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Robust Stabilization of Non-Minimum Phase Nonlinear Systems Using Extended High Gain Observers

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***Abstract*— A robust, stabilizing output feedback controller for systems in the normal form, which could potentially include systems with unstable zero dynamics, is presented. The control scheme adopted herein incorporates "smoothed" sliding mode control–chosen for its robustness properties as well as its ability to prescribe or constrain the motion of trajectories in the sliding phase–and an extended high gain observer to estimate one of the unknown functions. Stabilization in the case of an unknown control coefficient and uncertain constant parameters is shown.**

I. INTRODUCTION

The problem of stabilizing nonlinear systems via output feedback has drawn much attention in recent years. On the basis of certain assumptions made about the system structure, various control schemes have achieved global and semi-global results. A common assumption in many of these results, however, calls for the zero dynamics of the system under consideration to be either input-to-state stable, or globally asymptotically stable. While stabilization results for various classes of minimum phase nonlinear systems abound, there have been a few notable developments in the case of non-minimum phase systems in recent times, as well. Some examples include the work by Karagiannis *et al* [7], which uses a reduced order observer in conjunction with backstepping and the small-gain theorem; Marino and Tomei [10], which provides a stability result on a class of systems that is required to be minimum phase with respect to some linear combination of its states, though it may be non-minimum phase with respect to its output; and also a result by Isidori [3] that guarantees robust, semiglobal practical stability for a significantly large class of nonlinear systems that could be non-minimum phase. This work is particularly noteworthy because it provides a simple and very useful design tool for a broad class of nonlinear systems. The basic idea of an auxiliary system, derived from the original system to help solve the stabilization problem, forms the basis of this paper, and was set forth in [3].

In most applications, the full system state is not available for feedback, and an observer is required to estimate the states from output measurements. Furthermore, observers are often employed to estimate disturbances or uncertainties and to design a controller to reject these disturbances. High-gain observers are desirable in these scenarios because they are often simpler designs compared to other kinds of observers. In the presence of uncertainties, it is also often possible to incorporate an extended high-gain observer that estimates an extra derivative of the output and then use this additional information to cancel disturbances or uncertain parameters in the system, or to estimate a Lyapunov function derivative, etc. For this reason, an extended high-gain observer is utilized in this paper and the analysis of the output feedback system shares many similarities with Freidovich and Khalil [2]. Extended high-gain observers were also utilized in [8]. Furthermore, the output feedback system under consideration in this paper has properties very similar to that of systems addressed in Atassi and Khalil [1], and consequently, all the results from that work pertaining to the separation of controller and observer designs, and of asymptotic performance recovery when utilizing observers with "high enough" gains, are also applicable here.

II. PROBLEM STATEMENT

A single-input, single-output system with relative degree ρ, under a suitable diffeomorphism, can be expressed in the following normal form.

$$\begin{aligned} \dot{\eta} &= \phi(\eta, \xi, \theta), && (1)\\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1, && (2)\\ \dot{\xi}_\rho &= b(\eta, \xi, \theta) + a(\eta, \xi, \theta)u, && (3)\\ y &= \xi_1, && (4) \end{aligned}$$

where $\eta \in D_\eta \subset \mathbb{R}^{n-\rho}$, $\xi \in D_\xi \subset \mathbb{R}^\rho$ and $\theta \in \Theta \subset \mathbb{R}^p$ is a vector of constant parameters. We have the following assumptions about the (possibly uncertain) functions $a(\cdot), b(\cdot)$ and $\phi(\cdot)$.

Assumption 1: The functions $a(\cdot)$ and $b(\cdot)$ are continuously differentiable with locally Lipschitz derivatives, and $\phi(\cdot)$ is locally Lipschitz. In addition, $a(0, 0, \theta) \neq 0$, $b(0, 0, \theta) = 0$ and $\phi(0, 0, \theta) = 0$.

In this paper, the objective is to find a robust, stabilizing output feedback controller for systems of the form (1)–(3), which could potentially include systems with unstable zero dynamics. A similar problem was investigated by Isidori [3], [9], without considering the possibility of an uncertainty in the control coefficient $a(\cdot)$. In the technique developed by Isidori, the stabilization problem can be solved, provided an auxiliary system–defined below–can be globally stabilized

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by a dynamic feedback controller. The auxiliary system is defined by ([3], [9]),

$$\begin{aligned}
\dot{\eta} &= \phi(\eta, \xi_1, \ldots, \xi_{\rho-1}, u_a, \theta), && (5)\\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 2, && (6)\\
\dot{\xi}_{\rho-1} &= u_a, && (7)\\
y_a &= b(\eta, \xi_1, \ldots, \xi_{\rho-1}, u_a, \theta). && (8)
\end{aligned}$$

Equations (5)–(7) come from (1)–(2) by viewing ξ_ρ as the control input u_a, while the term $b(\eta, \xi, \theta)$ on the right-hand side of (3) is taken as the measured output.

This auxiliary problem is assumed to have a stabilizing dynamic controller of the form

$$\begin{aligned}
\dot{z} &= L(z, \xi_1, \ldots, \xi_{\rho-1}) + M(z, \xi_1, \ldots, \xi_{\rho-1}) y_a, && (9)\\
u_a &= N(z, \xi_1, \ldots, \xi_{\rho-1}), && (10)
\end{aligned}$$

where $z \in D_z \subset \mathbb{R}^r$. Under this assumption, it was shown by Isidori that a dynamic feedback law exists, which can robustly stabilize the original system [3, equation (27)], [9]. In this paper, a combination of a "smoothed" sliding mode control and an extended high gain observer is utilized to stabilize the original system using only measurement of the output y. The sliding mode control is designed to force the system trajectories to a manifold within a finite time, along which the system response coincides with a perturbed version of the auxiliary system, and hence the performance of the auxiliary system may be recovered under certain conditions and constraints on the model uncertainties.

Following [9], new variables are defined to help make the equations more compact. Let

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\rho \end{pmatrix}, \quad x_a = \begin{pmatrix} \eta \\ \xi_1 \\ \vdots \\ \xi_{\rho-2} \\ \xi_{\rho-1} \end{pmatrix}, \quad \zeta = \begin{pmatrix} x_a \\ z \end{pmatrix},$$

$$f_a(x_a, u_a, \theta) = \begin{pmatrix} \phi(\eta, \xi_1, \ldots, \xi_{\rho-1}, u_a, \theta) \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \\ u_a \end{pmatrix},$$

$$\begin{aligned}
h_a(x_a, u_a, \theta) &= b(\eta, \xi_1, \ldots, \xi_{\rho-1}, u_a, \theta), && (11)\\
F(\zeta, \xi_\rho, \theta) &= \begin{pmatrix} f_a(x_a, \xi_\rho, \theta) \\ L(\cdot) + M(\cdot) b(\cdot) \end{pmatrix}. && (12)
\end{aligned}$$

We can now write the state equation (1)–(3) as

$$\begin{aligned}
\dot{x}_a &= f_a(x_a, \xi_\rho, \theta), && (13)\\
\dot{\xi}_\rho &= h_a(x_a, \xi_\rho, \theta) + a(\eta, \xi, \theta) u, && (14)
\end{aligned}$$

while the auxiliary system now looks like

$$\begin{aligned}
\dot{x}_a &= f_a(x_a, u_a, \theta), && (15)\\
y_a &= h_a(x_a, u_a, \theta). && (16)
\end{aligned}$$

Thus, upon feeding back the output of the stabilizing dynamic controller to the system input, the resulting closed loop auxiliary system can be written as

$$\dot{\zeta} = F(\zeta, N(\cdot), \theta), \qquad (17)$$

wherein $\xi_\rho = N(z, \xi_1, \ldots, \xi_{\rho-1})$.

III. Controller Design for the State Feedback System

The controller to be designed utilizes "smoothed" sliding mode control. To motivate this choice, we consider the auxiliary system, and equations (7) and (10) in particular. Since u_a replaced ξ_ρ in going from the original problem to the auxiliary system, it stands to reason that if we are able to force the state ξ_ρ to the function $N(\cdot)$, then equations (1) and (2) will match the auxiliary system (17) precisely. Thus, the sliding manifold is taken as

$$s = \xi_\rho - N(z, \xi_1, \ldots, \xi_{\rho-1}), \qquad (18)$$

and the controller for this system is now taken to be

$$\begin{aligned}
\dot{z} &= L(\cdot) + M(\cdot)(b + \Delta_a l(\xi, z)), && (19)\\
u &= -\frac{\beta(\xi, z)}{\hat{a}(\xi)} \operatorname{sat}\left(\frac{\xi_\rho - N(z, \xi_1, \ldots, \xi_{\rho-1})}{\mu} \right) && (20)\\
&\triangleq l(\xi, z),
\end{aligned}$$

where $\operatorname{sat}(\cdot)$ is the standard saturation function, $\hat{a}(\xi)$ is a known function used in lieu of $a(\eta, \xi, \theta)$, $\Delta_a = a(\cdot) - \hat{a}(\cdot)$, and $\beta(\xi, z)$ is to be determined. We note that $a(\cdot) \neq 0$ and $\hat{a}(\cdot) \neq 0$ in the set D_ξ. We also have the following assumption about $a(\cdot)$.

Assumption 2: Suppose that $a(\eta, \xi, \theta)/\hat{a}(\xi) \ge k_0 > 0$, for all $(\eta, \xi) \in D_\eta \times D_\xi$ and $\theta \in \Theta$. Moreover, $\hat{a}(\xi)$ is locally Lipschitz in ξ over the domain of interest and globally bounded in ξ.

The term $b + \Delta_a l$ in (19) is based on the anticipation that, with the extended high-gain observer, we will be able to estimate this term arbitrarily closely. Roughly speaking, this can be seen from (3) as follows. If the derivative $\dot{\xi}_\rho = y^{(\rho)}$ and $a(\cdot)$ were available, we could calculate b from the expression $b = \dot{\xi}_\rho - au$. If only an estimate $\hat{a}(\cdot)$ were available for the uncertain term $a(\cdot)$, we would use $\hat{b} = \dot{\xi}_\rho - \hat{a}u$, which results in the error $\hat{b} - b = (a - \hat{a})u = \Delta_a u$. By requiring $\hat{a}$ to depend only on ξ, we have set up the problem such that the variables that need to be estimated are the derivatives of the output y up to the ρth order, which will be the role of the extended high-gain observer. Note that if $a(\cdot)$ were a known function of ξ, we could take $\hat{a} = a$ and drop Δ_a from (19).

Any dynamic feedback law of the form (19)–(20) is allowable to stabilize this system, provided it satisfies the following properties.

Assumption 3:

1) $L(\cdot)$, $M(\cdot)$ and $\beta(\cdot)$ are locally Lipschitz functions in their arguments over the domain of interest, $L(0, 0, \ldots, 0, 0) = 0$, $M(0, 0, \ldots, 0, 0) = 0$ and $\beta(0, 0) = 0$.

2) $L(\cdot)$, $M(\cdot)$ and $\beta(\cdot)$ are globally bounded functions of ξ.
3) The origin ($\eta = 0, \xi_1 = 0, \ldots, \xi_{\rho-1} = 0, z = 0$) is an asymptotically stable equilibrium point of the closed-loop auxiliary system (17).

IV. Analysis of the State Feedback System

The closed loop system is obtained by combining (13), (14), (19) and (20), and can now be written as

$$\begin{aligned}\dot{\zeta} &= \tilde{F}(\zeta, s + N(\cdot), \theta), && (21)\\ \dot{\xi}_\rho &= h_a(x_a, \xi_\rho, \theta) \\ &\quad - \frac{a(\eta, \xi, \theta)}{\hat{a}(\xi)} \beta(\zeta, \xi_\rho) \operatorname{sat}\left(\frac{s}{\mu}\right), && (22)\end{aligned}$$

In the equations above, $\tilde{F}(\zeta, \xi_\rho, \theta)$ is a perturbation of $F(\zeta, \xi_\rho, \theta)$, in that the function $b(\cdot)$ from the latter has been replaced with $b + \Delta_a l(\xi, z)$ in the former. The derivative of $s = \xi_\rho - N(z, \xi_1, \ldots, \xi_{\rho-1})$ is given by

$$\begin{aligned}\dot{s} &= h_a(x_a, \xi_\rho, \theta) - \frac{\partial N}{\partial \zeta} \tilde{F}(\zeta, \xi_\rho, \theta) \\ &\quad - \frac{a(\eta, \xi, \theta)}{\hat{a}(\xi)} \beta(\zeta, \xi_\rho) \operatorname{sat}\left(\frac{s}{\mu}\right). \qquad (23)\end{aligned}$$

We choose the function $\beta(\cdot)$ such that

$$\left| \frac{h_a(\cdot) - \frac{\partial N}{\partial \zeta} \tilde{F}(\cdot)}{a(\cdot)/\hat{a}(\cdot)} \right| \leq \beta(\xi, z) - \beta_0, \qquad (24)$$

for some $\beta_0 > 0$. We now consider the Lyapunov function $V_0(s) = s^2/2$. Outside the boundary layer, i.e. when $|s| > \mu$, we have

$$\begin{aligned}\dot{V}_0 = s\dot{s} &= \left(h_a(\cdot) - \frac{\partial N}{\partial \zeta} \tilde{F}(\cdot) \right) s - \frac{a(\cdot)}{\hat{a}(\cdot)} \beta(\cdot) |s| \\ &\leq -\beta_0 a(\cdot)/\hat{a}(\cdot) |s| \leq -\beta_0 k_0 |s|. \qquad (25)\end{aligned}$$

Hence, whenever $|s(0)| > \mu$, $|s(t)|$ will decrease until it reaches the set $\{|s| < \mu\}$ in finite time and will remain inside that set thereafter.

We would now like to study the behavior of the remaining states. We write the equations for these states as follows.

$$\dot{\zeta} = F(\zeta, s + N(\cdot), \theta) + G(\zeta, s + N(\cdot), \theta), \qquad (26)$$

where

$$G(\cdot) = \tilde{F}(\cdot) - F(\cdot) = \begin{pmatrix} 0 \\ M(\cdot) \end{pmatrix} \Delta_a(\cdot) l(\cdot).$$

Equation (26) can be regarded as a perturbation of the following system.

$$\dot{\zeta} = F(\zeta, s + N(\cdot), \theta). \qquad (27)$$

In order to proceed with the analysis, we require the following assumption that calls for the existence of a Lyapunov function for the above system.

Assumption 4: There exists a continuously differentiable Lyapunov function $V(\zeta, \theta)$ such that

$$\begin{aligned}\alpha_{11}(\|\zeta\|) \leq V(\zeta, \theta) &\leq \alpha_{12}(\|\zeta\|), && (28)\\ \frac{\partial V}{\partial \zeta} F(\zeta, s + N(\cdot), \theta) &\leq -\alpha_{13}(\|\zeta\|), && (29)\\ &\qquad \forall \|\zeta\| \geq \gamma(\|s\|),\end{aligned}$$

for all $(\zeta, s) \in D \subset \mathbb{R}^{n+r}$, where $\alpha_{11}(\cdot), \alpha_{12}(\cdot), \alpha_{13}(\cdot)$ and $\gamma(\cdot)$ are class $\mathcal{K}$ functions.

Remark 1: Inequality (29) is equivalent to regional input-to-state stability of the system (27) with s viewed as an input. Let us now consider the set

$$\Omega \triangleq \{V(\zeta, \theta) \leq c_0\} \times \{|s| \leq c\}, \qquad (30)$$

where $c > \mu$ and $c_0 \geq \alpha_{12}(\gamma(c))$. As shown in the development preceding [5, Theorem 14.1 (§14.1.2)], Ω is a positively invariant set for the system (27) when $\Delta_a = 0$. Now, to study the effect of Δ_a, we have, due to Assumption 3,

$$\|G(\cdot)\| \leq k_1 |\Delta_a l(\cdot)| \leq k_1 \left| \frac{a - \hat{a}}{\hat{a}} \right| \beta,$$

$\forall (\zeta, s) \in \Omega$. The derivative of V along the trajectories of (26) is given by

$$\dot{V} = \frac{\partial V}{\partial \zeta} F(\zeta, s + N(\cdot), \theta) + \frac{\partial V}{\partial \zeta} G(\zeta, s + N(\cdot), \theta).$$

On the boundary $V = c_0$, we have

$$\frac{\partial V}{\partial \zeta} F(\zeta, s + N(\cdot), \theta) \leq -\alpha_{13}(\|\zeta\|) \leq \alpha_{13}(\alpha_{12}^{-1}(c_0)).$$

We note that $\partial V / \partial \zeta$ and β are both bounded on Ω. Hence, $\left\| \frac{\partial V}{\partial \zeta} G(\cdot) \right\| \leq k_2 \left| \frac{a - \hat{a}}{\hat{a}} \right|$. Thus on $V = c_0$,

$$\dot{V} \leq -\alpha_{13}(\alpha_{12}^{-1}(c_0)) + k_2 \left| \frac{a - \hat{a}}{\hat{a}} \right|.$$

Assuming that the inequality

$$\left| \frac{a - \hat{a}}{\hat{a}} \right| \leq k_a \leq \frac{1}{k_2} \alpha_{13}(\alpha_{12}^{-1}(c_0)) \qquad (31)$$

holds over the set Ω, we see that

$$\dot{V} < 0 \text{ on } V = c_0.$$

Hence, Ω is a positively invariant set for the system (26). We note that k_2 is dependent on c_0.

As a consequence of the preceding analysis, all trajectories starting in Ω, stay in Ω and reach the boundary layer $\{|s| \leq \mu\}$ in finite time. Inside the boundary layer, we have

$$\begin{aligned}\dot{V} &\leq -\alpha_{13}(\|\zeta\|) + k_2 \left| \frac{a - \hat{a}}{\hat{a}} \right|, \ \forall \|\zeta\| \geq \gamma(\mu) \\ &\leq -\alpha_{13}(\|\zeta\|) + k_2 k_a \\ &\leq -\frac{1}{2} \alpha_{13}(\|\zeta\|), \ \forall \|\zeta\| \geq \max\left\{\gamma(\mu), \alpha_{13}^{-1}(2k_2k_a)\right\}.\end{aligned}$$

The inverse function $\alpha_{13}^{-1}(2k_2k_a)$ is well-defined when k_a is small enough, except when $\alpha_{13}(\|\zeta\|)$ is class $\mathcal{K}_\infty$. In the latter case, the inverse is defined for any k_2k_a.

The above analysis shows that the trajectories reach the positively invariant set

$$\begin{aligned}\Omega_\mu &= \left\{ V(\zeta, \theta) \leq \alpha_{12}\left(\max\left\{\gamma(\mu), \alpha_{13}^{-1}(2k_2k_a)\right\}\right)\right\} \\ &\quad \times \{|s| \leq \mu\},\end{aligned}$$

in finite time. Inside the set Ω_μ, we can use singular perturbation analysis to show that the origin of (21), (23) is asymptotically stable for sufficiently small μ and $|\Delta_a|$. Some additional regularity assumptions will be needed. The details are omitted due to space limitations.

V. Output Feedback Design Using an Extended High-Gain Observer

The output feedback design relies upon the estimates of the states and of $b(\eta, \xi, \theta)$ that are obtained using an extended high gain observer for the system (1)–(3), which is taken as

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + (\alpha_i/\varepsilon^i)(\xi_1 - \hat{\xi}_1),\ 1 \le i \le \rho - 1, && (32)\\ \dot{\hat{\xi}}_\rho &= \hat{\sigma} + \hat{a}(\hat{\xi})u + (\alpha_\rho/\varepsilon^\rho)(\xi_1 - \hat{\xi}_1), && (33)\\ \dot{\hat{\sigma}} &= (\alpha_{\rho+1}/\varepsilon^{\rho+1})(\xi_1 - \hat{\xi}_1), && (34) \end{aligned}$$

where ε is a positive constant to be specified, and the positive constants α_i are chosen such that the roots of $s^{\rho+1} + \alpha_1 s^\rho + \ldots + \alpha_\rho s + \alpha_{\rho+1} = 0$ are in the open left-half plane. It is apparent from (32)–(34) that the $\hat{\xi}_1, \ldots, \hat{\xi}_\rho$ are used to estimate the output and its first ρ derivatives, while $\hat{\sigma}$ is intended to provide an estimate for $b(\cdot)$. With the aid of these estimates, the output feedback controller for the original system can be taken as

$$\begin{aligned} \dot{z} &= L(z, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}, \hat{\sigma}), && (35)\\ u &= -\frac{\beta(\hat{\xi}, z)}{\hat{a}(\hat{\xi})} \operatorname{sat}\left(\frac{\hat{\xi}_\rho - N(z, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1})}{\mu}\right). && (36) \end{aligned}$$

In order to protect the system from peaking during the observer's transient response, we saturate the control outside the compact set of interest as given in (30). Now, let

$$K > \max_{(\zeta,s)\in\Omega} \left|\frac{\beta(\xi, z)}{\hat{a}(\xi)}\right|. \qquad (37)$$

Saturating the control at $\pm K$, we obtain the output feedback controller

$$u = K \operatorname{sat}(l(\hat{\xi}, z)/K). \qquad (38)$$

The closed loop system under output feedback can now be expressed as

$$\begin{aligned} \dot{\eta} &= \phi(\eta, \xi, \theta), && (39)\\ \dot{\xi} &= A\xi + B[b(\eta, \xi, \theta) \\ &\quad + a(\eta, \xi, \theta) K \operatorname{sat}(l(\hat{\xi}, z)/K)], && (40)\\ \dot{z} &= L(z, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) \\ &\quad + M(z, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1})\hat{\sigma}, && (41)\\ \dot{\hat{\xi}} &= A\hat{\xi} + B\left[\hat{\sigma} + \hat{a}(\hat{\xi}) K \operatorname{sat}(l(\hat{\xi}, z)/K)\right] \\ &\quad + H(\varepsilon)(y - C\hat{\xi}), && (42)\\ \dot{\hat{\sigma}} &= (\alpha_{\rho+1}/\varepsilon^{\rho+1})(y - C\hat{\xi}), && (43)\\ y &= C\xi, && (44) \end{aligned}$$

where A, B and C describe a chain of integrators, $H(\varepsilon) = \begin{pmatrix}\alpha_1/\varepsilon & \alpha_2/\varepsilon^2 & \ldots & \alpha_\rho/\varepsilon^\rho\end{pmatrix}^T$ and $\alpha_1, \ldots, \alpha_\rho, \alpha_{\rho+1}$ are real scalars such that the polynomial $s^{\rho+1} + \alpha_1 s^\rho + \ldots + \alpha_{\rho+1}$ is Hurwitz.

We now introduce a change of variables [2],

$$\chi_i \triangleq (\xi_i - \hat{\xi}_i)/\varepsilon^{\rho+1-i}, \quad \text{for } 1 \le i \le \rho, \qquad (45)$$

$$\begin{aligned} \chi_{\rho+1} &\triangleq b(\eta, \xi, \theta) - \hat{\sigma} \\ &\quad + \Delta_a(\varepsilon, \eta, \xi, \chi, \theta) K g_\varepsilon(l(\xi, z)/K), && (46) \end{aligned}$$

where $\Delta_a(\varepsilon, \eta, \xi, \chi, \theta) = a(\eta, \xi, \theta) - \hat{a}(\hat{\xi})$, and $g_\varepsilon(\cdot)$ is an odd function defined by

$$g_\varepsilon(y) = \begin{cases} y, & 0 \le y \le 1,\\ y + (y-1)/\varepsilon - 0.5(y^2 - 1)/\varepsilon, & 1 < y < 1 + \varepsilon,\\ 1 + 0.5\varepsilon, & y \ge 1 + \varepsilon. \end{cases}$$

The function $g_\varepsilon(\cdot)$ is nondecreasing, continuously differentiable with a locally Lipschitz derivative, bounded uniformly in ε on any bounded interval of ε, and satisfies $|g'_\varepsilon| \le 1$ and $|g_\varepsilon(y) - \operatorname{sat}(y)| \le \varepsilon/2$ for all $y \in \mathbb{R}$. The closed loop system can now be expressed in terms of (39)–(41), (44) and

$$\varepsilon\dot{\chi} = \Lambda\chi + \varepsilon[\bar{B}_1\Delta_1(\xi, \eta, \chi, \theta, \varepsilon) + \bar{B}_2\Delta_2(\xi, \eta, \chi, \theta, \varepsilon)], \qquad (47)$$

where Λ is a Hurwitz matrix, and the functions Δ_1 and Δ_2 are locally Lipschitz in their arguments and bounded from above by affine-in-$\|\eta\|$ functions, uniformly in ε. We can use (45) and the definition of g_ε to show that Δ_0/ε is locally Lipschitz [2]. The slow dynamics can be written as

$$\dot{\bar{\zeta}} = \begin{pmatrix}\dot{\zeta}\\ \dot{s}\end{pmatrix} = f_r(\bar{\zeta}, \chi, \varepsilon, \theta), \qquad (48)$$

where $\bar{\zeta} \triangleq \begin{pmatrix}\zeta\\ s\end{pmatrix}$ and

$$f_r(\bar{\zeta}, \chi, \varepsilon, \theta) \triangleq \begin{pmatrix} \phi(\eta, \xi, \theta)\\ \xi_2\\ \vdots\\ \xi_\rho\\ L(z, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}) + M(z, \hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1})\hat{\sigma}\\ b(\cdot) - \frac{\partial N}{\partial \zeta}\tilde{F}(\zeta, s + N(\cdot), \theta) + a(\cdot) K \operatorname{sat}\left(\frac{l(\hat{\xi}, z)}{K}\right) \end{pmatrix}.$$

Then, the reduced system is given by

$$\begin{aligned} \dot{\bar{\zeta}} &= f_r(\bar{\zeta}, 0, 0, \theta) = \\ &= \begin{pmatrix} \tilde{F}(\zeta, s + N(\cdot), \theta)\\ \begin{cases} b(\eta, \xi, \theta) - \frac{\partial N}{\partial \zeta}\tilde{F}(\zeta, s + N(\cdot), \theta)\\ + a(\eta, \xi, \theta) K \operatorname{sat}(l(\xi, z)/K) \end{cases} \end{pmatrix}, && (49) \end{aligned}$$

where, as before, $\tilde{F}(\zeta, s + N(\cdot), \theta)$ is a perturbation of $F(\zeta, s + N(\cdot), \theta)$, in that the function $b(\cdot)$ has been replaced with the quantity $\hat{\sigma} = b + \Delta_a(\cdot) K \operatorname{sat}(l(\hat{\xi}, z)/K)$. We note that the reduced system (49) is identical to the closed-loop system (21), (23).

The boundary layer system is given by

$$\frac{\partial \chi}{\partial \tau} = \Lambda\chi. \qquad (50)$$

In the subsections below, we state some results pertaining to the recovery of the performance of the system (21), (23) in the case of output feedback with sufficiently small ε. These results follow quite directly from the work of Atassi and Khalil [1] because equations (39)–(41) and (47) are precisely of the same form as equations (10)–(13) in [1]. It should also be noted that while the structure of system (39)–(41), (47) is very similar to that of the closed-loop system studied by Freidovich and Khalil [2], a key difference between this work and the former is the fact that both the nonlinear terms in the fast dynamics (47) are $O(\varepsilon)$ in this paper, while there is one term in [2] that is not so. The reason for such a term to appear in [2] was the fact that the estimate $\hat{\sigma}$ was used in the control law, while this quantity is not utilized directly in the control, in this paper. We can see this by examining (38) and (41)—$\hat{\sigma}$ appears in the compensator dynamics, but u is not explicitly dependent upon the former.

A. Boundedness, Ultimate Boundedness and Trajectory Convergence

Let $(\bar{\zeta}(t,\varepsilon),\chi(t,\varepsilon))$ denote the trajectory of the system (48), (47) starting from $(\bar{\zeta}(0),\chi(0))$. Also, let $\bar{\zeta}_r(t)$ be the solution of (49) starting from $\bar{\zeta}(0)$. Suppose the system (21), (23) has an asymptotically stable equilibrium point at the origin, and let $\mathcal{R}$ be its region of attraction. Let $\mathcal{S}$ be a compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be a compact subset of $\mathbb{R}^{\rho+1}$. The recovery of boundedness of the trajectories, the fact that they will be ultimately bounded, and the fact that $\bar{\zeta}(t,\varepsilon)$ converges to $\bar{\zeta}_r(t)$ as $\varepsilon \to 0$, uniformly in t, for all $t \geq 0$, is established by the following theorem [1, Theorems 1, 2 and 3], provided $\bar{\zeta}(0) \in \mathcal{S}$ and $\hat{\xi}(0) \in \mathcal{Q}$.

Theorem 1: Let Assumptions 1 through 4 hold, and suppose that the origin of (21), (23) is asymptotically stable. Moreover, let $\bar{\zeta}(0) \in \mathcal{S}$ and $\hat{\xi}(0) \in \mathcal{Q}$. Then,

1) there exists $\varepsilon_1^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the trajectories $(\bar{\zeta},\chi)$ of the system (48), (47), starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t \geq 0$.
2) given any $\delta > 0$, there exist $\varepsilon_2^* = \varepsilon_2^*(\delta) > 0$ and $T_1 = T_1(\delta)$ such that, for every $0 < \varepsilon \leq \varepsilon_2^*$, we have
$$\left\|\bar{\zeta}(t,\varepsilon)\right\| + \|\chi(t,\varepsilon)\| \leq \delta, \quad \forall t \geq T_1. \tag{51}$$
3) given any $\delta > 0$, there exists $\varepsilon_3^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_3^*$, we have
$$\left\|\bar{\zeta}(t,\varepsilon) - \bar{\zeta}_r(t)\right\| \leq \delta, \quad \forall t \geq 0. \tag{52}$$

B. Recovery of Exponential Stability of the Origin

We consider the case where the origin of (49) is exponentially stable. We then have the following result, due to [1, Theorem 5]. We note that the Lipschitz conditions imposed by Assumptions 1, 3 and the assumption of exponential stability of the state feedback system are the key requirements for this result.

Theorem 2: Let Assumptions 1 through 4 hold, and suppose the vector field $f_r(\bar{\zeta},0,0,\theta)$ is continuously differentiable around the origin. Furthermore, assume the origin of the full closed-loop system (21) and (23) is exponentially stable. Then, there exists $\varepsilon_5^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_5^*$, the origin of system (48), (47) is exponentially stable.

VI. An Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= \tan x_1 + x_2, & (53)\\ \dot{x}_2 &= x_1 + u, & (54)\\ y &= x_2. & (55) \end{aligned}$$

We note that this system has relative degree one, and is already in the normal form. The auxiliary system is

$$\begin{aligned} \dot{x}_1 &= \tan x_1 + u_a, & (56)\\ y_a &= x_1. & (57) \end{aligned}$$

The design of the stabilizing controller is now carried out by first considering the auxiliary system and proceeding in a step-by-step fashion, as shown in sections 3 and 5.

A. Design of the Stabilizing Controller

A dynamic compensator for the auxiliary system (56), (57) is designed using a low-pass filter and feedback linearization. This gives the controller

$$\begin{aligned} \dot{z} &= (y_a - z)/\varepsilon_a, & (58)\\ u_a &= -\tan z - z. & (59) \end{aligned}$$

Now, based on our knowledge of the stabilizing dynamic controller (58), (59) for the auxiliary system (56), (57), we choose our sliding manifold for the partial state feedback system, and the resulting control as

$$\begin{aligned} s &= x_2 + z + \tan z, & (60)\\ \dot{z} &= (x_2 - z)/\varepsilon_a, & (61)\\ u &= -\frac{\beta(z,x_2)}{\hat{a}} \operatorname{sat}\left(\frac{s}{\mu}\right), & (62) \end{aligned}$$

where $\beta(z,x_2)$ satisfies the inequality (24) in a compact set of interest, and $\hat{a}$ is a nominal value of the control coefficient $a = 1$ in the original plant.

To stabilize the system using only output feedback and an estimate of $b(x)$, we have the following extended high-gain observer.

$$\begin{aligned} \dot{\hat{x}}_2 &= \hat{\sigma} + \hat{a}u + (\alpha_1/\varepsilon)(y - \hat{x}_2) & (63)\\ \dot{\hat{\sigma}} &= (\alpha_2/\varepsilon)(y - \hat{x}_2). & (64) \end{aligned}$$

Hence, the output feedback controller is given by

$$\begin{aligned} \dot{z} &= (\hat{\sigma} - z)/\varepsilon_a, & (65)\\ u &= -\frac{\beta(z,x_2)}{\hat{a}} \operatorname{sat}\left(\frac{y + z + \tan z}{\mu}\right). & (66) \end{aligned}$$

B. Numerical Simulations

The step-by-step tuning procedure leading to performance recovery is illustrated by Figures 1 and 2. β was chosen to be 55. The compensator parameter ε_a was fixed at 0.1 and the width of the boundary layer was reduced in the partial state feedback system from $\mu = 1$ down to 0.01, and Figure 1 shows the recovery of the auxiliary system performance. Next, μ was fixed at 0.01 and the extended high-gain observer parameter ε was reduced from $\varepsilon = 10^{-3}$ to 10^{-4}, and Figure 2 shows the recovery of the performance of the state feedback system.

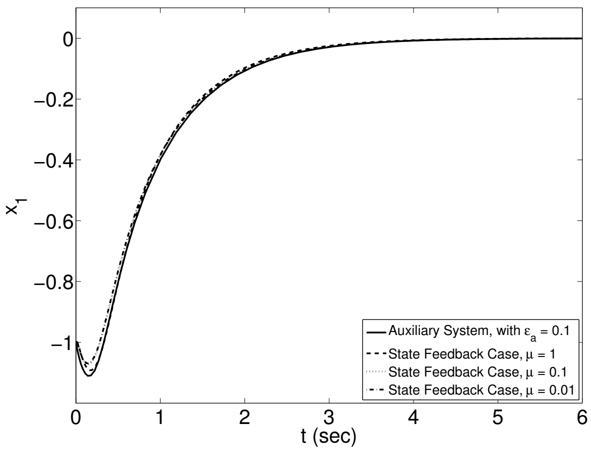

Fig. 1. Recovery of the Auxiliary System performance.

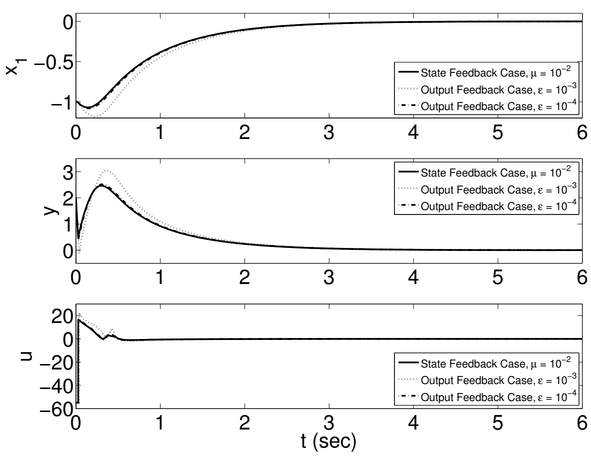

Fig. 2. Recovery of the Partial State Feedback System performance by decreasing ε.

VII. Conclusion

A robust, stabilizing output feedback controller for systems in the normal form, which could potentially include systems with unstable zero dynamics, was presented. The control scheme adopted herein incorporated "smoothed" sliding mode control and an extended high gain observer to estimate one of the unknown functions. Stabilization in the case of an unknown control coefficient and uncertain constant parameters was shown. The main result in the output feedback case was practical stabilization of the system and closeness of trajectories, while exponential stability is achievable provided the auxiliary problem is also exponentially stable.

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