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Perturbation analysis of eigenvalues of a class of self-adjoint operators

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Abstract— **We consider a class of spatially invariant systems whose coefficients are perturbed by spatially periodic functions. We analyze changes in transient behavior under the effect of such perturbations. This is done by performing a spectral analysis of the state transition operator at every point in time. Computational complexity is significantly reduced by using a procedure that captures the influence of the perturbation on only the largest singular values of the state transition operator. Furthermore, we show that the problem of computing corrections of all orders to the maximum singular values collapses to that of finding the eigenvalues of a set of finite dimensional matrices. Finally, we demonstrate the predictive power of this method via an example.**

Index Terms— **perturbation analysis; spatially periodic systems; transient response.**

I. Introduction

Perturbation theory of linear operators has been well studied over the last 50 years starting from the works of Rayleigh and Schrodinger [1]. It is a tool for efficiently approximating the influence of small perturbations on different properties of the unperturbed operator [1], [2]. In this paper, we study the effect of a special class of perturbations on the eigenvalues of a set of self-adjoint operators. This class of operators are in close relation with systems with spatially periodic coefficients.

Over the last decade, there has been a lot of excitement in analysis of periodic systems [3]. Systems with periodic coefficients in space arise in many important control problems. Fluid systems controlled by applying periodic body forcing or by imposing periodic boundary conditions in space are just an example of such systems [4]. A detailed analysis of such systems is given in [5]–[7].

It is shown in [7] that frequency response of systems with periodic coefficients in space takes a bi-infinite form. The simplest approach towards analysis of such systems is to approximate the bi-infinite operators using truncation. In the case where the system is defined in multiple spatial dimensions, numerical approximations would result in large matrices whose elements are themselves large matrices. Therefore the problem transforms into analysis of a large-scale system with at least several thousand states. From a computational point of view, the analysis of such systems is very expensive. Thus the mentioned approach is not efficient especially when one intends to perform a parametric study.

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In this paper we utilize *reduction theory* [1], [2] to analyze the spectral behavior of the state transition operator of the spatially distributed system. This method allows us to focus on only the singular values of the state transition operator that are responsible for the largest transient growth. Thus, reduction theory effectively helps to collapse the original infinite dimensional problem to one of finite dimensional matrices. In [8] reduction theory was used to investigate the stability properties of marginally stable spatially invariant systems under spatially periodic perturbations. Reference [8], differs from this work in that it investigated the effect of perturbation in the eigenvalues of the A-operator of the distributed system, as opposed to the singular values of the state transition operator.

The paper is organized as follows: We highlight the relevance of perturbation analysis of eigenvalues of certain self-adjoint operators in § II and give a brief introduction to reduction theory in § II-A. The main contribution of the paper is contained in § III where we describe the class of spatially distributed systems under consideration, their perturbation, and their frequency domain representation. Reduction theory is then applied to singular value analysis of these systems. The theory is demonstrated using an illustrative example in § IV and we conclude with some remarks in § V.

II. Preliminaries

Consider the spatially distributed system

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi \;=\; \mathcal{A}_0\,\psi \,+\, \mathcal{B}\,u, \quad \phi \;=\; \mathcal{C}\,\psi, \tag{1}$$

where ψ, u, and ϕ denote the spatio-temporal system state, input, and output, respectively and $\mathcal{A}_0, \mathcal{B}, \mathcal{C}$ are partial differential operators. We assume that $\mathcal{A}_0$ generates a strongly continuous semigroup [9], and that the evolution operator is exponentially stable.

In the analysis of linear systems we are often interested in certain scalar quantities that capture the system response to initial conditions and deterministic or stochastic inputs. For example, a relevant quantity in transient response analysis is the worst-case amplification of all possible initial conditions as a function of time. Another quantity of interest is the worst-case steady state gain of harmonic deterministic inputs. Both these quantities can be obtained by finding the largest eigenvalues of certain self-adjoint operators. Two such operators are

- $\mathcal{W}_0(t) = \mathcal{T}(t)\,\mathcal{T}(t)^*$, where $\mathcal{T}(t)$ is the state transition operator and $\mathcal{T}(t)^*$ is its transpose. In a finite dimensional setting we have $\mathcal{T}(t) := \mathrm{e}^{\mathcal{A}_0\,t}$. We will show later in § IV that the maximum eigenvalue of $\mathcal{W}_0(t)$ can be interpreted as the worst-case amplification of initial conditions at time t. Therefore, its application in transient response analysis of linear systems is imminent.

- $\mathcal{W}_0(\omega) = \mathcal{H}(\omega)\,\mathcal{H}(\omega)^*$, where $\mathcal{H}(\omega)$ is the frequency response of (1), i.e. $\mathcal{H}(\omega) := \mathcal{C}\,(\mathrm{j}\,\omega\,\mathcal{I} - \mathcal{A}_0)^{-1}\,\mathcal{B}$. It is a standard fact that the maximum eigenvalue of $\mathcal{W}_0(\omega)$ determines the largest steady-state system gain of a deterministic input with frequency ω. In fact, the H_∞ norm of (1) can be obtained by taking "sup" over the maximum eigenvalue of $\mathcal{W}_0(\omega)$.

In both the above cases, we are interested in computing maximal eigenvalues of $\mathcal{W}_0$. Our objective is to develop a method suitable for computing the effect of a particular class of perturbations on the eigenvalues of $\mathcal{W}_0$. This class of perturbations is motivated by the structures that arise in systems with periodic coefficients in space. Following a brief review of perturbation theory in § II-A, we utilize these structures to develop more specific results for spatially-periodic systems in § III.

A. Perturbation theory and reduction process

We consider a self-adjoint operator $\mathcal{W}_0$ with eigenvalue λ_0 of multiplicity m. Since $\mathcal{W}_0$ is self-adjoint, λ_0 is semi-simple meaning that it has a full set of corresponding independent eigenvectors g_0^i where $i = 1, 2, \ldots, m$. Also consider the perturbed operator $\mathcal{W}(\epsilon)$

$$\mathcal{W}(\epsilon) \;=\; \mathcal{W}_0 + \sum_{r=1}^{\infty} \epsilon^r\,\mathcal{W}_r, \quad 0 < \epsilon < \epsilon_0,$$

where each $\mathcal{W}_r$ is a self-adjoint operator itself.

Theorem 2.1: [1], [2] For sufficiently small values of ϵ_0 and in the case of the above self-adjoint perturbations, eigenvalues and eigenvectors of $\mathcal{W}(\epsilon)$ can be written in the form of a perturbation series

$$\begin{aligned} \mathcal{W}(\epsilon)\,g^i(\epsilon) \;&=\; \lambda^i(\epsilon)\,g^i(\epsilon), \quad i = 1, 2, \ldots, m,\\ \lambda^i(\epsilon) \;=\; \lambda_0 + \sum_{r=1}^{\infty} \epsilon^r\,\lambda_r^i, &\quad g^i(\epsilon) \;=\; g_0^i + \sum_{r=1}^{\infty} \epsilon^r\,g_r^i. \end{aligned} \tag{2}$$

Note that we have accounted for the fact that λ_0 may split into m distinct eigenvalues λ^i as a result of the perturbation. By g_r^i and λ_r^i we denote the rth order correction to g^i and λ^i, respectively.

We follow the development of Kato [1] and Baumgartel [2] to solve for the unknown coefficients in problem (2). The reduction process gives an iterative procedure for computing higher order correction coefficients in the perturbation series (2) for λ^i.

Let $e_1, e_2, ..., e_m$ be the set of orthonormal eigenfunctions corresponding to λ_0 and let $\mathcal{L}_0$ be the space spanned by these eigenfunctions, i.e. $\mathcal{L}_0 := \text{span}\,\{e_1, e_2, ..., e_m\}$. Also let $\mathcal{P}_0$, called the eigenprojection of λ_0, be the operator that projects the entire space onto the space $\mathcal{L}_0$, i.e., $\mathcal{P}_0 = \sum_{i=1}^m e_i\,e_i^*$. Let S_0 be the reduced resolvent operator determined from [2]

$$(\lambda\,\mathcal{I} - \mathcal{W}_0)^{-1} \;=\; \frac{\mathcal{P}_0}{\lambda - \lambda_0} + \sum_{i=0}^{m} \mathcal{S}_0^{i+1}\,(\lambda - \lambda_0)^i.$$

The step by step procedure is given below. Finding each correction term involves three steps [2].

Notation: By $\mathcal{M} \restriction \mathcal{N}$ we denote restriction of $\mathcal{M}$ to the space projected by $\mathcal{N}$. In other words, $\mathcal{M} \restriction \mathcal{N}$ acts on an element from $\mathcal{N}\,X$ and maps the result back to $\mathcal{N}\,X$, where X is the appropriate Hilbert space.

- First order correction:
  1) Let $\mathcal{P}_0$ be eigenprojection of λ_0
  2) Define $\mathcal{B}_0 = \mathcal{P}_0\,\mathcal{W}_1\,\mathcal{P}_0$.
  3) λ_1^i's are eigenvalues of $\mathcal{B}_0 \restriction \mathcal{P}_0$.
- Second order correction: Repeat the steps from first order correction with the following modifications
  1) If $\lambda_1^i \neq 0$, let $\mathcal{Q}_0^i$ be eigenprojection of λ_1^i. If $\lambda_1^i = 0$, let $\mathcal{Q}_0^i = \mathcal{P}_0$ from Iteration 1.
  2) Define $\mathcal{C}_0^i = \mathcal{Q}_0^i\,(\mathcal{W}_2 + \mathcal{W}_1\,\mathcal{S}_0\,\mathcal{W}_1)\,\mathcal{Q}_0^i$.
  3) λ_2^i's are eigenvalues of $\mathcal{C}_0^i \restriction \mathcal{Q}_0^i$.

The n-th order correction can be obtained in a similar way [2].

III. Spectrum perturbation of periodic systems

The frequency representation of linear systems with spatially periodic coefficients is completely addressed in [7], [8]. As an example, consider the following system with periodic coefficients in spatial variable x

$$\partial_t\,\psi(x,t) \;=\; \mathcal{A}(\epsilon)\,\psi(x,t),$$

where $\mathcal{A}(\epsilon) := \mathcal{A}_0 + \epsilon\,(\mathcal{A}_1\,\mathrm{e}^{\mathrm{j}\,\Omega\,x} + \mathcal{A}_{-1}\,\mathrm{e}^{-\mathrm{j}\,\Omega\,x})$ and $\mathcal{A}_0, \mathcal{A}_1$, and $\mathcal{A}_{-1}$ denote invariant operators in the x direction. The spatial frequency representation of this system is parameterized by $\theta \in [0, \Omega)$ and is given by [7], [8]

$$\partial_t\,\psi_\theta(t) \;=\; \mathcal{A}_\theta(\epsilon)\,\psi_\theta(t),$$

where $\mathcal{A}_\theta(\epsilon) := \mathcal{A}_{0\theta} + \epsilon\,\mathcal{A}_{1\theta}$, and $\mathcal{A}_{0\theta}$, and $\mathcal{A}_{1\theta}$ are bi-infinite operator-valued matrices.

Note that the spatial wave-number in the x direction, k_x, is determined by $k_x := \theta + n\,\Omega$ for any pair of (n, θ) where $\theta \in [0, \Omega)$ and $n \in \mathbb{Z}$. We note that as θ and n vary in their domains, k_x assumes all the values in $\mathbb{R}$.

The operators $\mathcal{A}_{0\theta}$, and $\mathcal{A}_{1\theta}$ are determined from

$$\begin{aligned} \mathcal{A}_{0\theta} \;&:=\; \begin{bmatrix} \ddots & & & & \\ & A_0(n-1) & & & \\ & & A_0(n) & & \\ & & & A_0(n+1) & \\ & & & & \ddots \end{bmatrix},\\ \mathcal{A}_{1\theta} \;&:=\; \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & 0 & A_{-1}(n) & 0 & \\ & A_1(n-1) & 0 & A_{-1}(n+1) & \\ & 0 & A_1(n) & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}, \end{aligned} \tag{3}$$

with $A_0(n) := \mathcal{A}_0(k_x = \theta + n\Omega)$, $A_1(n) := \mathcal{A}_1(k_x = \theta + n\Omega)$, and $A_{-1}(n) := \mathcal{A}_{-1}(k_x = \theta + n\Omega)$.

Although the frequency representation of these systems takes a bi-infinite form, the underlying operator-valued matrices have nice structures. For instance, for the case of the spatially periodic system above, $\mathcal{A}_{0\theta}$ is block-diagonal and $\mathcal{A}_{1\theta}$ has nonzero blocks only on the first upper and lower sub-diagonals. The C_0-semigroup $\mathcal{T}_\theta$ generated by $\mathcal{A}_\theta := \mathcal{A}_{0\theta} + \epsilon \mathcal{A}_{1\theta}$ satisfies

$$\partial_t \mathcal{T}_\theta \;=\; \mathcal{A}_\theta \mathcal{T}_\theta, \qquad \mathcal{T}_\theta(0) \;=\; \mathcal{I}. \tag{4}$$

Furthermore, for sufficiently small values of ϵ, $\mathcal{T}_\theta$ can be written as the following perturbation series

$$\mathcal{T}_\theta \;=\; \mathcal{T}_{0\theta} + \sum_{r=1}^{\infty} \epsilon^r \mathcal{T}_{r\theta}, \quad 0 < \epsilon \ll 1,$$

where the coefficients $\mathcal{T}_{r\theta}$ can be solved for from the following set of equations obtained by factoring out terms with equal powers of ϵ in (4)

$$\begin{array}{lllll} \epsilon^0: & \partial_t \mathcal{T}_{0\theta} & = & \mathcal{A}_{0\theta} \mathcal{T}_{0\theta}, & \mathcal{T}_{0\theta}(0) \;=\; \mathcal{I}, \\ \epsilon^r: & \partial_t \mathcal{T}_{r\theta} & = & \mathcal{A}_{0\theta} \mathcal{T}_{r\theta} + \mathcal{A}_{1\theta} \mathcal{T}_{r-1,\theta}, & \mathcal{T}_{r\theta}(0) \;=\; 0. \end{array}$$

Also, one can write $\mathcal{W}_\theta := \mathcal{T}_\theta^* \mathcal{T}_\theta := \mathcal{W}_{0\theta} + \sum_{r=1}^{\infty} \epsilon^r \mathcal{W}_{r\theta}$. By inspection, one sees that matrices $\mathcal{T}_{r\theta}$ and therefore $\mathcal{W}_{r\theta}$ inherit structures similar to those of $\mathcal{A}_{0\theta}$ and $\mathcal{A}_{1\theta}$. For instance, $\mathcal{W}_{0\theta}$ has nonzero blocks only on the main diagonal, $\mathcal{W}_{1\theta}$ has nonzero blocks only on the first upper and lower sub-diagonals, $\mathcal{W}_{2\theta}$ has nonzero blocks only on the main diagonal and second upper and lower sub-diagonals and so on. For notational convenience, we define

$$\mathcal{W}_{0\theta} \;:=\; \begin{bmatrix} \ddots & & & \\ & W_{0,0}(n-1) & & \\ & & W_{0,0}(n) & \\ & & & W_{0,0}(n+1) \\ & & & & \ddots \end{bmatrix},$$

$$\mathcal{W}_{1\theta} \;:=\; \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & 0 & W_{1,-1}(n) & 0 & \\ & W_{1,1}(n-1) & 0 & W_{1,-1}(n+1) & \\ & 0 & W_{1,1}(n) & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix},$$

$$\mathcal{W}_{2\theta} \;:=\; \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & W_{2,0}(n-1) & 0 & W_{2,-2}(n+1) & \\ & 0 & W_{2,0}(n) & 0 & \\ & W_{2,2}(n-1) & 0 & W_{2,0}(n+1) & \ddots \\ & & & \ddots & \ddots \end{bmatrix},$$

where $W_{r,l}$ denotes elements on the l-th subdiagonal of $\mathcal{W}_{r\theta}$.

The structures discussed above can be exploited to a great extent. It will be shown that general results about correction coefficients to the eigenvalues of $\mathcal{W}_\theta$ can be derived. Also, we show that the problem size can be reduced significantly. To this end, we first look at the structure of $\mathcal{P}_{0\theta}$. Recall that $\mathcal{P}_{0\theta}$ is the eigenprojection corresponding to eigenvalue λ_0 of $\mathcal{W}_{0\theta}$. In other words, $\mathcal{P}_{0\theta}$ is the space spanned by corresponding eigenvectors of λ_0. Since $\mathcal{W}_{0\theta}$ is block-diagonal, $\mathcal{P}_{0\theta}$ is also block-diagonal

$$\mathcal{P}_{0\theta} \;:=\; \begin{bmatrix} \ddots & & & \\ & P_0(n-1) & & \\ & & P_0(n) & \\ & & & P_0(n+1) \\ & & & & \ddots \end{bmatrix}.$$

By looking at the structure of $\mathcal{P}_{0\theta}$ and $\mathcal{W}_{1\theta}$, it is easy to show the following lemma.

Lemma 3.1: A necessary condition for λ_1^i, the first order correction to eigenvalue λ_0 of $\mathcal{W}_{0\theta}$ after perturbation, to be nonzero is that λ_0 be an eigenvalue of at least two subsequent blocks in $\mathcal{W}_{0\theta}$.

Proof: Recall from § II-A that λ_1^i is given by eigenvalues of $\mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta}$, where $\mathcal{B}_{0\theta} := \mathcal{P}_{0\theta} \mathcal{W}_{1\theta} \mathcal{P}_{0\theta}$. But, $\mathcal{B}_{0\theta}$ is equal to zero unless $\mathcal{P}_{0\theta}$ has adjacent nonzero blocks. ■

Remark 1: The above condition is not sufficient, because $\mathcal{B}_{0\theta}$ can be equal to zero even when $\mathcal{P}_{0\theta}$ satisfies the mentioned condition.

Example 1: Assume that λ_0 has multiplicity $m = 2$ and is an eigenvalue of the blocks $W_{0,0}(n-1)$ and $W_{0,0}(n)$ in $\mathcal{W}_{0\theta}$. Then only the adjacent blocks $P_0(n-1)$ and $P_0(n)$ in $\mathcal{P}_{0\theta}$ are nonzero and we have

$$\mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta} \;=\; \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix},$$

$$\begin{array}{lll} B_{12} & := & P_0(n-1)\, W_{1,-1}(n)\, P_0(n), \\ B_{21} & := & P_0(n)\, W_{1,1}(n-1)\, P_0(n-1). \end{array}$$

Note that $\mathrm{tr}(\mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta}) = 0$ and therefore, $\lambda_1^1 = -\lambda_1^2$. Thus, up to first order of correction, we have

- Case 1: $W_{1,1}(n-1) = (W_{1,-1}(n))^* = 0$;
  $\mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta} = 0 \quad \Rightarrow \quad \lambda_1^1 = \lambda_1^2 = 0$.
  In other words, λ_0 does not split.
- Case 2: $W_{1,1}(n-1) = (W_{1,-1}(n))^* \neq 0$;
  $\lambda_1^1 = -\lambda_1^2 \neq 0$.
  In other words, λ_0 splits into two eigenvalues that move in opposite directions along the real axis.

Similar results can be obtained for higher order corrections to the eigenvalues of $\mathcal{W}_\theta$. Although the number of terms present in equations for higher order corrections increase, the structures remain simple.

Note that the size of $\mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta}$ is only m times (twice in the case of Example 1) the size of each of the blocks in the bi-infinite matrix $\mathcal{W}_\theta$.

Remark 2: In fact, at each iteration level $\{\epsilon, \epsilon^2, \cdots\}$ of the reduction process, the *maximum* size of the problem is equal to the size of the constructive blocks of $\mathcal{W}_\theta$ times the multiplicity of $\{\lambda_0, \lambda_1^i, \cdots\}$ in $\{\mathcal{W}_{0\theta}, \mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta}, \cdots\}$, respectively. The problem size can be much smaller when $\{\mathcal{B}_{0\theta} \restriction \mathcal{P}_{0\theta}, \mathcal{C}_{0\theta}^i \restriction \mathcal{Q}_{0\theta}^i, \cdots\}$ is block-diagonal.

Therefore, by using perturbation analysis, the correction coefficients to the eigenvalues of $\mathcal{W}_\theta$ can be obtained by computing eigenvalues of a set of significantly smaller matrices compared to the the case where eigenvalues of $\mathcal{W}_\theta$ are computed after large-scale truncation of $\mathcal{W}_\theta$.

IV. Example: Transient Response Analysis of a Spatially Periodic System

We use the results of § III in the transient response analysis of an exponentially stable spatially periodic system. Systems that motivate transient response analysis are non-normal systems. These systems can have large transient growth before eventual decay. Consider the following non-normal system

$$\dot{\psi} \;=\; \mathcal{A}\,\psi, \qquad\qquad \mathcal{A}\,\mathcal{A}^* \;\neq\; \mathcal{A}^*\,\mathcal{A}. \tag{5}$$

The response of this system to initial condition ψ_0 is obtained by acting the C_0-semigroup generated by $\mathcal{A}$ on ψ_0

$$\psi(t) \;=\; \mathcal{T}(t)\,\psi_0.$$

A relevant quantity to consider in transient response analysis of (5) is the ratio between the norm of $\psi(t)$ and ψ_0 at a fixed time t

$$\frac{\|\psi(t)\|^2}{\|\psi_0\|^2} \;=\; \frac{\langle\psi(t),\psi(t)\rangle}{\langle\psi_0,\psi_0\rangle} \;=\; \frac{\langle\psi_0, \mathcal{T}^*(t)\,\mathcal{T}(t)\,\psi_0\rangle}{\langle\psi_0,\psi_0\rangle},$$
$$\sup_{\psi_0} \frac{\|\psi(t)\|^2}{\|\psi_0\|^2} \;=\; \lambda_1\{\mathcal{T}^*(t)\,\mathcal{T}(t)\} \;=\; \sigma_1^2\{\mathcal{T}(t)\},$$

where λ_1 and σ_1 denote the largest eigenvalues and singular values, respectively.

Therefore, the maximum eigenvalue of $\mathcal{W} := \mathcal{T}^*(t)\,\mathcal{T}(t)$ is equal to the supremum of the ratio between the norm of the solution at a fixed time t to that of the initial condition over all initial conditions. In other words, the maximum eigenvalue of $\mathcal{W}$ captures the worst-case amplification of initial conditions by the linear system at a fixed time t.

Now consider the following spatially periodic system motivated by channel flow systems. The system has two distributed states with the following state equations

$$\partial_t\,\psi(x,t) \;=\; (\mathcal{A}_0 \,+\, \epsilon\mathcal{A}_1)\,\psi(x,t), \quad \mathcal{A}_1 \;=\; 2\,\mathcal{L}\,\cos(\Omega\,x),$$

where

$$\mathcal{A}_0 \;=\; \begin{bmatrix} \frac{1}{R}(\partial_x^2 - c) & 0 \\ \partial_x & \frac{1}{R}(\partial_x^2 - c) \end{bmatrix}, \quad \mathcal{L} \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

One can think of $\epsilon\,\mathcal{A}_1\,\psi$ as a state feedback control with a spatially periodic gain. The frequency representation of this system is given in the beginning of § III by

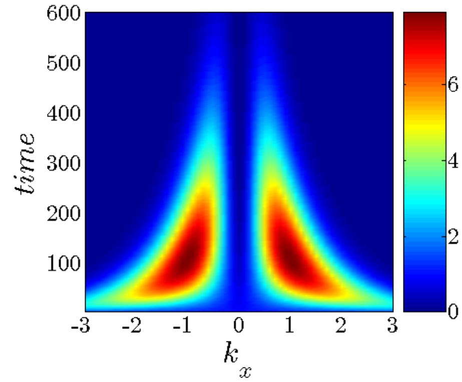

Fig. 1. The maximum eigenvalues of the unperturbed (uncontrolled) matrix $\mathcal{W}_0(k_x)$ for $R = 15, c = 1$

$$\partial_t\,\psi_\theta(t) \;:=\; \mathcal{A}_\theta(\epsilon)\,\psi_\theta(t) \;=\; (\mathcal{A}_{0\theta} \,+\, \epsilon\,\mathcal{A}_{1\theta})\,\psi_\theta(t),$$
$$A_0(n) \;:=\; \begin{bmatrix} -\frac{1}{R}((\theta+n\Omega)^2+c) & 0 \\ \mathrm{j}\,(\theta+n\Omega) & -\frac{1}{R}((\theta+n\Omega)^2+c) \end{bmatrix},$$
$$A_1(n) \;:=\; A_{-1}(n) \;:=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \forall\, n$$

We use perturbation analysis of the maximum eigenvalue of $\mathcal{W}_\theta$ generated by $\mathcal{A}_{0\theta} + \epsilon\mathcal{A}_{1\theta}$ in order to analyze the effect of the control parameter Ω on the transient response of the system. We verify the perturbation results by computing the maximum eigenvalues of the truncated $\mathcal{W}_\theta$.

Fig. 1 shows the maximum eigenvalues of the unperturbed matrix $\mathcal{W}_0(k_x)$ for $R = 15, c = 1$. The horizontal and vertical axes denote spatial frequency and time, respectively. We note that the solution, at a fixed frequency, shows transients as large as 8 times the norm of initial conditions before eventually decaying to zero. Also, at any given fixed time, the maximum eigenvalue has multiplicity $m = 2$.

By choosing Ω for the perturbed system, we sample the continuous spectrum in k_x by samples separated by integer multiples of Ω (recall that $k_x := \theta + n\,\Omega$). Thus, the smaller the frequency, the finer the sampling grid. In order for us to be able to analyze the effect of perturbation on the maximum eigenvalue of $\mathcal{W}_0$, we need to make sure that we sample the frequencies at which these maximum eigenvalues occur when sampling over k_x. This is done by choosing the appropriate value for θ for any choice of Ω.

We will compare the effect of two choices of Ω on transient response of the controlled system. Figs. 2(a) and 2(b) show the maximum eigenvalues of the uncontrolled matrix given in Fig. 1 with the horizontal axis changed to n instead of k_x to emphasize the sampling. We note that n and samples of k_x (separated by integer multiples of Ω) are equivalent once the pair (Ω, θ) is specified.

Fig. 2(a) shows the spectrum for $(\Omega, \theta) = (2, 1)$. We note that in this case, the maximum eigenvalues (corresponding to the red regions) occur at $n = -1, 0$. Therefore, the eigenprojection matrix of this eigenvalue with multiplicity

2, has two adjacent nonzero blocks on the main diagonal

$$\begin{array}{l}\Omega = 2 \\ \theta = 1\end{array} \qquad \mathcal{P}_{0\theta} = \begin{bmatrix} \ddots & & & & & \\ & 0 & & & & \\ & & P_0(n-1) & & & \\ & & & P_0(n) & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix}.$$

Fig. 2(b) shows the spectrum for $(\Omega, \theta) = (2/3, 1/3)$. Note that with a smaller Ω, the maximum eigenvalues (corresponding to the red regions) occur at $n = -2, 1$. As a result, in this case, the eigenprojection matrix of this eigenvalue with multiplicity 2, has two nonzero blocks separated by two zero blocks on the main diagonal

$$\begin{array}{l}\Omega = \frac{2}{3} \\ \theta = \frac{1}{3}\end{array} \qquad \mathcal{P}_{0\theta} = \begin{bmatrix} \ddots & & & & & & & \\ & 0 & & & & & & \\ & & P_0(n-2) & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & P_0(n+1) & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{bmatrix}.$$

Figs. 3(a) and 3(b) show the results of perturbation analysis of the maximum eigenvalue for the choices of the pair (Ω, θ) discussed above. The cumulative sum of the perturbation series for the maximum eigenvalue up to first, second, and third order of correction are plotted versus time. The perturbation parameter, ϵ, is 0.01. Eigenvalues of the unperturbed system (zeroth order correction) are also plotted to show the effect of control on transient response of the system. Finally, the maximum eigenvalues obtained from large scale truncation of $\mathcal{W}_\theta$ are given to compare with the results obtained by perturbation analysis.

Fig. 3(a) shows the transient response when $(\Omega, \theta) = (2, 1)$. We showed earlier that this choice of (Ω, θ) amounts to eigenprojection of the maximum eigenvalues of the unperturbed matrix that has adjacent nonzero blocks. Thus, from Lemma 3.1, we expect that the control affects the maximum eigenvalues of the unperturbed operator at the order of ϵ. The results shown in Fig. 3(a) agree with the expected response. First order correction is nonzero and higher order corrections converge to the results obtained by truncation. We note that the essential trends in this case are captured by the first order correction.

Fig. 3(b) shows the transient response for $(\Omega, \theta) = (2/3, 1/3)$. It can be seen from Fig. 2(b) that this choice of (Ω, θ) results in an eigenprojection matrix of the maximum eigenvalues of the unperturbed matrix that does not have adjacent nonzero blocks for $t < 270$. For this range of t, from Lemma 3.1, we expect to see the effect of control on the maximum eigenvalues of the unperturbed operator only at the order of ϵ^2. The results shown in Fig. 3(b) agree with the expected response. It can be seen that for $t < 270$, the first order correction is zero. Higher order corrections are relatively small and converge to the results obtained by truncation. Thus, for the selected control amplitude ($\epsilon = 0.01$), the control does not have significant effect on the

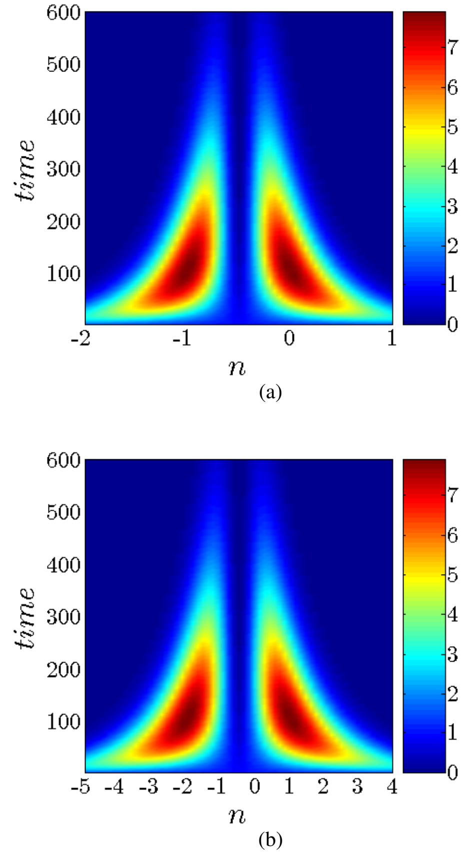

Fig. 2. The maximum eigenvalues of the unperturbed (uncontrolled) matrix $\mathcal{W}_{0\theta}$ for $R = 15, c = 1$. (a) $(\Omega, \theta) = (2, 1)$, (b) $(\Omega, \theta) = (2/3, 1/3)$.

response of the uncontrolled system. This result is important since it reveals which perturbation frequencies influence the uncontrolled system the most.

However, in Fig. 3(b), the type of response seen for $t > 270$ is completely different from that of $t < 270$. This is due to the fact that for the former range of t, the maximum eigenvalue of the unperturbed operator has a different location and occurs at $n = -1, 0$, see Fig. 2(b). Therefore, the structure of the problem for this range of t is similar to that of the case where $(\Omega, \theta) = (2, 1)$. Thus, it is not surprising that the type of solution is generically different. We note that this result is totally non-intuitive and we were able to explain it due to the predictive power of perturbation theory even at the level of the first order correction.

The perturbation results are in perfect agreement with truncation results except for a small range $234 < t < 270$. This is simply the consequence of the fact that we have performed perturbation analysis only on the maximum eigenvalue of the unperturbed matrix. In other words, we are tracking the effect of perturbation only on the maximum eigenvalue of the unperturbed matrix. I general, there may exist eigenvalues close to the maximum eigenvalues of the unperturbed matrix that are influenced more by the perturbation than the larger (maximum) eigenvalues. This can be specially the case when the larger and smaller eigenvalues lead to different structures in the viewpoint of perturbation analysis. As can be seen in this example, the maximum eigenvalues of the unperturbed matrix for $234 < t < 270$ still occur at $n = -2, 1$ and therefore are not affected by the control at the order of ϵ. However, for this time interval, the second largest eigenvalues occur at $n = -1, 0$ and are thus influenced by the control at the level of ϵ. Since the effect of control is increasing these eigenvalues for the perturbed system they result in the largest eigenvalue, although the largest unperturbed eigenvalue remains almost unchanged. Therefore perturbation analysis of only the largest eigenvalues cannot capture transition trends between two sets of largest eigenvalues. Had we studied the effect of perturbation on the second largest eigenvalues as well as the maximum eigenvalues, we could have captured the truncation results by taking the maximum of the responses obtained by perturbation analysis of the two sets of largest eigenvalues. In other words, one should make sure that the non-maximum eigenvalues that are not tracked are not influenced by the perturbation in a way that become the maximum eigenvalues of the perturbed matrix.

V. Concluding Remarks

We used perturbation theory to compute the correction coefficients for the eigenvalues of certain operators of interest in the transient response analysis of a class of spatially periodic systems. We utilized the structure of the frequency representation of systems with periodic coefficients to develop specific results for the spectral perturbation of these systems. We showed that the frequency of the perturbation is of integral importance in the behavior of the perturbed system.

We showed that the maximum singular value of the state transition operator at a fixed time can be interpreted as the worst-case amplification of all possible initial conditions. In an example, we utilized perturbation theory in order to find the maximum singular values of the state transition operator in time. We were able to capture the effect of control frequency on the system's transient response.

We showed that this type of analysis significantly reduces the computational effort. More importantly, by exploiting structures of the matrices involved in this analysis, one can get general results and intuitions as to how to select control parameters (here Ω) that influence the uncontrolled system the most and predict their effect in a systematic way.

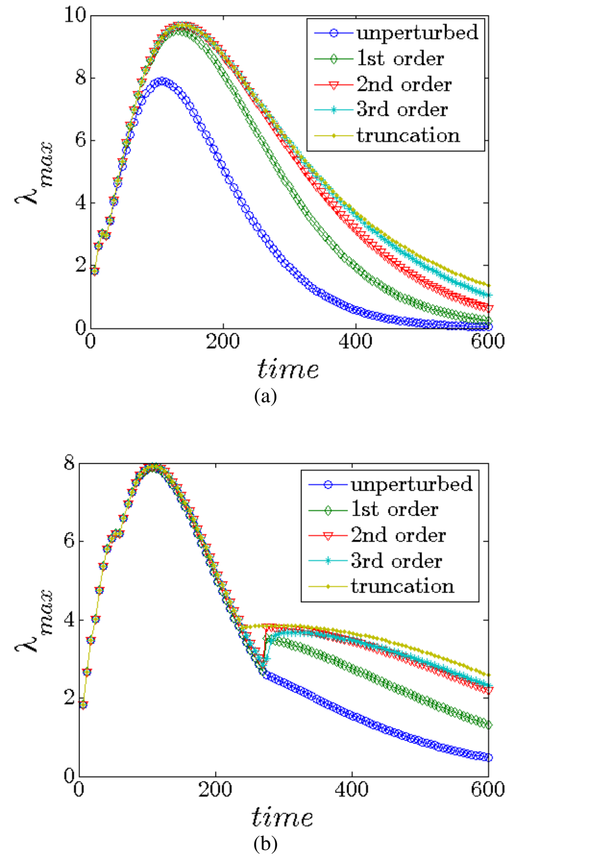

Fig. 3. Perturbation results for the maximum eigenvalues of the perturbed (controlled) matrix for $R = 15, c = 1, \epsilon = 0.01$ (a) $(\Omega, \theta) = (2, 1)$, (b) $(\Omega, \theta) = (2/3, 1/3)$.

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