uncertain time delay systems," *Automatica*, vol. 41, no. 8, pp. 1375-1383, 2005.

[14] S. Oucheriah, "Adaptive robust control of a class of dynamic delay systems with unknown uncertainty bounds," *International Journal of Adaptive Control and Signal Processing*, vol. 15, pp. 53-63, 2001.

[15] X. Sun and J. Zhao, "Robust adaptive control for a class of nonlinear uncertain neutral delay systems," *in Proceedings of American Control Conference*, vol. 1, pp. 609-613, 2004.