2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Adaptive Output Feedback Control for Complex-Valued Reaction-Advection-Diffusion Systems

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***Abstract*— We study a problem of output feedback stabilization of complex-valued reaction-advection-diffusion systems with parametric uncertainties (these systems can also be viewed as coupled parabolic PDEs). Both sensing and actuation are performed at the boundary of the PDE domain and the unknown parameters are allowed to be spatially varying. First, we transform the original system into the form where unknown functional parameters multiply the output, which can be viewed as a PDE analog of observer canonical form. Input and output filters are then introduced to convert a dynamic parametrization of the problem into a static parametrization where a gradient estimation algorithm is used. The control gain is obtained by solving a simple complex-valued integral equation online. The solution of the closed-loop system is shown to be bounded and asymptotically stable around the zero equilibrium. The results are illustrated by simulations.**

I. Introduction

We consider a problem of boundary stabilization of unstable complex-valued reaction-advection-diffusion systems with parametric uncertainties. The existing results on adaptive control for distributed parameter systems [3], [4], [5], [8], [11], besides dealing with real-valued plants, typically rely on full-state measurements. In this paper we deal with complex-valued plants with only boundary measurements. We assume that plant parameters are unknown functions of spatial variable. Therefore, both the state space and the parameter space are infinite dimensional, while the input and the output are scalar.

In a recent paper [10], adaptive output feedback controllers were introduced for parabolic PDEs with measurements on the opposite sides of the PDE domain. We extend the result of [10] in two ways. First, we consider PDEs with complex-valued state, which brings about several model equations important in applications, most notably the Schrödinger equation and the Ginzburg-Landau equation [12].

Second, we deal with the case of Neumann measurements, common in fluid problems, where one typically measures pressure and shear stress. In [10], the adaptive controllers were designed for Dirichlet sensing, typical in thermal/chemical problems where temperature or concentration are measured.

These two extensions make the result of this paper applicable to the problem of suppression of vortex shedding off of a bluff body in a fluid flow, for which Ginzburg-Landau PDE is a very good approximate model [1]. The control design would be implemented with pressure sensors located on the bluff body and downstream micro-jet actuators. The non-adaptive backstepping design for this problem has been introduced in [1], and the observers have been developed in [2]. However, since the vortex shedding problem is fully modelled only by Navier-Stokes equations, the parameters of the Ginzburg-Landau model are very difficult to derive analytically and adaptive control design becomes crucial.

The key feature of our design is the transformation of the original plant into an infinite-dimensional analog of the observer canonical form. Our adaptive observers are infinite dimensional extensions of Kreisselmeier observers [6] and the controllers are designed using the backstepping method [9]. We employ swapping identifiers with a gradient update law.

Even though the exact identification of the parameters is not the objective of this paper, the numerical results in Section IX indicate that the steady-state profiles of the parameter estimates are close to the true unknown functions.

II. Reaction-Diffusion System

We consider a reaction-diffusion plant

$$A_t(x,t) = aA_{xx}(x,t) + b(x)A(x,t), \quad 0 < x < 1\,, \tag{1}$$

where A is a complex-valued function. The boundary conditions are

$$A(0,t) = 0 \tag{2}$$

$$A(1,t) = U(t)\,, \tag{3}$$

where $U(t)$ is the control input. We assume that only boundary value $A_x(0,t)$ is available for measurement (this corresponds to pressure sensing in vortex shedding problem). The parameter a is a known constant that satisfies $\text{Re}(a) > 0$ and $b(x)$ is an unknown complex-valued continuous function. Without loss of generality, we assume that $\text{Re}(a) = 1$ (one can always achieve that with appropriate scaling of time). The open-loop system is unstable when $b(x)/a$ is large in absolute value. The objective is to stabilize the zero equilibrium of the plant.

A more general plant can be handled as outlined in Section VIII.

III. Transformation Into the Observer Canonical Form

We start by transforming the plant into the so-called observer canonical form, in which the unknown parameters multiply the output. Consider the transformation[1]

$$B(x) = A(x) - \int_0^x p(x,y)A(y)\,\mathrm{dy} \tag{4}$$

[1] In the rest of the paper time dependence is omitted when this does not lead to confusion.

where the complex valued function $p(x,y)$ is the solution of the PDE

$$p_{xx}(x,y) - p_{yy}(x,y) = \frac{b(y)}{a}\, p(x,y) \quad (5)$$

$$p(1,y) = 0 \quad (6)$$

$$p(x,x) = \frac{1}{2a}\int_x^1 b(s)\,\mathrm{ds}\ . \quad (7)$$

It is straightforward to verify that the transformation (4) maps (1)–(3) into

$$B_t(x,t) = aB_{xx}(x,t) + \theta(x)B_x(0,t) \quad (8)$$

$$B(0,t) = 0 \quad (9)$$

$$B(1,t) = U(t)\,, \quad (10)$$

where the new unknown parameter is $\theta(x) = a\,p(x,0)$. Our design will be now pursued for the new system (8)-(10). Note that $B_x(0,t) = A_x(0,t)$ and $B(1,t) = A(1,t)$, i.e. $B_x(0,t)$ is measured and the controller designed for (8)-(10) is applied to the original system without change. Since the PDE (5)–(7) has a twice continuously differentiable solution [10], the transformation (4) is invertible and therefore asymptotic stability of B implies asymptotic stability of A. Since θ will be estimated, there is actually no need to solve the PDE (5)-(7) for the controller implementation.

IV. Non-Adaptive State Feedback Controller

First we present the nominal control design in the assumption that $\theta(x)$ is known and the full state measurement is available.

Consider the transformation

$$W(x) = B(x) - \int_0^x k(x,y)B(y)\,\mathrm{dy}\ . \quad (11)$$

One can show that if the kernel $k(x,y)$ is the solution of the complex-valued PDE

$$k_{xx}(x,y) = k_{yy}(x,y) \quad (12)$$

$$a\,k(x,x) = -\theta(0) \quad (13)$$

$$a\,k(x,0) = -\theta(x) + \int_0^x k(x,y)\theta(y)\,\mathrm{dy}\ , \quad (14)$$

then (11) and the control law

$$U = B(1) = \int_0^1 k(1,y)B(y)\,\mathrm{dy} \quad (15)$$

map the system (8)-(10) into the exponentially stable system

$$W_t(x) = aW_{xx}(x) \quad (16)$$

$$W(0) = W(1) = 0\,. \quad (17)$$

Exponential stability of (16)–(17) along with bounded invertibility of the transformation (11) ensure that the closed-loop system (8)–(9), (15) is exponentially stable around zero equilibrium.

Note that the solution of (12)–(14) is easier to compute if we set $k(x,y) = \eta(x-y)$ (which satisfies (12)) and then solve the one dimensional integral equation

$$a\,\eta(x) = -\theta(x) + \int_0^x \eta(x-y)\theta(y)\,\mathrm{dy}\ .$$

V. Observer

The next step is to design input and output filters that will provide the estimation of the system state and will also drive the update law for the parameter estimation.

Let us consider the following output filter (which can be viewed as a family of filters in the parameter $\xi \in [0,\,1]$)

$$\phi_t(x,\xi) = a\phi_{xx}(x,\xi) + \delta(x-\xi)B_x(0) \quad (18)$$

$$\phi(0,\xi) = 0 \quad (19)$$

$$\phi(1,\xi) = 0 \quad (20)$$

and the input filter

$$\psi_t(x) = a\psi_{xx}(x) \quad (21)$$

$$\psi(0) = 0 \quad (22)$$

$$\psi(1) = B(1)\,. \quad (23)$$

These two filters can be used to build an observer of the state, with error

$$e(x) = B(x) - \psi(x) - \int_0^1 \theta(\xi)\phi(x,\xi)\,\mathrm{d}\xi\ . \quad (24)$$

The error $e(x)$ of the observer is governed by the exponentially stable complex heat equation

$$e_t(x) = ae_{xx}(x)\,, \quad e(0) = e(1) = 0\,.$$

as can be proved by simple substitution.

VI. Update Law

The observer error (24) provides us with the parametric model to build an estimator for $\theta(x)$. The parametric model is obtained by taking the spatial derivative of the error (24) on the boundary where the measurement is available:

$$e_x(0) = B_x(0) - \psi_x(0) - \int_0^1 \theta(\xi)\phi_x(0,\xi)\,\mathrm{d}\xi\ . \quad (25)$$

The estimation error

$$\hat{e}_x(0) = B_x(0) - \psi_x(0) - \int_0^1 \hat{\theta}(\xi)\phi_x(0,\xi)\,\mathrm{d}\xi \quad (26)$$

can then be used to drive the standard normalized gradient update law[2]

$$\hat{\theta}_t(x) = \gamma(x)\frac{\hat{e}_x(0)\phi_x^*(0,x)}{1+\|\phi_x(0,\cdot)\|^2}\,. \quad (27)$$

where $\gamma(x)$ is a positive adaptation gain function.

Lemma 1: The following properties hold:[3]

$$\frac{\hat{e}_x(0)}{\sqrt{1+\|\phi_x(0,\cdot)\|^2}}\,,\ \|\hat{\theta}_t\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty \quad (28)$$

$$\|e_x\|,\ \|\tilde{\theta}\| \in \mathcal{L}_\infty,\ \ |e_x(0)| \in \mathcal{L}_2\,. \quad (29)$$

Proof: Consider the Lyapunov function

$$V = V_1 + V_2\,,$$

[2]The superscript star indicates the complex conjugate.

[3]By $\mathcal{L}_2$ and $\mathcal{L}_\infty$ we denote the temporal norms on $[0,\infty)$.

where

$$V_1 = \frac{1}{2}\int_0^1 \frac{|\tilde{\theta}(\xi)|^2}{\gamma(x)}\,\mathrm{d}\xi\ , \quad V_2 = \frac{1}{2}\int_0^1 |e_x(\xi)|^2\,\mathrm{d}\xi\ . \quad (30)$$

The time derivatives of V_1 and V_2 are

$$\begin{aligned}\dot{V}_1 &= \mathrm{Re}\left\{\int_0^1 \frac{\tilde{\theta}(\xi)^*\tilde{\theta}_t(\xi)}{\gamma(\xi)}\,\mathrm{d}\xi\right\} = \\ &= -\mathrm{Re}\left\{\frac{-\hat{e}_x(0)}{1+\|\phi_x(0,\cdot)\|^2}\int_0^1 \tilde{\theta}^*\phi_x^*(0,\xi)\,\mathrm{d}\xi\right\} = \\ &= -\mathrm{Re}\left\{\frac{-\hat{e}_x(0)}{1+\|\phi_x(0,\cdot)\|^2}\,[\hat{e}_x^*(0)-e_x^*(0)]\right\} \le \\ &\le -\frac{|\hat{e}_x(0)|^2}{1+\|\phi_x(0,\cdot)\|^2} + \frac{|\hat{e}_x(0)||e_x(0)|}{\sqrt{1+\|\phi_x(0,\cdot)\|^2}}\ ,\end{aligned}$$

and

$$\begin{aligned}\dot{V}_2 &= \mathrm{Re}\left\{\int_0^1 e_x(\xi)e_{tx}^*(\xi)\,\mathrm{d}\xi\right\} = \\ &= \mathrm{Re}\left\{e_x(\xi)e_t^*(\xi)|_0^1 - \int_0^1 e_{xx}(\xi)e_t^*(\xi)\,\mathrm{d}\xi\right\} = \\ &= -\mathrm{Re}(a)\|e_{xx}\|^2 = -\|e_{xx}\|^2\,.\end{aligned}$$

Using the inequality $|e_x(0)| \le \|e_{xx}\|$ and Young's inequality we get

$$\begin{aligned}\dot{V}\ &\le \frac{-|\hat{e}_x(0)|^2}{1+\|\phi_x(0,\cdot)\|^2} + \frac{|\hat{e}_x(0)||e_x(0)|}{\sqrt{1+\|\phi_x(0,\cdot)\|^2}} - \|e_{xx}\|^2 \\ &\le -\frac{1}{2}\left[\frac{|\hat{e}_x(0)|^2}{1+\|\phi_x(0,\cdot)\|^2} + \|e_{xx}\|^2\right]\end{aligned}$$

and from this we conclude that signals $\frac{\hat{e}_x(0)}{\sqrt{1+\|\phi_x(0,\cdot)\|^2}}$, $\|e_{xx}\|$, and $|e_x(0)|$ are in $\mathcal{L}_2$. From the definitions (30) we get boundedness of $\|\tilde{\theta}\|$, $\|\hat{\theta}\|$, and $\|e_x\|$. Moreover, as $\hat{e}_x(0) = e_x(0) - \int_0^1 \tilde{\theta}(\xi)\phi_x(0,\xi)\,\mathrm{d}\xi$, we get $\frac{\hat{e}_x(0)}{\sqrt{1+\|\phi_x(0,\cdot)\|^2}} \in \mathcal{L}_\infty$ and from the update law (27) $\|\hat{\theta}_t\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. ■

VII. MAIN RESULT

The main result of the paper is stated in the following theorem.

Theorem 2: Consider the system (1)–(2) and the controller

$$U = \int_0^1 \hat{k}(1,y)\left[\psi(y) + \int_0^1 \hat{\theta}(\xi)\phi(y,\xi)\,\mathrm{d}\xi\right]\,\mathrm{d}y$$

where the kernel $\hat{k}(x,y) = \hat{\eta}(x-y)$ is the solution of

$$a\,\hat{\eta}(x) = -\hat{\theta}(x) + \int_0^x \hat{\eta}(x-y)\hat{\theta}(y)\,\mathrm{d}y\ ,$$

the filters ψ and ϕ are defined by (18)–(20), (21)-(23) and the estimate $\hat{\theta}$ is updated according to (27). If the closed loop system has a solution $(A,\psi,\phi,\hat{\theta}) \in C([0,\infty), H_1(0,1))$, then for any $\hat{\theta}(x,0)$ and any initial conditions $A(\cdot,0)$, $\psi(\cdot,0)$, $\phi(\cdot,\xi,0) \in H_1(0,1)$, the signals $\hat{\theta}$, ϕ, ψ and A are bounded for all $x \in [0,1]$ and

$$\lim_{t\to\infty}\max_{x\in[0,1]} A(x,t) = 0\,. \quad (31)$$

Proof: Consider the backstepping transformation

$$w(x) = T[h](x) = h(x) - \int_0^x \hat{k}(x,y)h(y)\,\mathrm{d}y \quad (32)$$

applied to the function

$$h(x) = \psi(x) + \int_0^1 \hat{\theta}(\xi)\phi(x,\xi)\,\mathrm{d}\xi\ . \quad (33)$$

The inverse transformation of (32) is

$$h(x) = w(x) + \int_0^x \hat{l}(x,y)w(y)\,\mathrm{d}y\ , \quad (34)$$

where $\hat{k}(x,y)$ and $\hat{l}(x,y)$ are related by

$$\hat{k}(x,y) - \hat{l}(x,y) = -\int_y^x \hat{l}(x,\xi)\hat{k}(\xi,y)\,\mathrm{d}\xi\ , \quad (35)$$

which is satisfied by

$$\hat{l}(x,y) = \frac{-\hat{\theta}(x-y)}{a}\,. \quad (36)$$

We can then derive the following system for w:

$$\begin{aligned} w_t(x) &= aw_{xx}(x) - a\hat{k}(x,0)\hat{e}_x(0) - \\ &- \int_0^x w(y)\left[\hat{l}_t(x,y) - \int_y^x \hat{l}_t(x,\xi)\hat{k}(\xi,y)\,\mathrm{d}\xi\right]\,\mathrm{d}y + \\ &+ T\left[\int_0^1 \hat{\theta}_t(\xi)\phi(x,\xi)\,\mathrm{d}\xi\right] \quad (37)\\ w(0) &= w(1) = 0\,. \quad (38)\end{aligned}$$

Rewriting the filter (18)–(20) in the form

$$\begin{aligned}\phi_t(x,\xi) &= a\phi_{xx}(x,\xi) + \delta(x-\xi)(w_x(0)+e_x(0)) \quad (39)\\ \phi(0,\xi) &= \phi(1,\xi) = 0\,, \quad (40)\end{aligned}$$

we obtain two interconnected systems driven by signals $e_x(0)$ and $\hat{\theta}_t$, which are "asymptotically small" in a sense (28)-(29).

In the rest of the proof we are going to need the bounds on $\hat{k}$, $\hat{l}$, and $\hat{l}_t$. From (35)-(36) and using Gronwall inequality, we have

$$\left|\hat{l}(x,y)\right| \le L_0, \quad \left|\hat{k}(x,y)\right| \le K_0 = L_0 e^{L_0}\ , \quad (41)$$

where $L_0 = \frac{1}{|a|}\max_{t\ge 0}\|\hat{\theta}\|_\infty$. Moreover, we have $\left|\hat{l}_t(x,y)\right| \le \left|\frac{\hat{\theta}_t(x-y)}{a}\right|$, which according to (28) is bounded and square integrable.

In the rest of the proof, we give the stability estimates without the intermediate steps due to the lack of space. We start with a Lyapunov function

$$V_1 = \frac{1}{2}\|\phi\|^2\,.$$

One can estimate its time derivative as

$$\begin{aligned}\dot{V}_1 =\le &-\left(1 - c_1 - \frac{1}{2c_3}\right)\|\phi_x\|^2 + l_1 + \\ &+ l_1\|\phi_x\|^2 + c_2\,\|\phi_{xx}\|^2 + \frac{c_3}{2}\|w_{xx}\|^2\,,\end{aligned}$$

where l_1 is a generic function of time in $\mathcal{L}_1$. In the above estimate we used the inequality $\int_0^1 \phi(x,x)\mathrm{dx} \le \|\phi_x\|$ as well as Cauchy-Schwartz and Poincare inequalities and properties (28)–(29). Here and later by c_i we denote arbitrary positive constants that will be chosen at the end of the proof.

Consider another Lyapunov function

$$V_2 = \frac{1}{2}\|w\|^2 .$$

Its time derivative $\dot{V}_2 = \mathrm{Re} \int_0^1 w_t(x)w^*(x)\ \mathrm{dx}$ can be evaluated separately for the different terms on the right-hand side of (37):

Term 1:

$$\mathrm{Re} \int_0^1 a w_{xx}(x) w^*\ \mathrm{dx} = -\|w_x\|^2 .$$

Term 2:

$$\begin{aligned}&\mathrm{Re} \int_0^1 -a\hat{k}(x,0)\hat{e}_x(0)w^*(x)\ \mathrm{dx} \\ &\le c_4\|w\|^2 + c_5\|\phi_{xx}\|^2 + l_1\|w\|^2 + l_1 .\end{aligned}$$

Term 3:

$$\begin{aligned}&\mathrm{Re} \int_0^1 w^*(x)\int_0^x w(y)\left[\hat{l}_t(x,y) - \int_y^x \hat{l}_t(x,\xi)\hat{k}(\xi,y)\ \mathrm{d}\xi\right]\mathrm{dydx} \\ &\le l_1\|w\|^2 + c_6\|w\|^2 .\end{aligned}$$

Term 4:

$$\begin{aligned}&\mathrm{Re} \int_0^1 T\left[\int_0^1 \hat{\theta}_t(\xi)\phi(x,\xi)\ \mathrm{d}\xi\right] w^*(x)\ \mathrm{dx} \\ &\le l_1\|w\|^2 + c_7\|\phi\|^2 .\end{aligned}$$

We have

$$\begin{aligned}\dot{V}_2 \le& -\|w_x\|^2 + (c_4+c_6)\|w\|^2 + c_7\|\phi\|^2 + \\ &+ c_5\|\phi_{xx}\|^2 + l_1\|w\|^2 + l_1 .\end{aligned}$$

Using a third Lyapunov function

$$V_3 = \frac{1}{2}\|\phi_x\|^2$$

we get

$$\dot{V}_3 \le -\left[1 - c_8 - \frac{c_9}{2}\right]\|\phi_{xx}\|^2 + l_1\|\phi_x\|^2 + \frac{1}{2c_9}\|w_{xx}\|^2.$$

The fourth, and last, Lyapunov function is

$$V_4 = \frac{1}{2}\|w_x\|^2 .$$

Its time derivative is

$$\begin{aligned}\dot{V}_4 \le& -[1-c_{10}]\|w_{xx}\|^2 + c_{11}\|\phi_{xx}\|^2 + l_1\|\phi_x\|^2 \\ &+ l_1\|w\|^2 + l_1\|\phi\|^2 + l_1 .\end{aligned}$$

Taking the total Lyapunov function as

$$V = V_1 + V_2 + V_3 + \frac{4}{3}V_4$$

and choosing $c_3 = 3/2$, $c_4+c_6 = 1/9$, $c_2+c_5 = c_8 = c_{10} = c_7 = 1/18$, $c_{11} = 2/27$, $c_1 = 1/9$, $c_9 = 4/3$, we obtain

$$\dot{V} \le -\frac{1}{9}V + l_1 V + l_1 . \tag{42}$$

Boundedness

Once we proved the inequality (42), by Lemma A.1 we conclude that all the signals $\|w\|$, $\|\phi\|$, $\|w_x\|$ and $\|\phi_x\|$ belong to $\mathcal{L}_2\cap\mathcal{L}_\infty$. Using the inverse transformation (34) and the bound (41), we get that $\|h\|$ too belongs to $\mathcal{L}_2\cap\mathcal{L}_\infty$. From the definition (33) of h, this implies that the state of the input filter ψ is also in $\mathcal{L}_2\cap\mathcal{L}_\infty$. Finally, from the equation of the observer (24) and from the transformation (4), boundedness of $B(x)$ and $A(x)$ follows. Note that by Agmon inequality we also get pointwise (in space) boundedness of all the signals.

Regulation to zero

Using Lemma A.2, we get that $\|w\|$, $\|w_x\|$, $\|\phi\|$ and $\|\phi_x\|$ go to zero as $t\to\infty$. Using the relation, directly derived from (34),

$$h_x(x) = w_x(x) + \int_0^x \hat{l}_x(x,y)w(y)\ \mathrm{dy} + \hat{l}(x,x)w(x)$$

and the bound (41), we can then state than $\|h_x\|\to 0$ when $t\to\infty$. From the definition (33) we obtain the expression for $h_x(x)$

$$h_x(x) = \psi_x(x) + \int_0^1 \hat{\theta}(\xi)\phi_x(x,\xi)\ \mathrm{d}\xi$$

which implies that $\|\psi_x\|$ also goes to zero. From the expression of the spatial derivative of the observer error

$$e_x(x) = B_x(x) - \psi_x(x) - \int_0^1 \theta(\xi)\phi_x(x,\xi)\ \mathrm{d}\xi$$

we obtain that $\|B_x\|\to 0$ as $t\to\infty$. Since $\|B\|$ is bounded, by Agmon inequality this implies that $B(x,t)\to 0$ for all $x\in[0,1]$ as $t\to\infty$. Since (4) is an invertible transformation, we get

$$A(x,t)\to 0 \quad \text{as } t\to\infty,\ \forall x\in[0,1] .$$

■

VIII. Reaction - Advection - Diffusion System

Our approach extends in a straightforward manner to a more general system in which an advection term appears together with the diffusion and the reaction terms. Consider the plant

$$\breve{A}_t(x,t) = a_1\breve{A}_{xx}(x,t) + a_2(x)\breve{A}_x(x,t) + a_3(x)\breve{A}(x,t) \tag{43}$$

with boundary conditions

$$\begin{aligned}\breve{A}(0,t) &= 0\\ \breve{A}(1,t) &= U_c(t)\end{aligned}$$

where $U_c(t)$ is the control signal. We assume that only $\breve{A}_x(0)$ is available for measurement. The functions $a_2(x)$ and $a_3(x)$ are unknown and the constant a_1 is known and satisfies $\mathrm{Re}(a_1) > 0$. With a transformation

$$A(x,t) = \breve{A}(x,t)e^{\frac{1}{2a_1}\int_0^x a_2(s)\,\mathrm{ds}}$$

we eliminate the advection term from (43) at the cost of multiplying the input with a new unknown parameter. The transformed system is

$$\begin{aligned} A_t(x,t) &= aA_{xx}(x,t) + b(x)A(x,t) \\ A(0,t) &= 0 \\ A(1,t) &= \theta_1 U_c(t)\,, \end{aligned}$$

where $a = a_1$ and

$$\begin{aligned} b(x) &= a_3(x) - \frac{a_2^2(x)}{4a_1} - \frac{a_2'(x)}{2} \\ \theta_1 &= e^{\frac{1}{2a_1}\int_0^1 a_2(s)\,\mathrm{d}s}\,. \end{aligned}$$

The only difference compared to the design for reaction-diffusion system is the need to estimate the constant θ_1 along with the function $\theta(x)$. The estimation error (26) becomes

$$\hat{e}_x(0) = B_x(0) - \hat{\theta}_1\psi_x(0) - \int_0^1 \hat{\theta}(\xi)\phi_x(0,\xi)\,\mathrm{d}\xi$$

and the update laws become

$$\hat{\theta}_t(x) = \gamma(x)\frac{\hat{e}_x(0)\phi_x^*(0,x)}{1 + \|\phi_x(0,\cdot)\|^2 + |\psi_x(0)|^2}$$

and

$$\dot{\hat{\theta}}_1 = \gamma_1\,\frac{\hat{e}_x(0)\psi_x^*(0)}{1 + \|\phi_x(0,\cdot)\|^2 + |\psi_x(0)|^2}\,.$$

As the control signal has to be multiplied by $1/\hat{\theta}_1$ before being applied to the original plant, projection is required to keep $\hat{\theta}_1$ positive.

IX. Simulation

We now present the results of numerical implementation of the controllers developed in the paper. The simulations are done using the finite-difference scheme with a 100-step spatial discretization. In Fig. 1 $b(x)$ is shown (solid line), while $a = 0.0234(1 - j)$. In Fig. 2 the open-loop unstable behavior of the system has been plotted.

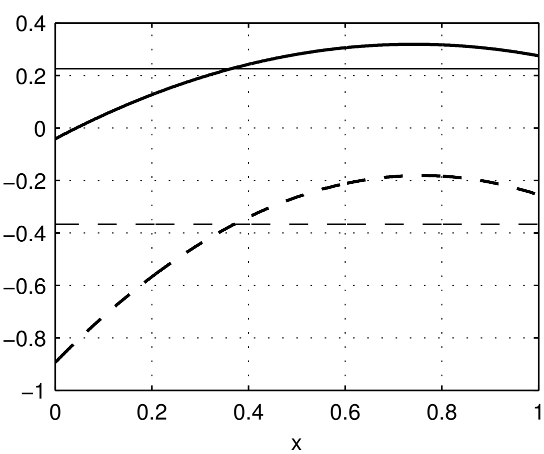

Fig. 1. Real part (solid) and imaginary part (dashed) of the parameter $b(x)$ (bold line) together with the approximate constant $\bar{b}$ (thin line).

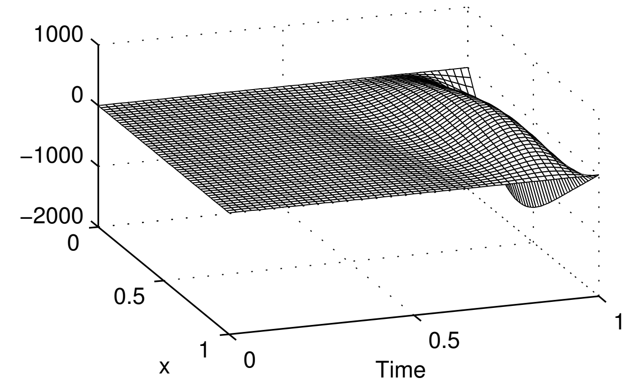

Fig. 2. Real part of the state of the open-loop system. The imaginary part is qualitatively the same.

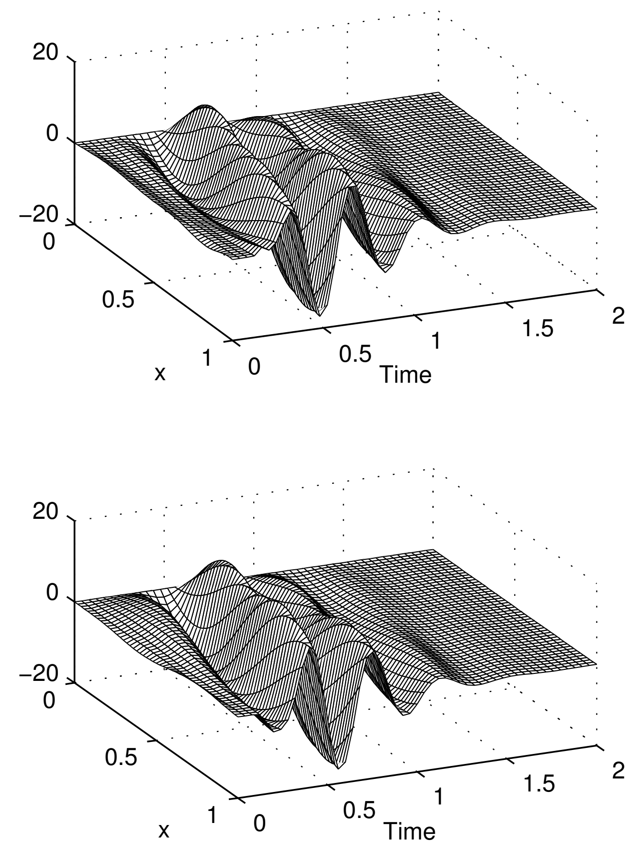

Fig. 3. Real part (top) and imaginary part (bottom) of the state of the closed-loop system.

The evolution of the closed-loop system with adaptive output feedback controller (with $\hat{\theta}$ initialized to zero) is shown in Fig. 3. The control signal is shown in Fig. 4.

Even though the objective of the control design is stabilization and not necessarily the identification of the parameters of the system, the final profile of the parameter estimate $\hat{\theta}$ turns out to be close to the true parameter $\theta(x)$ as can be seen in Fig. 5. This suggests that the plant needs a rather accurate knowledge of the system parameter $b(x)$ to be controlled to

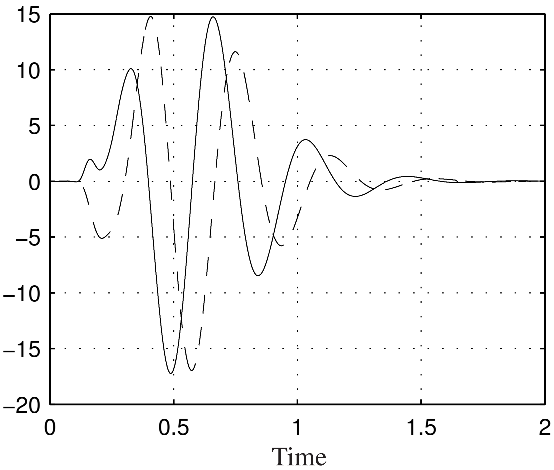

Fig. 4. Real part (solid) and imaginary part (dashed) of the control signal in the controlled case.

zero. To verify this intuitive statement, we approximated the unknown function $b(x)$ by a its mean value $\bar{b}$ (plotted as a thin line in Fig. 1). The controller designed on the basis of this approximation fails to stabilize the plant, as illustrated in Fig. 6.

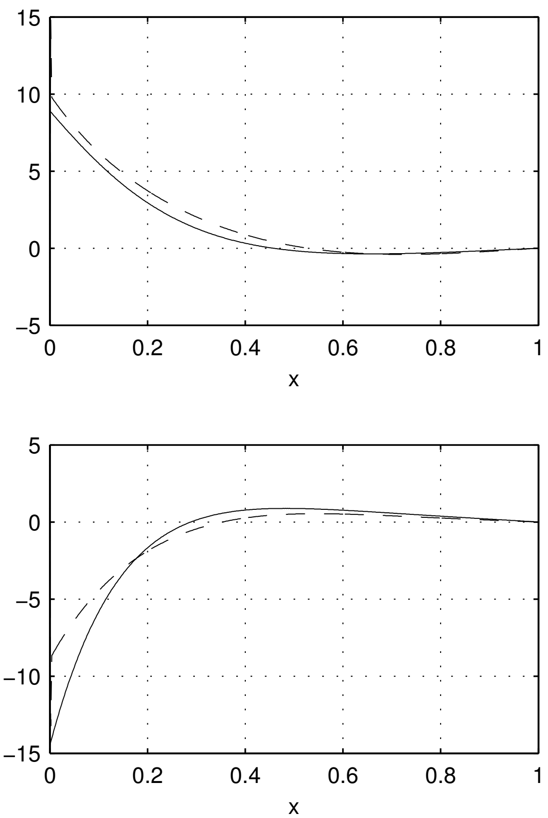

Fig. 5. Real part (top) and imaginary part (bottom) of the parameter θ (solid) with the final profile of its estimate $\hat{\theta}$ (dashed).

APPENDIX

Lemma A.1 (Lemma B.6 in [6]): Let v, l_1, and l_2 be real valued functions defined on $\mathbb{R}_+$, and let c be a positive

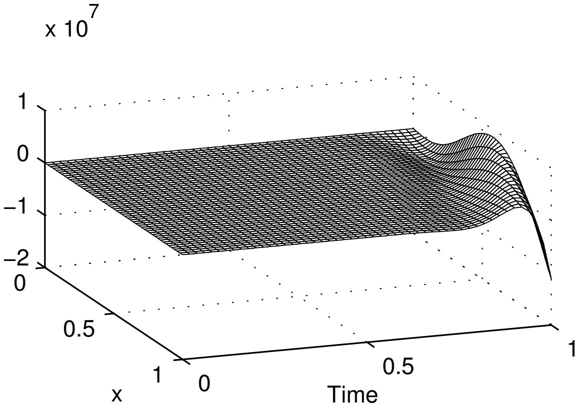

Fig. 6. Real part of the state of the closed-loop system with non-adaptive controller designed on the basis of $\bar{b}$. The imaginary part is qualitatively the same.

constant. If l_1 and l_2 are nonnegative and in $\mathcal{L}_1$ and satisfy the differential inequality

$$\dot{v} \leq -cv + l_1(t)v + l_2(t)\,, \quad v(0) \geq 0$$

then $v \in \mathcal{L}_1 \cap \mathcal{L}_\infty$.

Lemma A.2 (Lemma 3.1 in [7]): Suppose the function $f(t)$ defined on $[0, \infty)$ satisifes the following conditions:

a. $f(t) \geq 0$
b. $f(t)$ is differentiable and there exists a constant M such that $f'(t) \leq M$, $\forall t \geq 0$
c. $f \in \mathcal{L}_1$

Then $\lim_{t\to\infty} f(t) = 0$.

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