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Existence of Optimal Homoclinic Orbits

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***Abstract*— The problem of optimal periodic control is considered from a geometric point of view. The objective is to determine the conditions under which a given optimal control problem admits a homoclinic orbit as an extremal solution. The analysis is performed on the Hamiltonian dynamical system obtained from the application of Pontryagin Maximum Principle. Assuming the existence of nondegenerate control, the existence problem is studied through the dynamical structure of the associated critical Hamiltonian dynamical system. A key tool used in the present development is the application of Morse theory in the context of symplectic geometry. The main result of the paper follows from the study of the critical points of the Hamiltonian function. An application example is provided to illustrate the method.**

I. INTRODUCTION

The usual task of optimal control is to compute the extremal trajectory that steers a controlled dynamical system to a desired point of operation with respect to a given objective functional. In some applications a desired steady-state point of operation might be unreachable or not minimizing when compared to dynamic operation. In those cases, the optimal target may lie on a trajectory, for example a periodic orbit. This situation occurs for example in drug scheduling optimization [18] and cyclic operation of chemical reactors [11]. The optimal periodic control (OPC) problem considered in this paper consists in the minimization of a functional of the type

$$\begin{aligned} \min_{u(\cdot)} J &= \frac{1}{\tau}\int_0^\tau L(x,u)dt && (1)\\ \text{s.t. } \dot{x} &= f(x,u), && (2) \end{aligned}$$

$x \in \mathbb{X} \subset \mathbb{R}^n$, $u \in \mathbb{U} \subset \mathbb{R}$, $\tau > 0$ is the period and f and L are sufficiently smooth. The problem considered here is to find necessary conditions under which the optimization problem admits a homoclinic orbit as the optimal solution, i.e. the trajectory along the optimal flow joins a saddle critical point to itself.

Variational approaches to deal with this problem were introduced in [11]. Second order conditions of optimality for periodic orbits were given around known steady-state of operation in [17], based on the properties of the monodromy matrix at the static optimal point of operation. More generally, if an optimal steady-state of operation to the problem is known, a local test for existence of improving periodic perturbation, the π-test, can be used to decide on the existence of an optimal periodic orbit [2]. An account of this approach to the problem is given in [4]. Two main causes for such optimal dynamical property can be distinguished: first, the critical steady-state of operation solving Eqs. (1-2) may be unstable (or more than one such point may exists) or the cost-function may be non-convex.

The above class of approaches requires the computation of periodic solutions to the Riccati differential equation (see [17] and references therein). Also, it requires the knowledge of a feasible steady-state point of operation minimizing the functional (1). Minimizing static points of operation may fail to exist in some applications, for example in drug scheduling applications. Recently, constructive methods admitting periodic orbits as target sets to solve the OPC problem were introduced in [19]. Assuming flatness of the system dynamic, optimal periodic outputs are constructed using off-line optimization methods. Based on this result, an extremum-seeking procedure was proposed in [10], where optimal periodic flat outputs are computed and tracked online.

In the present paper, existence of periodic solutions to the optimal problem is studied from a geometric point of view. The objective is to characterize the Hamiltonian system of differential equations from the application of the Pontryagin Maximum Principle

$$\begin{aligned} \dot{x} &= f(x,u), \quad x(0) = x_0 && (3)\\ \dot{\lambda} &= -\frac{\partial H}{\partial x}(x,\lambda,u), \quad \lambda(\tau) = 0 && (4) \end{aligned}$$

where $\lambda \in \mathbb{R}^n$ are the costates and the $\mathbb{R}$-valued function $H(x,\lambda,u)$ is the Hamiltonian defined by

$$H(x,\lambda,u) = \lambda^T f(x,u) + L(x,u). \quad (5)$$

Throughout the paper, it is assumed that an explicit optimal control $u^*(x,\lambda)$ can be computed. From the discussion in [14], it requires that the Hessian of the Hamiltonian function with respect to u is non-singular in a neighborhood of u^*, i.e. $\frac{\partial^2 H}{\partial u^2} \neq 0$, and that the first-order condition for optimality

$$\frac{\partial H}{\partial u}(x,\lambda,u^*) = 0 \quad (6)$$

leads to a unique solution $u^*(x,\lambda)$ for the optimal control problem.

Hence, in the present paper, the existence of stable periodic solutions to the optimal control problem will be studied on a $2n$-dimensional dynamical system

$$\begin{aligned} \dot{x} &= f(x,u^*(x,\lambda)), \quad x(0) = x_0 && (7)\\ \dot{\lambda} &= -\frac{\partial H}{\partial x}(x,\lambda,u^*(x,\lambda), \quad \lambda(\tau)) = 0, && (8) \end{aligned}$$

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along with the critical Hamiltonian function

$$H(x,\lambda,u^*(x,\lambda)) = \lambda^T f(x,u^*(x,\lambda)) + L(x,u^*(x,\lambda)). \quad (9)$$

The strategy to be employed here consists in studying the structure of the solution manifold by characterizing the critical points of the one-form dH associated with the Hamiltonian function along the symplectic vector field (see [1] for applications of symplectic geometry in control). One approach to characterize the critical points of dH is Morse theory, that gives an understanding of the global structure of the solution manifold. Morse theory-based analysis of vector fields leading to existence conditions for asymptotic cycles were introduced in [15]. The idea was then extended in terms of differential forms in [8], using the concept of Lyapunov one-forms for manifolds carrying a gradient structure. The later approach parallels the approach taken in [5] where Morse theory of smooth functions was used. In the present paper, assuming that the dynamic has one nondegenerate hyperbolic critical point and using a result from [16] for smooth Hamiltonian system in $\mathbb{R}^{2n}$, conditions for existence of optimal homoclinic orbits of critical controlled Hamiltonian structures are derived and illustrated. The main advantage of the method lies in the fact that using Morse theory for differential forms, the characterization of the solution manifold is greatly simplified when compared to convexity and variational methods.

The paper is divided as follows. Mathematical preliminaries on exterior calculus and critical point theory for Hamiltonian differential forms are given in Section II. The main result of the paper on existence of homoclinic orbit for the critical Hamiltonian dynamical system presented above is developed in Section III. An optimal periodic control application example treated previously in [17] is studied in Section IV. Conclusions are given in Section V.

II. MATHEMATICAL PRELIMINARIES

A. Exterior Calculus

In this section, the basic elements of exterior calculus on $\mathbb{R}^n$ are recalled. A complete account of exterior calculus on general smooth manifolds can be found in [6]. We will work exclusively in standard (Euclidean) coordinates $(x^1,\ldots,x^n)$ on $\mathbb{R}^n$. We denote the ring of smooth real-valued functions on $\mathbb{R}^n$ by $\mathcal{C}^\infty(\mathbb{R}^n)$.

For all $x\in\mathbb{R}^n$, let the tangent space to $\mathbb{R}^n$ at x be denoted by $T_x\mathbb{R}^n$. For any $x\in\mathbb{R}^n$, $T_x\mathbb{R}^n$ is canonically isomorphic to $\mathbb{R}^n$, so it can be endowed with a n-dimensional real vector space structure. For any $x\in\mathbb{R}^n$, a tangent vector $v\in T_x\mathbb{R}^n$ can be written as

$$v = \sum_{i=1}^{n} v^i \partial_{x_i}|_x, \quad (10)$$

where $v^i\in\mathbb{R}$ for all $i\in\{1,\ldots,n\}$. The tangent vectors $\partial_{x_i}|_x$, $i\in\{1,\ldots,n\}$, comprise the standard basis for $T_x\mathbb{R}^n$. For any $f\in\mathcal{C}^\infty(\mathbb{R}^n)$, the action of the tangent vector $\partial_{x_i}|_x$ on f is defined by $\partial_{x_i}|_x(f) = (\partial f/\partial x_i)|_x$. Let $\pi_{T\mathbb{R}^n} : T\mathbb{R}^n \to \mathbb{R}^n$ denote the tangent bundle over $\mathbb{R}^n$, and let $\Gamma^\infty(T\mathbb{R}^n)$ denote the real vector space of all smooth sections of $T\mathbb{R}^n$, *i.e.*, the collection of all smooth maps $X:\mathbb{R}^n\to T\mathbb{R}^n$ satisfying $\pi_{T\mathbb{R}^n}\circ X|_x = x$ for all $x\in\mathbb{R}^n$. A smooth vector field on $\mathbb{R}^n$ is an element $X\in\Gamma^\infty(T\mathbb{R}^n)$. Any $X\in\Gamma^\infty(T\mathbb{R}^n)$ can be written as

$$X|_x = \sum_{i=1}^{n} v^i(x)\partial_{x_i}|_x, \quad (11)$$

where $v^i\in\mathcal{C}^\infty(\mathbb{R}^n)$ for all $i\in\{1,\ldots,n\}$. For each $x\in\mathbb{R}^n$, let the cotangent space to $\mathbb{R}^n$ be denoted by $T_x^*\mathbb{R}^n$. Recall that for a given $x\in\mathbb{R}^n$, $T_x^*\mathbb{R}^n$ is the algebraic dual of $T_x\mathbb{R}^n$, *i.e.*, the set of all linear functionals on $T_x\mathbb{R}^n$. For each $x\in\mathbb{R}^n$, the standard (dual) basis for $T_x^*\mathbb{R}^n$ is thus comprised of the cotangent vectors $dx_i|_x$, $i\in\{1,\ldots,n\}$, determined by the relations $dx_i|_x(\partial_{x_j}|_x) = \delta_j^i$ for all $i,j\in\{1,\ldots,n\}$, where δ_j^i is the Kronecker delta. Hence, for any $x\in\mathbb{R}^n$, a cotangent vector $w\in T_x^*\mathbb{R}^n$ can be written as

$$w = \sum_{i=1}^{n} \omega_i dx_i|_x, \quad (12)$$

where $\omega_i\in\mathbb{R}$ for all $i\in\{1,\ldots,n\}$. Let $\pi_{T^*\mathbb{R}^n}:T^*\mathbb{R}^n\to\mathbb{R}^n$ denote the cotangent bundle over $\mathbb{R}^n$, and let $\Omega^1(\mathbb{R}^n)$ denote the real vector space of all smooth sections of $T^*\mathbb{R}^n$, *i.e.*, the collection of all smooth maps $\omega:\mathbb{R}^n\to T^*\mathbb{R}^n$ satisfying $\pi_{T^*\mathbb{R}^n}\circ\omega|_x = x$ for all $x\in\mathbb{R}^n$. A differential one-form on $\mathbb{R}^n$ is an element $\omega\in\Omega^1(\mathbb{R}^n)$. Any $\omega\in\Omega^1(\mathbb{R}^n)$ can be written as

$$\omega|_x = \sum_{i=1}^{n} w_i(x)dx^i|_x, \quad (13)$$

where $w_i\in\mathcal{C}^\infty(\mathbb{R}^n)$ for all $i\in\{1,\ldots,n\}$. For each $k\geq 0$, we denote by $\Lambda^k(\mathbb{R}^n)$ the real vector space of differential k-forms on $\mathbb{R}^n$. By convention, $\Lambda^k(\mathbb{R}^n)=\{0\}$ when $k<0$. Note that when $k=0$, $\Lambda^0(\mathbb{R}^n)=\mathcal{C}^\infty(\mathbb{R}^n)$; furthermore, when $k=1$, $\Lambda^1(\mathbb{R}^n)=\Omega^1(\mathbb{R}^n)$.

The exterior (wedge) product of two differential one-forms is the bilinear map $\wedge:\Omega^1(\mathbb{R}^n)\times\Omega^1(\mathbb{R}^n)\to\Lambda^2(\mathbb{R}^n)$, denoted $(\omega_1,\omega_2)\mapsto\omega_1\wedge\omega_2$, uniquely defined (up to a scalar factor) by the requirements that

$$\begin{aligned} dx_i\wedge dx_j &= -dx_j\wedge dx_i \\ dx_i\wedge f dx_j &= f dx_i\wedge dx_j \end{aligned}$$

for all $i,j\in\{1,\ldots,n\}$ and all $f\in\mathcal{C}^\infty(\mathbb{R}^n)$.

For each $0\leq k\leq n-1$, there is a linear map

$$d:\Lambda^k(\mathbb{R}^n)\to\Lambda^{k+1}(\mathbb{R}^n), \quad (14)$$

called the exterior derivative, uniquely defined by the following three properties:

1. $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta, \quad \forall\alpha,\beta\in\Lambda^k(\mathbb{R}^n)$;
2. $df = \sum_{i=1}^n(\partial_{x_i}(f))dx_i, \quad \forall f\in\mathcal{C}^\infty(\mathbb{R}^n)$;
3. $d\circ d\alpha = 0, \quad \forall\alpha\in\Lambda^k(\mathbb{R}^n)$.

For each $0\leq k\leq n-1$, the interior product $\lrcorner$ is the map

$$\lrcorner:\Gamma^\infty(T\mathbb{R}^n)\times\Lambda^k(\mathbb{R}^n)\to\Lambda^{k-1}(\mathbb{R}^n) \quad (15)$$

defined by the following four properties. For all $X \in \Gamma^\infty(T\mathbb{R}^n)$, $\lrcorner$ satisfies:

1. $X \lrcorner f = 0, \quad \forall f \in \mathcal{C}^\infty(\mathbb{R}^n)$ when $k = 0$;
2. $X \lrcorner \omega = \omega(X), \quad \forall \omega \in \Lambda^1(\mathbb{R}^n)$ when $k = 1$;
3. $X \lrcorner (\alpha + \beta) = X \lrcorner \alpha + X \lrcorner \beta, \quad \forall \alpha, \beta \in \Lambda^k(\mathbb{R}^n)$;
4. $X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta), \quad \forall \alpha, \beta \in \Lambda^k(\mathbb{R}^n)$.

With our notation established, we now consider special properties of Hamiltonian forms.

B. Hamiltonian Forms

Consider a Hamiltonian function $H(x, \lambda) \in T^*\mathbb{R}^n$. The associated vector field X_H in coordinates $(x, \lambda) \in \mathbb{R}^{2n}$ is defined by

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial \lambda_i} \partial_{x_i} - \frac{\partial H}{\partial x_i} \partial_{\lambda_i}. \tag{16}$$

The canonical symplectic closed two-form in $T^*\mathbb{R}^n$ [1] is given by

$$\Omega = \sum_{i=1}^{n} dx_i \wedge d\lambda_i \tag{17}$$

which is non-degenerate at every point [13]. The vector field X_H given above is said to be symplectic if it satisfies

$$\mathcal{L}_{X_H}(\Omega) = 0, \tag{18}$$

where $\mathcal{L}_{X_H}(\Omega)$ is the Lie derivative of the two-form with respect to the vector field X_H. By definition, the Lie derivative $\mathcal{L}$ of a differential form ω with respect to a vector field X is the linear operator defined by

$$\begin{aligned} \mathcal{L}_X \quad &: \quad \Lambda^k(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^n) \\ \mathcal{L}_X(\omega) \quad &= \quad X \lrcorner (d\omega) + d(X \lrcorner\, \omega). \end{aligned} \tag{19}$$

$\mathcal{L}_X$ satisfies the Leibniz rule, i.e. $\forall \alpha, \beta \in \Lambda(\mathbb{R}^n)$:

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta). \tag{20}$$

In other words, a symplectic vector field on $T^*\mathbb{R}^n$ preserves the symplectic two-form Ω. As a consequence,

$$\omega = X_H \lrcorner \Omega \tag{21}$$

is a closed one-form, i.e. $d\omega = 0$. By application of the Cartan formula [6],

$$d\omega = dX_H \lrcorner \Omega = (dX_H + X_H \lrcorner d)(\Omega) = \mathcal{L}_{X_H}(\Omega) = 0, \tag{22}$$

we see that ω defined above is closed if and only if X_H is a symplectic vector field. With X_H defined as above, it is possible to show that ω is indeed closed. Taking the interior product of the canonical two-form with respect to the vector field X_H yields to the one form

$$\omega = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \lambda_i} d\lambda_i + \frac{\partial H}{\partial x_i} dx_i \right) \tag{23}$$

which is the exterior derivative of the Hamiltonian function H. From the discussion in [8], since the one-form $\omega = X_H \lrcorner \Omega = dH$, it is thus exact, *i.e.* ω is generated by the exterior derivative of a zero form, in this case the Hamiltonian function.

One consequence of the above analysis is that there is a one-to-one correspondence between the closed one-form and the symplectic vector field X_H. Critical points of the vector field X_H hence coincide with the critical points of the one-form dH. This observation will be used in the sequel when critical points of the Hamiltonian function $H(x, \lambda)$ will be computed using the one-from $dH(x, \lambda)$.

C. Critical Points and Morse Theory

The notion of critical point of a smooth function translates to the notion of a zero of a closed one-form [8], i.e. for a given function f, the critical points p where $df(p) = 0$ are the ones for which the closed one-form $\omega = df$ vanishes, i.e. $\omega(p) = 0$. A critical point p is said to be nondegenerate if for some coordinate system $x_1, \ldots, x_n$ centered at p, the Hessian matrix is nonsingular,

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)(p) \neq 0. \tag{24}$$

Remark 2.1: The notion of nondegenerate critical point is independent of the choice of coordinate system (see [3]).

The index of a nondegenerate critical point is the number of negative eigenvalues in the Hessian matrix. A function is said to be of Morse type if all its critical points are nondegenerate. Assuming that the Hamiltonian critical function is of Morse type, it is possible to discriminate between the stable and unstable manifolds in a neighborhood of a critical point and hence, to seek for existence of cycles on given manifolds [7], [5]. It also has an importance in the definition of the cohomology class of the closed form. In particular, in [8] and in [15], existence theorem for orbits and gradient flows of general dynamical systems are given in terms of the de Rham cohomology class. In the present paper, we are interested in the local behaviour around the critical point. The main result of interest to be used in the sequel is the Morse lemma [12] that states that for a nondegenerate critical point p of a Morse function f with index r, there exists a coordinate charts $x_1, x_2, \ldots, x_n$ in a neighborhood U of p such that the function has the follwing representation

$$f(x) = f(p) - \sum_{i=1}^{r} x_i^2 + \sum_{i=r+1}^{n} x_i^2. \tag{25}$$

III. MAIN RESULT

The main result of the paper is now presented. Consider the critical Hamiltonian function $H(x, \lambda, u^*(x, \lambda))$, denoted in the sequel by $H(x, \lambda)$. Let X_H be defined as in Section II-B, i.e.

$$X_H = S \nabla H(x, \lambda), \quad S = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{26}$$

where $\nabla H = [\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}, \frac{\partial H}{\partial \lambda_1}, \ldots, \frac{\partial H}{\partial \lambda_n},]^T$. Leading to the particular formulation

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \lambda_i} \partial x_i - \frac{\partial H}{\partial x_i} \partial \lambda_i \right).$$

First, consider the following general existence result, due to Séré [16].

Lemma 3.1 ([16]): Let the following three conditions hold:

C1. Let $\Sigma = \{[x,\lambda]^T | H(x,\lambda) = 0\}$ be a compact set, with $H(x,\lambda)$ a smooth function defined on $\mathbb{R}^{2n}$, whose differential dH does not vanish on Σ except at one point p and with a nondegerate Hessian $A = d^2H(p)$.

C2. The matrix (SA) is hyperbolic, i.e. (SA) does not have pure imaginary eigenvalues.

C3. There is a C^1 vector field Y transverse to $\Sigma \setminus \{p\}$ such that $L_Y\Omega = \Omega$ everywhere in $\mathbb{R}^{2n}$.

Then the Hamiltonian system has at least one homoclinic solution, i.e. a non-constant and doubly asymptotic solution to the critical point p.

Remark 3.2: The above theorem is an analog to Weinstein Conjecture for homoclinic orbits (see also [20] where the result is stated in terms of closed characteristics).

The main result of this paper is the following.

Theorem 3.3: Let the critical Hamiltonian function $H(x,\lambda)$ on $\mathbb{R}^{2n}$ be a Morse function admitting only one nondegenerate critical point

Then, conditions C1-C3 are met and the Hamiltonian system has at least one homoclinic solution.

Proof: Let the vector field X_H, the two-form Ω, and the closed one-form ω be as defined in the preceding sections. By the assumptions on the critical Hamiltonian from lemma 3.1 and Morse lemma, conditions C1 and C2 are immediately met.

Condition C3 is met if one picks $Y = \nabla H$. By the definition of Hamiltonian vector field 26,

$$X_H = S\nabla H. \tag{27}$$

The vector field ∇H is therefore transverse to level sets of H, in particular to $\Sigma = \{H(x,\lambda) = 0\}$. Moreover, ∇H and X_H vanishes at the same critical point p. ■

IV. APPLICATION EXAMPLE

The optimal control problem formulated in [17] is now introduced to illustrate the technique developed in Section III. Consider the periodic optimal problem

$$\begin{aligned}\min_{u(\cdot)} J &= \frac{1}{\tau}\int_0^\tau \left(\frac{x_1^2}{2} + \frac{x_2^4}{4} - \frac{x_2^2}{2} + b\frac{u^2}{2}\right) dt & (28)\\ \text{s.t.}\quad \dot{x}_1 &= x_2 & (29)\\ \dot{x}_2 &= u & (30)\end{aligned}$$

with $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$. The Hamiltonian for this problem is given by

$$H(x,\lambda,u) = \lambda_1 x_2 + \lambda_2 u + \frac{x_1^2}{2} + \frac{x_2^4}{4} - \frac{x_2^2}{2} + \frac{bu^2}{2}. \tag{31}$$

From the analysis in [17], the problem admits periodic solutions for $0 < b < 1/4$. First note that the non-negativity condition can be easily derived using Legendre-Clebsch condition

$$\frac{\partial^2 H}{\partial u^2}|_{u^*} = b > 0. \tag{32}$$

The Hamiltonian dynamics are

$$\begin{aligned}\dot{x}_1 &= x_2,\ x_1(0) = x_{1,0} & (33)\\ \dot{x}_2 &= u,\ x_2(0) = x_{2,0} & (34)\\ \dot{\lambda}_1 &= -x_1,\ \lambda_1(\tau) = 0 & (35)\\ \dot{\lambda}_2 &= -\lambda_1 - x_2^3 + x_2,\ \lambda_2(\tau) = 0. & (36)\end{aligned}$$

Using the first-order optimality condition

$$\frac{\partial H}{\partial u}(x,\lambda,u^*) = 0, \tag{37}$$

$u^*(x,\lambda)$ is unique and is given by

$$u^*(x,\lambda) = -\frac{\lambda_2}{b}. \tag{38}$$

Equation (35) can be re-written in terms of λ_2 as

$$\dot{x}_2 = -\frac{\lambda_2}{b}. \tag{39}$$

Consider the critical Hamiltonian function

$$H(x,\lambda,u^*(x,\lambda)) = \lambda_1 x_2 - \frac{\lambda_2^2}{2b} + \frac{x_1^2}{2} + \frac{x_2^4}{4} - \frac{x_2^2}{2} \tag{40}$$

and the associated one-form

$$dH = x_1 dx_1 + (\lambda_1 + x_2^3 - x_2)dx_2 + x_2 d\lambda_1 + \frac{1}{b}\lambda_2 d\lambda_2. \tag{41}$$

The one-form $\omega = dH$ has one critical point, located at the origin. The Hessian of the function is

$$d^2H = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 3x_2^2 - 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{b}\end{pmatrix} \tag{42}$$

which is nondegenerate at the origin, hence $H(x,\lambda,u^*)$ is a Morse function. Evaluating the Hessian at the origin, we have

$$d^2H(0) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{b}\end{pmatrix}. \tag{43}$$

One can check that the dimension of the unstable manifold is 2, hence the Morse index of the critical point is $k = 2$. Note that d^2H is not positive definite everywhere and the function is not convex, hence one cannot rely on the existence result for periodic orbits of convex Hamiltonian systems derived in [21].

Multiplying the Hessian evaluated at the origin on the right by the matrix

$$S = \begin{pmatrix} 0_{2\times2} & I_{2\times2}\\ -I_{2\times2} & 0_{2\times2}\end{pmatrix}$$

leads to the following condition on the parameter b for all eigenvalues to have a non-zero real part,

$$(1 - 4b) > 0 \Rightarrow b < \frac{1}{4}, \tag{44}$$

recovering the condition for periodic orbits given in [17] using the π-test.

The main drawback of the method is that we do not have any knowledge or control on the optimality of the return time, *i.e.* the period $\tau > 0$ for a trajectory along the flow joining the critical point to itself is not taken into account. One possibility to alleviate this problem is to parameterize the time in the original optimal control problem Eqs (1-2) using $t = \frac{T}{\tau}$, we have a parameterized optimal periodic control problem:

$$\min_{u(\cdot)} J = \int_0^1 L(x,u)dT \tag{45}$$

$$\text{s.t.} \quad \frac{dx}{dT} = \tau \cdot f(x,u). \tag{46}$$

Applying the Pontryagin Maximum Principle, the following Hamiltonian function is obtained

$$H(x,\lambda,u,\tau) = L(x,u) + \tau \cdot \lambda f(x,u). \tag{47}$$

Hence, the period τ is lumped in the dynamics of the costates, *i.e.*, letting $\lambda' = \tau \cdot \lambda$, we have

$$H(x,\lambda',u) = L(x,u) + \lambda' f(x,u). \tag{48}$$

Finding the extremals of Eq. (47) with respect to τ would lead here to an extra conditions for optimality, an approach used originally in [11].

V. CONCLUSION

The problem of existence of stable optimal periodic orbits was solved for a class of nonlinear optimal control problems. Under the assumptions that the Hamiltonian was nondegenerate with respect to the control variable and possessed one nondegenerate critical point, the structure of the resulting Hamiltonian dynamics were used to decide on the existence of homoclinic orbits. Applying the conditions in the developed framework, and using the fact that the Hamiltonian one-form was exact in $\mathbb{R}^{2n}$, it was possible to recover, in an example, the conditions obtained for existence of periodic orbits using the π-test. Future work will focus on extending the method to other classes of critical dynamical structure that generate periodic behavior.

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