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Stability Analysis of Stochastic Systems using Polynomial Chaos

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Abstract— **A novel framework for stability analysis of linear and polynomial stochastic systems is presented. The framework is built on generalized polynomial chaos theory, which enables analysis of dynamical systems with probabilistic uncertainty on system parameters with various distributions. The theory allows for the transformation of stochastic problems into a higher dimensional deterministic problem, that is able to accurately approximate the evolution of uncertainty in the state trajectories due to stochastic system parameters. The developed theory is applied to analyze a linear flight control design for an F-16 aircraft model. The problem of generating stability certificates for stochastic polynomial systems is also considered.**

I. Introduction

Stability analysis of stochastic systems has been receiving much interest of late. Determination of stability for the nominal system itself is not often very useful. Instead we wish to analyze stability with uncertainty in the system dynamics. For linear systems, traditional linear robust control [1] addresses robust stability for frequency dependent, uniformly distributed uncertainty in the system dynamics. For nonlinear systems in general, stability have been addressed for deterministic systems with stochastic forcing [2], [3]. In this paper we restrict our attention to systems with stochastic parameters, i.e. systems with probabilistic uncertainty in system parameters. For such class of systems, sampling based approaches are often used to solve the stochastic problem in a deterministic setting. The drawback of this approach is that it can result in the solution of vary large problems for accurate characterization of uncertainty. For linear systems, the vertex set of the uncertainty polytope is often considered [4], but such analysis is restricted to uniform distributions. For nonlinear systems, it is not always possible to reduce the problem set in this manner.

In this paper we present a novel way to examine the stability of linear and nonlinear polynomial systems with probabilistic uncertainty in their parameters. Here, we analyze the stability problem by utilizing polynomial chaos theory which allows the transformation of stochastic dynamics into deterministic dynamics in higher dimension. While this increases the computational complexity of the system, the resulting complexity can be significantly lower than sampling based methods.

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Polynomial chaos (gPC) was first introduced by Wiener [5] where Hermite polynomials were used to model stochastic processes with Gaussian random variables. According to Cameron and Martin [6] such an expansion converges in the $\mathcal{L}_2$ sense for any arbitrary stochastic process with finite second moment. This applies to most physical systems. Xiu *et al.*[7] generalized the result of Cameron-Martin to various continuous and discrete distributions using orthogonal polynomials from the so called Askey-scheme [8] and demonstrate $\mathcal{L}_2$ convergence in the corresponding Hilbert functional space. This is popularly known as the generalized polynomial chaos (gPC) framework.

The gPC framework has been applied to applications including stochastic fluid dynamics [9], [10], [11], stochastic finite elements [12], and solid mechanics [13], [14]. However, application of gPC to control related problems has been surprisingly limited. The work of Hover *et al.*[15] addresses stability & control of a dynamical system with probabilistic uncertainty on the system parameters, but only addresses systems where dynamics appear bilinearly. In this work, we address not only stability of linear systems, but also the stability of nonlinear polynomial systems. We develop a methodology of generalizing the gPC framework to nonlinear polynomial systems of arbitrary dimension. The deterministic result of the gPC expansion for linear systems can be solved via the traditional Lyapunov equation. Stability of the polynomial equations is analyzed using sum-of-squares (SOS) techniques as described in [16]. In both cases, the stochastic stability problem can be solved by analyzing a set of deterministic equations.

II. Wiener-Askey Polynomial Chaos

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of the subsets of Ω, and P is the probability measure. Let $\Delta(\omega) = (\Delta_1(\omega), \cdots, \Delta_d(\omega)) : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}^d)$ be an $\mathbb{R}^d$-valued continuous random variable, where $d \in \mathbb{N}$, and $\mathcal{B}^d$ is the σ-algebra of Borel subsets of $\mathbb{R}^d$. A general second order process $X(\omega) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ can be expressed by polynomial chaos as

$$X(\omega) = \sum_{i=0}^{\infty} x_i \phi_i(\Delta(\omega)), \tag{1}$$

where ω is the random event and $\phi_i(\Delta(\omega))$ denotes the gPC basis of degree p in terms of the random variables $\Delta(\omega)$. The functions $\{\phi_i\}$ are a family of orthogonal basis in $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ satisfying the relation

$$\mathbf{E}[\phi_i \phi_j] = \mathbf{E}[\phi_i^2]\delta_{ij}, \tag{2}$$

where δ_{ij} is the Kronecker delta and $\mathbf{E}[\cdot]$ denotes the expectation with respect to the probability measure $dP(\omega) = f(\Delta(\omega))d\omega$ and probability density function $f(\Delta(\omega))$. Henceforth, we will use Δ to represent $\Delta(\omega)$.

For random variables Δ with certain distributions, the family of orthogonal basis functions $\{\phi_i\}$ can be chosen in such a way that its weight functions has the same form as the probability density function $f(\Delta)$. These orthogonal polynomials are the members of the Askey-scheme of polynomials [8], which forms a complete basis in the Hilbert space determined by their corresponding support. Table I summarizes the correspondence between the choice of polynomials for a given distribution of Δ. When the appropriate set of polynomials is used, the convergence has been shown to be exponential[7].

Random Variable Δ	$\phi_i(\Delta)$ of the Wiener-Askey Scheme
Gaussian	Hermite
Uniform	Legendre
Gamma	Laguerre
Beta	Jacobi

TABLE I

CORRESPONDENCE BETWEEN CHOICE OF POLYNOMIALS AND GIVEN DISTRIBUTION OF $\Delta(\omega)$.

III. STOCHASTIC LINEAR DYNAMICS AND POLYNOMIAL CHAOS

A. System Description

Define a linear stochastic system in the following manner

$$\dot{x}(t,\Delta) = A(\Delta)x(t,\Delta) + B(\Delta)u(t,\Delta), \tag{3}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$. The system has probabilistic uncertainty in the system parameters, characterized by $A(\Delta), B(\Delta)$, which are matrix functions of random variable $\Delta \equiv \Delta(\omega) \in \mathbb{R}^d$ with certain *stationary* distributions. Due to the stochastic nature of (A, B), the system trajectory will also be stochastic. The control $u(t)$ is however may be deterministic or stochastic depending upon the desired implementation.

Let us represent components of $x(t,\Delta), A(\Delta)$ and $B(\Delta)$ as,

$$x(t,\Delta) = [x_1(t,\Delta) \cdots x_n(t,\Delta)]^T, \tag{4}$$

$$A(\Delta) = \begin{bmatrix} A_{11}(\Delta) & \cdots & A_{1n}(\Delta) \\ \vdots & & \vdots \\ A_{n1}(\Delta) & \cdots & A_{nn}(\Delta) \end{bmatrix}, \tag{5}$$

$$B(\Delta) = \begin{bmatrix} B_{11}(\Delta) & \cdots & B_{1m}(\Delta) \\ \vdots & & \vdots \\ B_{n1}(\Delta) & \cdots & B_{nm}(\Delta) \end{bmatrix}. \tag{6}$$

By applying the Wiener-Askey gPC expansion to $x_i(t,\Delta), A_{ij}(\Delta)$ and $B_{ij}(\Delta)$, we get

$$x_i(t,\Delta) = \sum_{k=0}^{p} x_{i,k}(t)\phi_k(\Delta) = \mathbf{x}_i(t)^T \Phi(\Delta), \tag{7}$$

$$u_i(t,\Delta) = \sum_{k=0}^{p} u_{i,k}(t)\phi_k(\Delta) = \mathbf{u}_i(t)^T \Phi(\Delta) \tag{8}$$

$$A_{ij}(\Delta) = \sum_{k=0}^{p} a_{ij,k}\phi_k(\Delta) = \mathbf{a}_{ij}^T \Phi(\Delta), \tag{9}$$

$$B_{ij}(\Delta) = \sum_{k=0}^{p} b_{ij,k}\phi_k(\Delta) = \mathbf{b}_{ij}^T \Phi(\Delta), \tag{10}$$

where $\mathbf{x}_i(t), \mathbf{a}_{ij}, \mathbf{b}_{ij}, \Phi(\Delta) \in \mathbb{R}^p$ are defined by

$$\begin{aligned} \mathbf{x}_i(t) &= [x_{i,0}(t) \cdots x_{i,p}(t)]^T, \\ \mathbf{u}_i(t) &= [u_{i,0}(t) \cdots u_{i,p}(t)]^T, \\ \mathbf{a}_{ij} &= [a_{ij,0}(t) \cdots a_{ij,p}(t)]^T, \\ \mathbf{b}_{ij} &= [b_{ij,0}(t) \cdots b_{ij,p}(t)]^T, \\ \Phi(\Delta) &= [\phi_0(\Delta) \cdots \phi_p(\Delta)]^T. \end{aligned}$$

When $u_i(t,\Delta)$ is a feedback control, it follows that it must also be probabilistic (depending on the implementation), and if the control is not probabilistic, this implies $u_i(t) = u_{i,0}(t)$ with all other coefficients as zero.

The number of terms p is determined by the dimension d of Δ and the order r of the orthogonal polynomials $\{\phi_k\}$, satisfying $p+1 = \frac{(d+r)!}{d!r!}$. The coefficients $a_{ij,k}$ and $b_{ij,k}$ are obtained via Gelarkin projection onto $\{\phi_k\}_{k=0}^p$ given by

$$a_{ij,k} = \frac{\langle A_{ij}(\Delta), \phi_k(\Delta)\rangle}{\langle \phi_k(\Delta)^2\rangle}, \tag{11}$$

$$b_{ij,k} = \frac{\langle B_{ij}(\Delta), \phi_k(\Delta)\rangle}{\langle \phi_k(\Delta)^2\rangle}. \tag{12}$$

The $n(p+1)$ time varying coefficients, $\{x_{i,k}(t)\}; i = 1,\cdots,n; k = 0,\cdots,p$, are obtained by substituting the approximated solution in the governing equation (eqn.(3)) and conducting Gelarkin projection onto $\{\phi_k\}_{k=0}^p$, to yield $n(p+1)$ *deterministic* linear differential equations, given by

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}, \tag{13}$$

with $\mathbf{X} \in \mathbb{R}^{n(p+1)}$; $\mathbf{A} \in \mathbb{R}^{n(p+1)\times n(p+1)}$; $\mathbf{B} \in \mathbb{R}^{n(p+1)\times m}$ and

$$\mathbf{X} = [\mathbf{x}_1^T \ \mathbf{x}_2^T \ \cdots \ \mathbf{x}_n^T]^T, \tag{14}$$

$$\mathbf{U} = [\mathbf{u}_1^T \ \mathbf{u}_2^T \ \cdots \ \mathbf{u}_m^T]^T. \tag{15}$$

While it is possible to derive many forms for the $\mathbf{A}$ and $\mathbf{B}$ matrices, a convenient form can be obtained in the following manner. Define $\hat{e}_{ijk} = \frac{\langle \phi_i, \phi_j \phi_k\rangle}{\langle \phi_i^2\rangle}$. The linear equations of motion can be expressed as

$$\begin{aligned} \dot{x}_{i,l} = & \sum_{j=1}^{n}\sum_{k=0}^{p}\sum_{q=0}^{p} a_{ij,k} x_{j,q} \hat{e}_{lkq} + \\ & \sum_{j=1}^{m}\sum_{k=0}^{p}\sum_{q=0}^{p} b_{ij,k} u_{j,q} \hat{e}_{lkq}. \end{aligned}$$

Define the matrix Φ_k as

$$\Phi_k = \begin{bmatrix} \hat{e}_{1k1} & \hat{e}_{1k2} & \cdots & \hat{e}_{1kp} \\ \hat{e}_{1k2} & \hat{e}_{2k2} & \cdots & \hat{e}_{2kp} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{1kp} & \hat{e}_{2kp} & \cdots & \hat{e}_{pkp} \end{bmatrix}. \tag{16}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1n} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \cdots & \mathbf{A}_{nn} \end{bmatrix}, \tag{17}$$

$$\mathbf{A}_{ij} = \sum_{k=0}^{p} a_{ij,k}\Phi_k, \tag{18}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1m} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n1} & \mathbf{B}_{n2} & \cdots & \mathbf{B}_{nm} \end{bmatrix}, \tag{19}$$

$$\mathbf{B}_{ij} = \sum_{k=0}^{p} b_{ij,k}\Phi_k. \tag{20}$$

More convenient expressions for $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \sum_{k=0}^{p} A_k \otimes \Phi_k, \tag{21}$$

$$\mathbf{B} = \sum_{k=0}^{p} B_k \otimes \Phi_k, \tag{22}$$

where $\otimes$ is the Kronecker product and the A_k, B_k matrices are the projections of each matrix onto their respective polynomial subspaces. Therefore, transformation of a stochastic linear system with $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, with p^{th} order gPC expansion, results in a *deterministic* linear system with increased dimensionality equal to $n(p+1)$.

B. Stochastic Stability

By writing the stochastic system in a deterministic framework, we are able to obtain deterministic equations that can be analyzed to determine the stability properties of a system. This framework will yield one larger LMI as opposed to the many smaller LMI's required to show stability using a Monte-Carlo approach.

Proposition 1: The system in (13) with $u = 0$ is stable if and only if there exists a $P = P^T > 0$ such that

$$\mathbf{A}^T P + P\mathbf{A} \le 0$$

Proof: Choose $V = \mathbf{X}^T P\mathbf{X}$ and utilize the standard Lyapunov argument. ■

This result is presented to demonstrate the power of the approach to enable the study of system stability in terms of well known methodologies.

Remark 1: The number of polynomials should be chosen to minimize inaccuracy in the approximation as the validity of the stability arguments depend upon the accuracy of the polynomial approximation.

The closed-loop stability of a system can be analyzed by utilizing similar arguments.

Proposition 2: Given a feedback control $u(t,\Delta) = Kx(t,\Delta)$, the feedback gain asymptotically stabilizes the distribution of systems if the condition

$$\mathbf{A}^T P + P\mathbf{A} + (K^T \otimes I_p)\mathbf{B}^T P + P\mathbf{B}(K \otimes I_p) < 0$$

is satisfied for some $P = P^T > 0$.

Proof: First, let us look at $u(t,\Delta) = Kx(t,\Delta)$.

$$u_i(t,\Delta) = \sum_{l=0}^{p} u_{i,l}\phi_l = \sum_{j=1}^{n}\sum_{k=0}^{p} k_{ij}x_{j,k}\phi_k$$

By projecting, we find that

$$\mathbf{U} = (K \otimes I_p)\mathbf{X} \tag{23}$$

Therefore, the closed loop system is given by

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}(K \otimes I_p)\mathbf{X}$$

Using the Lyapunov function $V = \mathbf{X}^T P\mathbf{X}$, where $P = P^T > 0$, and taking its derivative implies that the system is asymptotically stable (exponentially stable since this is a linear system) if

$$\mathbf{X}^T \left(\mathbf{A}^T P + P\mathbf{A} + (K \otimes I_p)\mathbf{B}P + P\mathbf{B}(K \otimes I_p)\right)\mathbf{X} < 0.$$

This completes the proof. ■

This result allows us to test the stability of a control law for a family of systems by the analysis of a single deterministic system. It does not make sense to examine marginal stability ($\lambda(\mathbf{A}) = 0$) for these systems because any inaccuracies in the approximation of the distribution could lead to instability. Therefore, the amount of uncertainty in the distribution approximation should be considered when analyzing stability margins.

IV. Stochastic Polynomial Systems

We now consider the problem of analyzing the stability of polynomial systems with stochastic coefficients. The gPC methodology is useful because it preserves the order of polynomial systems. In other words, a q^{th}-order polynomial remains a q^{th}-order polynomial after the substitution. The only difference is the number of system states required to describe the stochastic system.

A. System Description

Consider a system of the form

$$\dot{x}_i(t,\Delta) = \sum_{j=1}^{m} a_{ij}(\Delta)\mathbf{x}^{\alpha_j}(t,\Delta), \tag{24}$$

where m represents the number of terms in the expression, $i = 1,\ldots,n$ represents the number of states, a_{ij} are the coefficients, $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$, and $\alpha_j = [\alpha_{j1} \ \cdots \ \alpha_{jn}]^T$ with $\alpha_{jk} \in \mathbb{N}^+$ is a vector containing the order of each term in the monomial. For example the term given by $x_1^2 x_2^3 x_3 = \mathbf{x}^\alpha$ with $\alpha = [2 \ 3 \ 1]^T$. Note that without loss of generality, this vector does not need to depend upon i because we can just add zeros to a_{ij} for any terms that do not appear in the equations of some state x_i. To apply the gPC expansion to this equation of motion, we write

$$x_i(t,\Delta) = \sum_{k=0}^{p} x_{i,k}(t)\phi_k(\Delta), \tag{25}$$

$$a_{ij}(\Delta) = \sum_{k=0}^{p} a_{ij,k}\phi_k(\Delta), \tag{26}$$

where these forms are familiar as they are identical to those of the linear system.

These expressions can be utilized to derive the equations of motion in a fashion similar to that utilized in the previous section. Consider the term

$$a_{ij}(\Delta)\mathbf{x}^{\alpha_j}(t,\Delta) = \sum_{k=0}^{p}\sum_{k_{11}=0}^{p}\cdots\sum_{k_{1\alpha_{j1}}}^{p}\sum_{k_{21}=0}^{p}\cdots\sum_{k_{n\alpha_{jn}}}^{p}\Big[a_{ij,k}x_{1,k_{11}}\cdots x_{1,k_{1\alpha_{j1}}}x_{2,k_{21}}\cdots x_{n,k_{n\alpha_{jn}}}\phi_k\cdots\phi_{k_{n\alpha_{jn}}}\Big].$$

While this expression involves a large number of summations (the number depends upon the order of each polynomial term), clearly, the order of the polynomial in terms of the vector $\mathbf{x}$ is preserved, however the number of terms in the polynomial has increased dramatically. As was done in the linear case, the equations of motion can be projected onto each polynomial subspace to obtain a system of ODE's in terms of our coefficients. Each equation of motion is then given by

$$\dot{x}_{i,q} = \sum_{j=1}^{m}\sum_{\mathbf{k}=0}^{p}\left[\hat{e}_{q,k,k_{11},\ldots,k_{n\alpha_{jn}}} a_{ij,k}\prod_{r=1}^{n}\prod_{m=1}^{\alpha_{jr}} x_{rk_m}\right], \tag{27}$$

where

$$\hat{e}_{q,k,k_{11},\ldots,k_{n\alpha_{jn}}} = \frac{1}{\langle\phi_q^2\rangle}\langle\phi_q\phi_k\phi_{k_{11}}\cdots\phi_{k_{n\alpha_{jn}}}\rangle,$$

and $\sum_{\mathbf{k}=0}^{p}[\cdot] = \sum_{k=0}^{p}\sum_{k_{11}=0}^{p}\cdots\sum_{k_{n\alpha_{jn}}=0}^{p}[\cdot]$. As an example, consider a polynomial of the form $ax_1^2x_2$ with x_1, x_2, and a as random variables. For this term, $\alpha = [2 \;\; 1]^T$. The gPC expansion of this term is written as

$$ax_1^2x_2 = \sum_{\mathbf{k}=0}^{p} a_k x_{1,k_{11}}x_{1,k_{12}}x_{2,k_{21}}\phi_k\phi_{k_{11}}\phi_{k_{12}}\phi_{k_{21}}$$

In general, we can write the expanded system in the following form

$$\dot{\mathbf{X}} = \sum_{j=1}^{\hat{m}} \hat{\mathbf{a}}_{ij}\mathbf{X}^{\hat{\alpha}_j}, \tag{28}$$

where $\mathbf{X}$ has been previously defined. The term, $\hat{m}$, represents the new number for terms based on the addition of more variables, $\hat{\mathbf{a}}_{ij}$ is the coefficient of each new term, and $\hat{\alpha}_j$ contains the orders of each of the monomials.

B. Stability

Now that it has been shown that the new deterministic system has become a polynomial system of the same order with $n(P+1)$ variables, we can utilize some techniques to talk about the stability of such a system. One such technique is Sum-of-Squares (SOS) programming. We can utilize the SOS framework to discuss the stability of these new polynomials. For details on this approach, see [16], [17], [18] Let $\mathcal{X}(\mathbf{X})$ be a vector of monomials with the property that $\mathcal{X} = 0$ if and only if $\mathbf{X} = 0$. Define a function

$$V = \mathcal{X}^T P \mathcal{X} \tag{29}$$

Furthermore, define a function $W(\mathbf{X})$ that is positive definite in X and is a sum-of-squares polynomial in terms of $\mathbf{X}$.

Proposition 3: The approximation of the family of polynomial systems is stable when a function, V, can be found such that

$$V(\mathbf{X}) - W(\mathbf{X}) \quad \text{is} \quad \text{SOS} \tag{30}$$

$$-\dot{V}(\mathbf{X}) \quad \text{is} \quad \text{SOS} \tag{31}$$

Proof: For proof see[16]. ∎

Remark 2: This result is straight-forward but powerful. It enables the analysis of uncertainty in nonlinear systems in an algorithmic manner that does not require case-by-case analysis of the various changes in the terms.

V. Examples

A. F-16 Control Verification

Here we consider a flight control problem, based on an F-16 aircraft model, where a feedback control K has been designed for the nominal system. We wish to verify the robustness of the controller in the presence of parametric uncertainty in the F-16 model. For simplicity, we assume that the variation in the system parameters are dependent on a single random variable, Δ, meaning that the variation in these parameters is not independent. In general, these parameters could be independent random variables. In this example, we consider the short-period approximation of an F-16. The model is given by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where the state vector $x = [\alpha \quad q \quad x_e]^T$; α is the angle of attack, q is the pitch rate, and x_e is an elevator state which captures actuator dynamics. The control, $u = \delta_{ec}$, is the elevator command in degrees. The matrix parameters are

$$\begin{aligned} A &= \begin{bmatrix} -0.6398 & 0.9378 & -0.0014 \\ (-1.5679) & (-0.8791) & (-0.1137) \\ 0 & 0 & -20.2000 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 & 20.2 \end{bmatrix}^T \\ C &= \begin{bmatrix} 0 & \frac{180}{\pi} & 0 \end{bmatrix} \end{aligned}$$

The values in parenthesis are assumed to be uniformly distributed with 10% deviation about their nominal values. A frequency-domain control has been designed based on feedback of q. The control is of the form

$$u = \frac{0.3122s + 0.5538}{s^2 + 2.128s + 1.132}q.$$

This control is a pitch-rate tracking control. This is converted to state-space to find system matrices A_c, and B_c, C_c. These

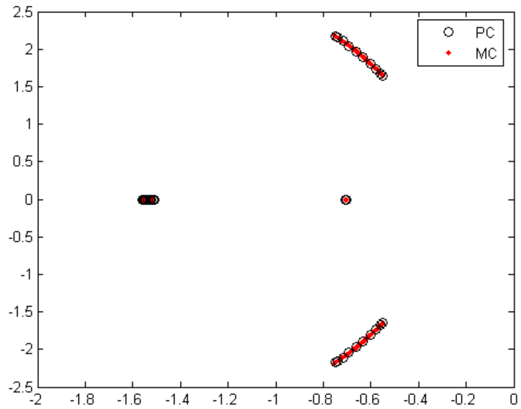

Fig. 1. Eigenvalue distributions of system for ±20%

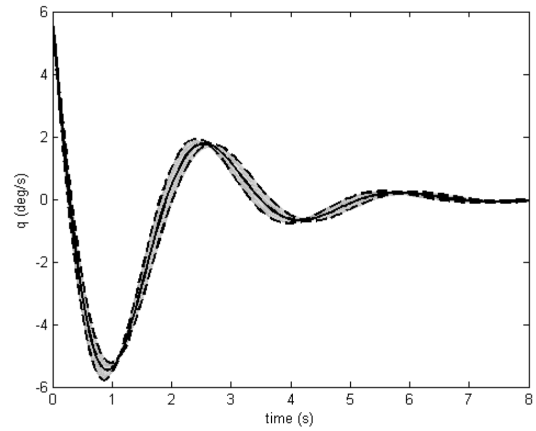

Fig. 2. Predicted and Monte-Carlo system response to ±10% parameter uncertainty

matrices are then augmented to the system to arrive at the closed loop system

$$\dot{x_a} = A_{cl} x_a + B_{cl} u,$$

where

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} \mathbf{0} \\ B_c \end{bmatrix} \end{aligned}$$

The equivalence of the deterministic gPC system and the stochastic system can be inferred from figure 1. The circles (black) represent the eigenvalues of the gPC system with ten terms. The solid (red) dots represent the eigenvalues of the system obtained by sampling the stochastic system over Δ. It is interesting to note that the distribution of eigenvalues of the stochastic system is similar to the set of eigenvalues of the gPC system. This gives us confidence in the use of polynomial chaos for stability analysis and control of stochastic dynamical systems. The gPC system replicates the Monte Carlo behavior in a deterministic framework, as we are able to obtain these eigenvalues all at once from the closed loop A_{cl} matrix. Furthermore, we are able to understand how the system trajectories evolve over time. Figure 2 shows the pitch rate response of the system in the presence of ±10% system uncertainty in the aforementioned parameters. The predicted mean and trajectory bounds from gPC are represented by the dark solid and dashed lines respectively. The Monte-Carlo responses of each system are depicted in gray. We observe that the bounds predicted by the gPC system are in excellent agreement with the responses of the Monte-Carlo simulations. In this manner, we are able to deterministically predict the statistical behavior of the system through examination of the gPC system. In essence, the use of gPC removes the need for a large number of repeated compuations (for Monte-Carlo) but trades repeated compuations for a single simulation over a higher dimensional space. The efficiency of this approach depends on the problem and the complexity of the system.

B. Nonlinear Example

For linear systems with parametric uncertainty appearing linearly in the parameters (as in the previous example), it is possible to determine system stability by examining the stability of the vertex set [4]. While for many nonlinear systems, this may be the case, one cannot in general assume that the stability of the vertex set implies stability of the nonlinear system over the entire range. As a result, it becomes even more important to ensure that stability is guaranteed for entire distribution of parameters. The gPC methodology, in this context, is very useful in the analysis of stability for uncertain nonlinear systems. Proof of stability for the gPC system ensures that the stochastic nonlinear system is stable for the entire distribution of parameter uncertainty. This is exemplified by the following analysis. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + a(\Delta) x_2^3, \end{aligned}$$

we want to understand the stability of this system when a is uncertain and its value is based on a uniform distribution around a mean value of -0.5. For this case, we consider the distribution to vary by ± 0.4 ($a(\Delta) \in [-0.9, -0.1]$). The nominal system is stable, and by utilizing SOSTOOLS (see[19], [20]), we are able to show stability and obtain a Lyapunov function of the form

$$V = .79602 x_1^2 + .70839 x_2^2.$$

To verify the stability of the system, we introduce the gPC expansion and determine the stability of the deterministic system. The deterministic system is another polynomial system of the same order, but with increased dimensionality. To demonstrate the methodology, stability certificates were generated for various values of p, the number of gPC expansions.

For a specific case of $p = 4$, the Lyapunov function is given by,

$$V = Z^T Q Z,$$

where $Z = [x_{23}\ \ x_{22}\ \ x_{21}\ \ x_{20}\ \ x_{13}\ \ x_{12}\ \ x_{11}\ \ x_{10}]^T$, and

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}.$$

The sub-matrices are given by

$$Q_{11} = \begin{bmatrix} 0.5230 & 0.1472 & -0.0659 & 0.0239 \\ 0.1472 & 0.6272 & 0.1509 & -0.0949 \\ -0.0659 & 0.1509 & 0.6814 & 0.1377 \\ 0.0239 & -0.0949 & 0.1377 & 0.7589 \end{bmatrix}$$

$$Q_{22} = \begin{bmatrix} 0.6581 & 0.0510 & -0.0242 & 0.0086 \\ 0.0510 & 0.7073 & 0.0590 & -0.0385 \\ -0.0242 & 0.0590 & 0.7376 & 0.0567 \\ 0.0086 & -0.0385 & 0.0567 & 0.7772 \end{bmatrix}$$

It is interesting to note that for this system, the structure of the Q matrix takes a block diagonal form. The Lyapunov function for the gPC system retains the original structure, i.e. it is also block diagonal. This suggests ways of examining stability and generating certificates for gPC systems. It is important to note that the number of terms in the certificate increases significantly as more coefficients are added. If the structure of the Lyapunov function is unknown, then guessing all possibilities of monomials can lead to problem formulations with large numbers of variables, which are extremely computationally intensive in the SOSTOOLS framework.

VI. Summary

In this paper we present a novel framework for stability analysis of linear and polynomial stochastic systems. We restrict ourselves to systems with probabilistic uncertainty in system parameters. The framework is built on generalized polynomial chaos theory, where stochastic dynamical systems are transformed into equivalent deterministic systems in higher dimensional space. The stability analysis for the stochastic system reduces to stability analysis of the deterministic system. The novelty of the paper is the accurate representation of the uncertainty in the state trajectories due to stochastic parameters and analysis of its stability in the Lyapunov sense. The developed theory is also applied for robustness verification of a linear flight control design for a stochastic F-16 aircraft model. A computational approach for of generating stability certificates for stochastic polynomial systems is also presented.

VII. Acknowledgements

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